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Cremona Maps of de Jonquières Type

Ivan Pan and Aron Simis

Abstract. This paper is concerned with suitable generalizations of a plane de Jonquières map to higher dimensional space $\mathbb{P}^n$ with $n \geq 3$. For each given point of $\mathbb{P}^n$ there is a subgroup of the entire Cremona group of dimension n consisting of such maps. We study both geometric and group-theoretical properties of this notion. In the case where $n = 3$ we describe an explicit set of generators of the group and give a homological characterization of a basic subgroup thereof.

Introduction

Let k denote an algebraically closed field of characteristic zero, and let $\mathbb{P}^n = \mathbb{P}^n_k$ denote the projective space of dimension n over k. A classical problem is to understand the structure of the k-automorphism group of the function field of $\mathbb{P}^n$ or, equivalently, its *Cremona group* $\mathrm{Cr}(\mathbb{P}^n)$ of birational maps.

An important subgroup of $\mathrm{Cr}(\mathbb{P}^n)$ is the group $\mathrm{PGL}(n+1, k)$ of projective (linear) transformations. For $n = 1$ one easily sees that $\mathrm{Cr}(\mathbb{P}^1) = \mathrm{PGL}(2, k)$, the so-called group of Möbius transformations over k. For $n = 2$ a celebrated result states that $\mathrm{Cr}(\mathbb{P}^2)$ is generated by $\mathrm{PGL}(3, k)$ and the standard quadratic map of $\mathbb{P}^2$. The first proof was given by M. Noether ([13, 15]). Unfortunately, the proof contained one gap. A complete proof was later given by G. Castelnuovo. For an interesting account of the history of this result, including the contributions by Castelnuovo and others, the reader is referred to [1, Chap. 8]. Now, in his version of the theorem, Castelnuovo gives an alternative approach by first proving that every plane Cremona transformation is a composite of de Jonquières maps, then by showing that any such map is a composite of projective transformations and the standard quadratic map. This shows the prominence of de Jonquières maps in the classical Cremona map theory.

A *de Jonquières map* (in honor of [12] where it was been first studied) is a plane Cremona map $\mathfrak{F}$, say, of degree $d \geq 2$, satisfying any one of the following equivalent conditions:

- $\mathfrak{F}$ has homaloidal type $(d\,; d-1, 1^{2d-2})$.
- There exists a point $o \in \mathbb{P}^2$ such that the restriction of $\mathfrak{F}$ to a general line passing through o maps it birationally to a line passing through o.

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- Up to projective coordinate change (source and target) $\mathfrak{F}$ is defined by d-forms $\{qx_0, qx_1, f\}$ such that $f, q \in k[x_0, x_1, x_2]$ are relatively prime x_2-monoids one of which at least has degree 1 in x_2.

The first alternative means that the base cluster of the map has one proper point of multiplicity $d-1$ and $2d-2$ simple, possibly infinitely near, points. The second alternative is a "dynamical" notion emphasizing the behavior of the map with respect to proper linear subspaces of $\mathbb{P}^2$. Finally, the third alternative stresses the shape of the defining forms or, as one might say, the underlying indeterminacy locus of $\mathfrak{F}$, with an emphasis on the monoid shape of intervening forms. We refer to [17, Proposition 2.2, Corollary 2.3] for a simple geometric proof, and to [10, Proposition 2.3, Remark 2.4] for a later argument stressing the algebraic fundamentals of plane Cremona maps.

Up to a projective change of coordinates, one can take $o = (0\!:\!0\!:\!1)$, and hence the explicit format delivered by the third alternative is

$$\big((c(x_0,x_1)x_2 + d(x_0,x_1))\,x_0 : (c(x_0,x_1)x_2 + d(x_0,x_1))\,x_1 : a(x_0,x_1)x_2 + b(x_0,x_1)\big),$$

$a, b, c, d \in k[x_0, x_1]$ are forms such that $ad - bc \neq 0$ and of degrees

$$\deg(a) = \deg(d) = \deg(b) - 1 = \deg(c) + 1.$$

Note that the fraction $(a(x_0,x_1)x_2 + b(x_0,x_1))/(c(x_0,x_1)x_2 + d(x_0,x_1))$ defines a Möbius transformation in the variable x_2 over the function field of $\mathbb{P}^1$, *i.e.*, the subfield of $k(x_0, x_1)$ consisting of homogeneous rational functions of degree 0.

