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EVEN COVERING PROPERTIES AND SOMEWHAT NORMAL SPACES

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ABSTRACT. A topological space X is said to be somewhat normal provided that for each open cover $\mathscr{C}$ of X $\{\mathrm{st}(x, \mathscr{C}) : x \in X\}$ is a normal cover of X. We show that a completely regular somewhat normal space need not be normal, thereby answering a question of W. M. Fleischman. We note that a collectionwise normal somewhat normal space need not be almost 2-fully normal, as had previously been asserted, and that neither the perfect image nor the perfect preimage of a somewhat normal space has to be somewhat normal.

1. **Introduction**. An open cover $\mathscr{C}$ of a topological space X is said to be *even* provided that there is a neighborhood V of the diagonal of X so that $\{V(x) | x \in X\}$ refines $\mathscr{C}$. It is natural to generalize well-known covering properties by restricting the requirements of the covering properties to even covers. Thus a space is called *evenly Lindelöf* provided that each of its even covers has a countable subcover, and a space is called *evenly (para)compact* provided that each of its even covers has a (locally) finite open refinement. In [5] a space X is called *somewhat normal* provided that for each open cover $\mathscr{C}$ of X, $\{\mathrm{st}(x, \mathscr{C}) | x \in X\}$ is a normal cover. (In subsequent papers, somewhat normal spaces are called spaces with property * by V. L. Kljušin [7] and K-spaces by M. W. Asanov [1]).

Even compactness has already been studied in a different guise in [5], where it is established that a Hausdorff space is evenly compact if and only if it is countably compact. Our first proposition implies immediately that every separable space is evenly Lindelöf and, in light of a technique from [11, Theorem 18], that every ω_1-compact space is evenly Lindelöf as well. It would, of course, be pleasant if every regular evenly Lindelöf space were evenly paracompact, but the Tychonoff Plank provides a counterexample. Nonetheless, every somewhat normal space is evenly paracompact and in a normal space even paracompactness and somewhat normality coincide [1]. W. M. Fleischman observes in [5] that a normal space that is either countably compact or almost 2-fully normal is somewhat normal, but he leaves open the problem whether normality implies, or is implied by, somewhat normality. In [1], Asanov extends

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Fleischman's results by showing that every regular somewhat normal space is completely regular and that every normal M-space is somewhat normal. Moreover, Fleischman's question appears to be answered negatively by Asanov, who asserts that $\omega_1 \times \beta\omega_1$ is a somewhat normal space that is not normal [1, Example 1.4]. We show, however, that $\omega_1 \times \beta\omega_1$ fails to be somewhat normal. Fleischman's question appears to be answered affirmatively by H. L. Shapiro and F. A. Smith who assert that a completely regular somewhat normal space is almost 2-fully normal ([12, Theorem 4.8], [13, Theorem 2.2] and [14, Theorem 2]). Unfortunately, it is easily seen that this assertion is also incorrect, since an example due to R. H. Bing is both collectionwise normal and somewhat normal but not almost 2-fully normal. In this paper we resolve Fleischman's problem by showing that the so-called Hausdorff-gap space, introduced by E. van Douwen in [3], is a somewhat normal space that fails to be normal. In light of the coincidence of somewhat normality and even paracompactness in a normal space, one interesting question remains: Is every normal evenly Lindelöf space somewhat normal? Throughout this note we assume that all spaces under consideration are Hausdorff spaces.

2. **Even covering properties**. We adopt the following notation: For each open cover $\mathcal{R}$ of a space X, $\mathcal{R}^*$ denotes the cover $\{\mathrm{st}(x, \mathcal{R}) | x \in X\}$. R. Engelking outlines a proof of the result stated in [5] that countable compactness and even compactness are equivalent properties [4, page 303]. The remaining implications of the diagram given below are either well known or are consequences of the following two propositions.