In general, the problems concerning the Cremona group have (at least) two facets: the group theoretic questions, such as booking generators and relations of some important subgroups, and the geometric questions that deal with classifying types according to the geometric properties or the group constituents. These two facets are interspersed, and it often happens that the geometric results help visualize the group structure. In this paper we deal with both aspects of the theory and, in addition, bring up the underlying commutative algebra in terms of the ideal theoretic and homological side of the so-called base ideals of the maps.

We will deal with suitable generalizations of de Jonquières maps to higher dimensional space $\mathbb{P}^n$ with $n \geq 3$. These generalizations will be subsumed under the general frame of *maps of de Jonquières type*. For $n \geq 3$ we will study elements of the Cremona group $\mathrm{Cr}(n) = \mathrm{Cr}(\mathbb{P}^n)$ satisfying a condition akin to the second alternative above. More precisely, for a point $o \in \mathbb{P}^n$ and a positive integer m we consider the Cremona transformations that map a general m-dimensional linear subspace passing through o onto another such subspace. Fixing the point o, these maps will form a subgroup $\mathrm{J}_o(m\,;\mathbb{P}^n) \subset \mathrm{Cr}(n)$. This subgroup is our main concern in this work.

Let us focus on the case $m = 1$. The strategy is based on an exact sequence of groups

$$1 \longrightarrow \mathrm{PGL}(2, k(\mathbb{P}^{n-1})) \longrightarrow \mathrm{J}_o(1;\mathbb{P}^n) \xrightarrow{\rho} \rho\,\mathrm{Cr}(\mathbb{P}^{n-1}) \longrightarrow 1,$$

where ρ is a natural homomorphism that one may define by thinking of $\mathbb{P}^{n-1}$ in $\mathrm{Cr}(\mathbb{P}^{n-1})$ as the set of lines passing through o (see Section 1 for a more precise definition). Hence $\mathrm{PGL}(2, k(\mathbb{P}^{n-1}))$ corresponds to the subgroup consisting of de Jonquières transformations that stabilize a general line passing through o. The elements of this subgroup are the Möbius transformations over the function field $k(\mathbb{P}^{n-1})$ of $\mathbb{P}^{n-1}$. Moreover, one can see that the sequence is right split (see Section1).

One does not know much about the structure of $\mathrm{J}_o(1;\mathbb{P}^n)$ when $n \geq 3$. For example, it is not known whether this group together with the projective linear transformations generate the entire Cremona group $\mathrm{Cr}(\mathbb{P}^n)$, as happens when $n = 2$. It has been proved in [16, Theorem 1] that if $k = \mathbb{C}$ (or, more generally, if k is an algebraically closed field of characteristic zero having uncountably many elements), then any set of group generators of $\mathrm{Cr}(\mathbb{P}^n)$ contains uncountably many non linear transformations. In this paper we show that $\mathrm{J}_o(1;\mathbb{P}^3)$ itself inherits this property, and yet we are still able to describe a complete set of families of maps that generates it.

The paper is divided into three sections. The first section is devoted to the basic definitions and the main properties of the subgroup $\mathrm{J}_o(m\,;\mathbb{P}^n) \subset \mathrm{Cr}(n)$.

The second section establishes the main group theoretic results of the paper, based on information coming from the geometric side. The first result states that any $\mathfrak{F} \in \mathrm{J}_o(1;\mathbb{P}^n)$ contracts a finite number of irreducible hypersurfaces each of which has geometric genus bounded by a number depending on $\deg(\mathfrak{F})$. This is then used to deduce, provided k is an uncountable field, that given any set $\mathcal{G}$ of generators of $\mathrm{J}_o(1;\mathbb{P}^n)$ and an integer $d \geq 2$, any subset $\mathcal{G}_0 \subset \mathcal{G}$ such that $\mathrm{PGL}(2, k(\mathbb{P}^{n-1}))$ is generated by elements of $\mathcal{G}_0$ contains uncountably many elements of degrees $\geq d$. Finally we focus on dimension 3 and show that $\mathrm{J}_o(1;\mathbb{P}^3)$ is generated by its subgroup $\mathrm{J}_o(2;\mathbb{P}^3)$ and by the cubic Cremona map $(x_0x_1x_2 : x_0^2x_2 : x_0^2x_1 : x_1x_2x_3)$.

In the third section of the paper we expand on certain algebraic aspects of rational maps akin to maps of de Jonquières type by stressing homological properties of a class of homogeneous ideals resembling the base ideals of such rational maps. The gist of this section is to take a more abstract view, with an emphasis on the ideal theoretic and homological properties of the base locus of a rational map. For most of the material of this part one can drop the requirement on the characteristic of k and any additional hypothesis on the transcendence degree of k over its prime field.

Constructs of a similar type have been considered in [11] as parametrizations of certain implicit monoid hypersurfaces (see also [6] for a related development).

One goal is to give an ideal theoretic characterization of the elements of a set of generators of the group $J_0(1;\mathbb{P}^n)$ (see Proposition 3.6); by this we mean making explicit the form and free resolution of the base ideals of those generators as rational maps.

Another goal is a homological characterization of the elements of the subgroup $J_0(2;\mathbb{P}^3) \subset J_0(1;\mathbb{P}^n)$ – this is the content of Theorem 3.8. This result hinges on the purely algebraic result of Theorem 3.7.

1 Maps of the Jonquières Type

We will be solely concerned with rational maps of $\mathbb{P}^n$ to itself. A rational map $\mathfrak{F} \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is defined by $n+1$ forms $\mathbf{f} = \{f_0, \dots, f_n\} \subset R := k[\mathbf{x}] = k[x_0, \dots, x_n]$ of the same degree $d \geq 1$, not all null. We often write $\mathfrak{F} = (f_0 : \cdots : f_n)$ to underscore the projective setup. Any rational map can without lost of generality be made to satisfy the condition that $\gcd\{f_0, \dots, f_n\} = 1$. In order to have a well-defined notion of *degree* of $\mathfrak{F}$, we will always assume the latter condition, which means that we will usually be identifying the maps $(f_0 : \cdots : f_n)$ and $(f f_0 : \cdots : f f_n)$ for any nonzero form $f \in R$.

Given a point $o \in \mathbb{P}^n$, we denote by $S_o(m,n)$ the Schubert cycle parameterizing the set of m-dimensional linear subspaces $L \subset \mathbb{P}^n$ containing o; it is known to be an irreducible variety. We will, as usual, identify a member L of the set with the corresponding point of $S_o(m,n)$.

Fix a point $o \in \mathbb{P}^n$ and an integer $1 \le m \le n-1$.

Definition 1.1 A Cremona map $\mathfrak{F}$ of $\mathbb{P}^n$ is a *de Jonquières map of type m with center o* if, given a nonempty open set $U \subset \mathbb{P}^n$ on which $\mathfrak{F}$ induces a biregular morphism onto its image, the following condition holds: $L \in S_o(m,n)$ with $L \cap U \neq \varnothing$ implies $\overline{\mathfrak{F}(L \cap U)} \in S_o(m,n)$, where over-line indicates Zariski closure.

In a more informal way, the condition is that the general member of the Schubert cycle $S_o(m,n)$ is mapped onto a member of $S_o(m,n)$. Note that since the union of contracted linear subspaces by a Cremona map is contained in a finite set of hypersurfaces, the restriction of the Cremona map to a general member of $S_o(m,n)$ is a birational map onto its image.

The set $\mathrm{J}_o(m;\mathbb{P}^n)$ of all de Jonquières maps of type m with center o is a subgroup of the whole Cremona group; it will be referred to as the *de Jonquières group of type m with center o*. We note that, with varying terminology, this notion has appeared elsewhere, *e.g.,* [17, Proposition 2.1] and [7, Section 4.3] where the author called *level* $n-m$ what we call *type m*.