PROPOSITION 2.1. *An open cover $\mathcal{C}$ of a space X is even if and only if there is an open cover $\mathcal{R}$ of X so that $\mathcal{R}^*$ refines $\mathcal{C}$.*

COROLLARY. *Every ω_1-compact space is evenly Lindelöf.*

PROOF. Let $\mathcal{C}$ be an open cover of an ω_1-compact space X. By [11, Theorem 18], there is a discrete set D so that $\{\mathrm{st}(x, \mathcal{C}) : x \in D\}$ refines $\mathcal{C}^*$, and since X is ω_1-compact, D is countable. □

PROPOSITION 2.2. [1, Remark 1.2a] *A normal space X is somewhat normal if and only if it is evenly paracompact.*

countably compact
$\updownarrow$
compact $\rightarrow$ evenly compact
$\downarrow$ $\downarrow$
Lindelöf $\rightarrow$ evenly Lindelöf $\leftarrow$ ω_1-compact or separable
a) $\downarrow$ b) $\downarrow$
paracompact $\rightarrow$ evenly paracompact
$\downarrow$ c) $\updownarrow$
normal somewhat normal

a) and regular b) and countably paracompact c) and normal

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It appears from the diagram that normality might imply somewhat normality; however a normal metacompact space that is not paracompact cannot be somewhat normal. (Such a space is given in [10, Example G].) Nonetheless there is some evidence that somewhat normality behaves like normality. Fleischman establishes that every pseudocompact somewhat normal space is countably compact [5, Proposition 8] and further evidence is provided by Proposition 2.3.

A topological space X is a cb-*space* if for each locally bounded real-valued function h on X there is a continuous real-valued function f on X so that $f \geq |h|$. A space X is a cb-space if and only if each countable increasing open cover of X has a locally finite refinement by co-zero sets [8, Theorem 1]. Thus every cb-space is countably paracompact.

PROPOSITION 2.3. *Every countably metacompact somewhat normal space is a cb-space.*

PROOF. Let $\mathcal{C}$ be a countable increasing open cover of a countably metacompact somewhat normal space X and let $\mathcal{R}$ be a point-finite open refinement of $\mathcal{C}$. Then $\mathcal{R}^*$ has a locally finite refinement $\mathcal{G}$ by co-zero sets. Since $\mathcal{C}$ is a directed cover and $\mathcal{R}$ is point finite we have that $\mathcal{R}^*$, and hence $\mathcal{G}$, is a refinement of $\mathcal{C}$. □

We consider the relationship between evenly Lindelöf spaces and evenly paracompact spaces. K. P. Hart has recently shown that M. E. Rudin's Dowker space is almost 2-fully normal and hence evenly paracompact [6]. We know of no evenly Lindelöf normal space that is not evenly paracompact. Note that such a space would have to be a Dowker space. The following example shows that a Tychonoff space that is evenly Lindelöf need not be evenly paracompact.

EXAMPLE 2.4. *Let T be the Tychonoff Plank (i.e., $T = [(\omega_1 + 1) \times (\omega + 1)] - \{(\omega_1, \omega)\}$). Since T is not countably compact, it is not evenly compact and, since every locally finite open cover of T is finite, it follows that T is not evenly paracompact. To see that T is evenly Lindelöf, let $\mathcal{C}$ be an even cover of T. By Proposition 2.1, there is an open cover $\mathcal{R}$ of T so that $\mathcal{R}^*$ refines $\mathcal{C}$. Without loss of generality, we assume that for each limit ordinal $x \in \omega_1$ there is a member of $\mathcal{R}$ of the form $(f(x), x] \times [n_x, \omega]$. By the Pressing-Down Lemma, there is $\beta \in \omega_1$ and a cofinal subset S of ω_1 so that $f(x) = \beta$ for each $x \in S$, and since S is uncountable we may also assume without loss of generality that there is $n \in \omega$, with $n_x = n$ for all $x \in S$. Let $p = (\beta + 1, \omega)$. Then $T - \mathrm{st}(p, \mathcal{R})$ is the union of a countable set and a compact set, and so $\mathcal{R}^*$ has a countable open refinement.*

We now consider whether somewhat normality implies normality. In [2], H. J. Cohen gives an example, attributed to R. H. Bing, of a collectionwise normal space that is not almost 2-fully normal. An argument similar to the one given above shows that Bing's space is evenly paracompact, thus refuting the assertion of Shapiro and Smith that every completely regular somewhat normal space is almost 2-fully normal.

According to Kljušin [7, Proposition 2], both the perfect image and the perfect pre-image of a somewhat normal space is somewhat normal. Using this proposition, Asanov concludes that $\omega_1 \times \beta\omega_1$ is an example of a non-normal somewhat normal space [1, Example 1.4]. We show, however, that this space fails to be somewhat normal and, in fact, we show subsequently that neither direction of Kljušin's proposition obtains.

EXAMPLE 2.5. *The space* $X = \omega_1 \times \beta\omega_1$ *is the perfect pre-image of the somewhat normal space* ω_1 *and yet fails to be somewhat normal. Let* $C_{\omega_1} = \omega_1 \times \omega_1$ *and for each* $\alpha < \omega_1$ *let* $C_\alpha = [0, \alpha] \times (\alpha, \omega_1]$. *Let* $\mathscr{C} = \{C_\alpha | \alpha \leq \omega_1\}$. *Suppose that* $\mathscr{L}$ *is a locally finite refinement of* $\mathscr{C}^*$ *by co-zero sets; since* $\omega_1 \times \beta\omega_1$ *is countably compact,* $\mathscr{L}$ *is finite. We list the members of* $\mathscr{L}$ *as* $L_1, L_2, L_3, \ldots, L_n$ *and choose* $G_1, G_2, G_3, \ldots, G_n$ *members of* $\mathscr{C}^*$ *so that* $L_i \subset G_i$ *for* $1 \leq i \leq n$. *Set* $\Delta = \{(\alpha, \alpha) | \alpha < \omega_1\}$, *set* $T = [0, \omega_1) \times \{\omega_1\}$, *set* $\mathscr{G} = \{G_i | 1 \leq i \leq n\}$, *set* $\mathscr{G}_1 = \{G \in \mathscr{G} | \Delta \subset G\}$ *and set* $\mathscr{G}_2 = \{G \in \mathscr{G} |$ *there is* $\alpha \in \omega_1$ *so that* $(\alpha, \omega_1) \times \{\omega_1\} \subset G\}$. *Then* $\mathscr{G}_1 \neq \emptyset$, $\mathscr{G}_2 \neq \emptyset$, $\mathscr{G} = \mathscr{G}_1 \cup \mathscr{G}_2$ *and both* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *are linearly ordered by inclusion. For notational convenience, we assume, without loss of generality, that* G_1 *is the largest member of* $\mathscr{G}_1$ *and* G_2 *is the largest member of* $\mathscr{G}_2$. *Set* $H_1 = \cup\{L \in \mathscr{L} | L \subset G_1\}$ *and set* $H_2 = \cup\{L \in \mathscr{L} | L \subset G_2\}$. *Then* $\{H_1, H_2\}$ *is a co-zero set cover of* X *and so there are open sets* S_1 *and* S_2 *so that* $\overline{S}_1 \subset H_1$, $\overline{S}_2 \subset H_2$ *and* $S_1 \cup S_2 = \omega_1 \times \beta\omega_1$ [4, Exercise 7b, Page 481]. *Then* $\Delta \subset S_1 \subset \overline{S}_1 \subset G_1$. *There is an* $\alpha \in \omega_1$ *so that* $(\alpha, \omega_1) \times \{\omega_1\} \subset \overline{S}_1$, *whereas* $G_1 \cap T$ *contains no cofinal subset of* T — *a contradiction.*

We note that Example 2.5 shows that a non-normal evenly (para)-compact space need not be somewhat normal. Thus normality cannot be replaced by complete regularity in Proposition 2.2. For use in our last example, we note here that the method of proof above establishes that the following space S is not somewhat normal. Let $S = \{(x, y) \in \omega_1 \times \beta\omega_1 | x \leq y\}$. The points of T and Δ are given their usual neighborhoods and the remaining points are isolated. The modified proof uses that the two copies of ω_1 in S are countably compact.