Since a general member of $S_o(m,n)$ is the intersection of two general members in $S_o(m+1,n)$, one easily deduces that $\mathrm{J}_o(m';\mathbb{P}^n) \subset \mathrm{J}_o(m;\mathbb{P}^n)$ for any $m' \ge m$.

Now we focus on the case $m = 1$. Let $H \subset \mathbb{P}^n$ be a hyperplane not containing the point o and let K stand for its field of rational functions over k; note that the projective space of lines passing through o may be identified with H by associating each such line with its intersection with H. By definition, an element $\mathfrak{F} \in \mathrm{J}_o(1;\mathbb{P}^n)$ is a Cremona map of $\mathbb{P}^n$ that acts birationally on the set of lines passing through o, hence $\mathfrak{F}$ induces a birational map $H \dashrightarrow H$. By identifying $H = \mathbb{P}^{n-1}$, we obtain a map

$$\rho\colon \mathrm{J}_o(1;\mathbb{P}^n) \to \mathrm{Cr}(\mathbb{P}^{n-1}), \tag{1.1}$$

which is clearly a group homomorphism.

The group $\mathrm{J}_o(1;\mathbb{P}^n)$ itself was treated in [17]. Here we provide further details about the above map.

For this, we introduce some additional notation, where we set $o = (0 : \cdots : 0 : 1)$ and $H : \{x_n = 0\}$. Given forms $a, b, c, d \in k[x_0, \dots, x_{n-1}] = \mathrm{Sym}_k[H^*]$ of degrees $r-1, r, r-2, r-1 \ge 1$, respectively, such that either $a \neq 0$ or $c \neq 0$ and satisfying $\gcd(ax_n + b, cx_n + d) = 1$, one considers the following two objects:

- The element $\mathfrak{f}_{a,b,c,d} := (ax_n + b)/(cx_n + d) \in k[x_0, \dots, x_{n-1}](x_n)$.
- The rational map $\mathfrak{F}_{a,b,c,d}\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n (n \ge 2)$ defined by

$$\big((cx_n + d)x_0 : \cdots : (cx_n + d)x_{n-1} : ax_n + b\big).$$

Let us emphasize that $ax_n+b \in (x_0,\dots,x_{n-1})k[x_0,\dots,x_{n-1}][x_n]$ always holds, while the same holds for cx_n+d except when its degree as a form is 1. We observe that $\mathfrak{F}_{a,b,c,d}$ is an element of $\mathrm{Cr}(\mathbb{P}^n)$ since the first n coordinates defines the identity map of $\mathrm{Cr}(H)$ up to the identification $H = \mathrm{Proj}(\mathrm{Sym}_k(k[H^*]))$ and the last coordinate is of degree at most 1 in the variable x_n ([17, Proposition 2.2]).

Forms such as ax_n+b, cx_n+d are called *x_n-monoids.*

Proposition 1.2 *Let $\rho\colon \mathrm{J}_o(1;\mathbb{P}^n) \to \mathrm{Cr}(\mathbb{P}^{n-1})$ be as in (1.1). Fix $o = (0\colon\cdots\colon 0\colon 1)$ and $H := \{x_n = 0\}$ and let K stand for the function field of $H \subset \mathbb{P}^n$. Then*

(i) *A Cremona map $\mathfrak{F} \in \mathrm{Cr}(\mathbb{P}^n)$ belongs to $\mathrm{J}_o(1;\mathbb{P}^n)$ if and only if as a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ it has the form $(qg_0\colon\cdots\colon qg_{n-1}\colon f)$, where $(g_0\colon\cdots\colon g_{n-1})$ defines a Cremona map of $H = \mathbb{P}^{n-1}$ and $q, f \in k[x_0,\dots,x_{n-1},x_n]$ are relatively prime x_n-monoids at least one of which has positive x_n-degree.*

(ii) *The group $\mathrm{PGL}(2,K)$ can be identified with the Möbius group whose elements have the form $\mathfrak{f}_{a,b,c,d}$.*

(iii) *The map $\mathfrak{f}_{a,b,c,d} \mapsto \mathfrak{F}_{a,b,c,d}$ is an injective group homomorphism $\psi\colon\ \mathrm{PGL}(2,K) \hookrightarrow \mathrm{J}_o(1;\mathbb{P}^n)$.*

(iv) *$\mathrm{im}(\psi) = \ker(\rho)$; in particular $\mathfrak{F}_{a,b,c,d}$ maps a general line passing through the point $o = (0\colon\cdots\colon 0\colon 1)$ birationally to itself.*

Proof (i) This was proved in [17, Proposition 2.2].