Our next example, due to van Douwen, shows that a regular somewhat normal space need not be normal. The example is somewhat involved, but we obtain lagniappe that the perfect image of a somewhat normal space need not be somewhat normal and that somewhat normality is not a closed hereditary property.

EXAMPLE 2.6. *A* separable *first countable* countably paracompact *(collectionwise)* Hausdorff locally compact zero-dimensional *somewhat normal space* X *that fails to be normal. We adopt the notation and terminology of* [3]. *All the properties of the example except somewhat normality are established in* [3], *and it follows from the non-italicized properties that the space is a completely regular evenly paracompact space. (Since* X *is not normal, Proposition 2.2 does not obtain.)*

The set X is $L_0 \cup L_1 \cup \omega$ where $L_0 = \{0\} \times (\omega_1 - \{0\})$ and $L_1 = \{1\} \times (\omega_1 - \{0\})$. In order to show that van Douwen's space X is somewhat normal it is sufficient to know that X is topologized in such a way that

(a) the points of ω are isolated,

(b) each point $x = (0, x')$ of L_0 has a countable open-closed neighborhood base $\{G(\alpha, x) | 0 < \alpha < x'\}$ where for each α $G(\alpha, x) = (\{0\} \times (\alpha, x']) \cup B_\alpha$ where $B_\alpha \subset \omega$,

(c) each point $x = (1, x')$ of L_1 has a countable open-closed neighborhood base $\{G(\alpha, x) | 0 < \alpha < x'\}$ where for each α $G(\alpha, x) = (\{1\} \times (\alpha, x']) \cup C_\alpha$ where $C_\alpha \subset \omega$, and

(d) L_0 and L_1 are disjoint closed sets that cannot be separated in X.

Let $\mathscr{C}$ be an even cover of X and let $\mathscr{H}$ be an open cover of X so that $\mathscr{H}^*$ refines $\mathscr{C}$. For each $x \in X$ choose a basic open and closed neighborhood G_x of x contained in some member of $\mathscr{H}$. Let $\mathscr{G} = \{G_x | x \in X\}$. In order to show that X is somewhat normal, it suffices to show that $\mathscr{G}^*$ has a locally finite refinement by co-zero sets. For $i = 0, 1$ set $L_i' = \{x' \in \omega_1 | (i, x') \in L_i\}$. By the Pressing-Down Lemma, there exist for $i = 0, 1$ ordinals β_i' and cofinal subsets S_i' of L_i' so that for $s = (i, s')$, $s' \in S_i'$ we have $\{i\} \times (\beta_i', s'] = G_s \cap L_i$. For $i = 0, 1$ set $S_i = \{(i, s') | s' \in S_i'\}$ and for each $n \in \omega$ set $f(n) = \sup \{\min \{s_0', s_1'\} | n \in G_{(0, s_0')} \cap G_{(1, s_1')}, (0, s_0') \in S_0, (1, s_1') \in S_1\}$.