(ii) This is an easy exercise passing to inhomogeneous coordinates $\frac{x_1}{x_0},\dots,\frac{x_{n-1}}{x_0}$.

(iii) By (i), ψ maps to $\mathrm{J}_o(1;\mathbb{P}^n)$. A straightforward computation gives the composition law

$$\mathfrak{F}_{a,b,c,d}\,\mathfrak{F}_{a',b',c',d'} = \mathfrak{F}_{aa'+bc',ab'+bd',ca'+dc',cb'+dc'},$$

which shows that ψ is a group homomorphism. For the injectivity, note that $\mathfrak{F}_{a,b,c,d}$ is the identity map if and only if $b = c = 0, a = d$.

(iv) Clearly ρ maps any $\mathfrak{F}_{a,b,c,d}$ to the identity map of $\mathrm{Cr}(\mathbb{P}^{n-1})$. Conversely, let $\mathfrak{F} \in \ker(\rho)$. By (i), $\mathfrak{F} = (qg_0\colon\cdots\colon qg_{n-1}\colon f)$, for suitable x_n-monoids $q, f \in k[x_0,\dots,x_{n-1},x_n]$ at least one of which has positive x_n-degree. But since ρ maps $\mathfrak{F}$ to the identity of $\mathrm{Cr}(\mathbb{P}^{n-1})$, $(t_0\colon\cdots\colon t_{n-1})$ must be the identity map. This shows that $\mathfrak{F} = \mathfrak{F}_{a,b,c,d}$, with $f = ax_n+b, q = cx_n+d$. ■

Let $o = (0\colon\cdots\colon 0\colon 1)$ and $H = \{x_n = 0\} \subset \mathbb{P}^n$ as before.

As a consequence of the above methods, we observe that $\mathrm{J}_o(1;\mathbb{P}^n)$ has two distinguished subgroups: one is the kernel $\ker(\rho)$, which we have shown to be exactly the subgroup of Cremona maps of the form $\mathfrak{F}_{a,b,c,d}$, for suitable forms $a, b, c, d \in k[x_0,\dots,x_{n-1}]$. Note that these fix a general hyperplane through the point o, since the first n coordinates of the map define the identity map on the fixed hyperplane H avoiding o. The other subgroup is $\mathrm{J}_o(n-1;\mathbb{P}^n) \subset \mathrm{J}_o(1;\mathbb{P}^n)$, whose elements map a general hyperplane through o birationally onto a hyperplane through o (not necessarily fixing the source hyperplane).

The simple geometry behind the relationship between these two subgroups asks for a group-theoretic formulation. And in fact, there is a simple one.

Proposition 1.3 *Let* $\mathrm{PGL}(n+1,k)_o$ *be the subgroup of linear automorphisms of* $\mathbb{P}^n$ *fixing o. Then* $\ker(\rho)$ *is a normal subgroup of* $\mathrm{J}_o(n-1;\mathbb{P}^n)$ *and the equality*

$$\mathrm{J}_o(n-1;\mathbb{P}^n)=\mathrm{PGL}(n+1,k)_o\ker(\rho)$$

holds.