We argue by contradiction that for some $n \in \omega, f(n) = \omega_1$. Suppose that $f(n) < \omega_1$ for each $n \in \omega$ and set $\alpha = \sup \{f(n) | n \in \omega\}$. Note that $\alpha < \omega_1$. For $i = 0, 1$ set $H_i = \cup\{G_s | s \in S_i, s = (i, s') \text{ and } s' > \alpha\}$. If $x \in H_0 \cap H_1$, there are $s_0 \in S_0$ and $s_1 \in S_1$ so that $s_0 = (0, s_0'), s_1 = (1, s_1'), s_0', s_1' > \alpha$ and $x \in G_{s_0} \cap G_{s_1} \cap \omega$. By definition, $\alpha < \min \{s_0', s_1'\} \leq f(x) \leq \alpha$ – a contradiction. Thus H_0 and H_1 are disjoint. Set $K_i = \{i\} \times (0, \beta_i']$ for $i = 0, 1$. Evidently for $i = 0, 1$, $L_i - K_i \subset H_i$ and since each K_i is compact, there are $M_i = \cup\{P_{ij} | 1 \leq j \leq n_i\}$, where each P_{ij} is a closed and open basic neighborhood of a point of K_i so that M_0 and M_1 are disjoint sets such that $K_0 \subset M_0$ and $K_1 \subset M_1$. Set $A_0 = [(X - M_1) \cap H_0] \cup M_0$ and set $A_1 = [(X - M_0) \cap H_1] \cup M_1$. Then A_0 and A_1 are disjoint open sets such that $L_0 \subset A_0$ and $L_1 \subset A_1$. This contradicts condition (d).

It follows that there is an $n \in \omega$ so that $f(n) = \omega_1$. For $i = 0, 1$ there are cofinal subsets A_i' of S_i' such that $n \in G_{(i, \alpha_i')}$ for all $\alpha_i' \in A_i'$. Set $B = [(\cup\{G_{(0, \alpha_0')} | \alpha_0' \in A_0'\}) \cup (\cup\{G_{(1, \alpha_1')} | \alpha_1' \in A_1'\})] - (M_0 \cup M_1)$. Evidently B is an open set and $(L_0 - K_0) \cup (L_1 - K_1) \subset B \subset \text{st}(n, \mathscr{G})$. Let $p \in \bar{B} - B$. Points of ω are isolated. Suppose, without loss of generality, that $p \in L_0$. Then $p \in K_0 \subset M_0$, which is an open set containing p that is disjoint from B – a contradiction. Thus B is closed as well as open. Since $K_0 \cup K_1$ is compact, there is a finite subcollection $\{G_{k_j}\}_{j=1}^m$ of $\mathscr{G}$ so that $K_0 \cup K_1 \subset \cup \{G_{k_j} | 1 \leq j \leq m\}$. Set $\mathscr{R}' = \{G_{k_j} | 1 \leq j \leq m\} \cup \{B\}$ and set $\mathscr{R}'' = \{\{x\} | x \notin \cup\mathscr{R}'\}$. Then $\mathscr{R}' \cup \mathscr{R}''$ is a locally finite refinement of $\mathscr{G}^*$ by open and closed sets. □

Since the space X described above is a locally compact countably paracompact separable Hausdorff space in which there are two disjoint copies L_0 and L_1 of ω_1 that cannot be separated by disjoint open sets, the space constructed from $X \times \omega$ by identifying points as described in [9, page 240] yields a countably paracompact locally compact separable Hausdorff space that is not a cb-space. Call the resulting space Y and let i denote the identification map. Then $i: X \times \omega \to Y$ is a perfect map, $X \times \omega$ is somewhat normal, and, by Proposition 2.3, Y is not somewhat normal. Thus we have seen that neither implication of [7, Proposition 2] obtains. Since every open-and-closed subspace of a somewhat normal space is somewhat normal, had Kljušin's proposition been correct, it would have followed that somewhat normality would be a closed hereditary property [4, Theorem 3.7.29].

EXAMPLE 2.7. *Let $Z = S \cup X$ be the disjoint union of the space S described in Example 2.5 and the space X of Example 2.6. Both S and X contain two copies of ω_1. We identify pointwise one copy of ω_1 in S with a copy of ω_1 in X and then identify pointwise the remaining two copies of ω_1, one in S and one in X. The resulting space is a somewhat normal space that contains a closed copy of S. We omit the proof that the resulting space is somewhat normal, since this proof mimics the arguments of Example 2.6.*

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