Proof Clearly, both $\ker(\rho)$ and $\mathrm{PGL}(n+1,k)_o$ are subgroups of $\mathrm{J}_o(n-1;\mathbb{P}^n)$ and the first is normal since it is normal in the larger group $\mathrm{J}_o(1;\mathbb{P}^n)$. Therefore, the product of the two subgroups is a subgroup of $\mathrm{J}_o(n-1;\mathbb{P}^n)$. Conversely, let $\mathfrak{F}\in\mathrm{J}_o(n-1;\mathbb{P}^n)$. Expressing it as an element of $\mathrm{J}_o(1;\mathbb{P}^n)$ we know from the previous part that $\mathfrak{F}=(qg_0:\cdots:qg_{n-1}:f)$, for a suitable $\mathfrak{G}=(g_0:\cdots:g_{n-1})\in\mathrm{Cr}(\mathbb{P}^{n-1})$ and certain x_n-monoids q,f. Since $\mathfrak{F}$ maps a general hyperplane through o to a hyperplane through o, the forms $g_0,\dots,g_{n-1}$ are necessarily linear forms in $k[x_0,\dots,x_{n-1}]$. Let $\mathfrak{A}\in\mathrm{PGL}(n+1,k)_o$ denote the inverse of the linear automorphism defined by $(g_0:\cdots:g_{n-1}:x_n)$. Then $\mathfrak{A}\mathfrak{F}=(qx_0:\cdots:qx_{n-1},f)$, which is a map of the type $\mathfrak{F}_{a,b,c,d}$, and hence it belongs to $\ker(\rho)$ by Proposition 1.2(iv). ■

2 Generators of the de Jonquières Group of Type 1

The following fact was established in [17, Proposition 2.1], for which an affine argument was given. We isolate it as a lemma for reference convenience and give a proof in terms of the projective geometry.

Lemma 2.1 *Consider the previous group homomorphism* $\mathrm{J}_o(1;\mathbb{P}^n)\xrightarrow{\rho}\mathrm{Cr}(\mathbb{P}^{n-1})$, *whose kernel is identified with* $\mathrm{PGL}(2,k(\mathbb{P}^{n-1}))$ *by Proposition 1.2. Then there is a map* $\sigma\colon\mathrm{Cr}(\mathbb{P}^{n-1})\to\mathrm{J}_o(1;\mathbb{P}^n)$ *such that* $\rho\circ\sigma$ *is the identity of* $\mathrm{Cr}(\mathbb{P}^{n-1})$. *In particular,* ρ *is surjective and* $\mathrm{J}_o(1;\mathbb{P}^n)$ *is isomorphic to the semi-direct product* $\mathrm{PGL}(2,k(\mathbb{P}^{n-1}))\rtimes\mathrm{Cr}(\mathbb{P}^{n-1})$.

Proof A splitting map σ is of course not uniquely defined. We choose one such map $\sigma\colon\mathrm{Cr}(\mathbb{P}^{n-1})\to\mathrm{J}_o(1;\mathbb{P}^n)$. Given $\mathfrak{t}:=(t_0:\cdots:t_{n-1})\in\mathrm{Cr}(\mathbb{P}^{n-1})$, let $\sigma(\mathfrak{t})$ be the rational map of $\mathbb{P}^n$ defined as follows:

$$\sigma(\mathfrak{t})=(x_0t_0(\mathbf{x}):x_0t_1(\mathbf{x}):\cdots:x_0t_{n-1}(\mathbf{x}):t_0(\mathbf{x})\,x_n),$$

where $\mathbf{x}=\{x_0,\dots,x_{n-1}\}$. It is clear that $\sigma(\mathfrak{t})\in\mathrm{J}_o(1;\mathbb{P}^n)$ by appealing to Proposition 1.2(i) with $\mathfrak{G}=(t_0(\mathbf{x}):\cdots:t_{n-1}(\mathbf{x}))$, $q=x_0$ and $f=t_0(\mathbf{x})\,x_n$.

Note that if $\mathfrak{t}'=(t'_0:\cdots:t'_{n-1})\in\mathrm{Cr}(\mathbb{P}^{n-1})$ has degree r and $\mathfrak{t}\circ\mathfrak{t}'=(s_0:\cdots:s_{n-1})$, then

$$\begin{aligned}\sigma(\mathfrak{t})\circ\sigma(\mathfrak{t}')&=\sigma(\mathfrak{t})\big(x_0t'_0(\mathbf{x}),\dots,x_0t'_{n-1}(\mathbf{x}),x_nt'_0(\mathbf{x})\big)\\&=\big(x_0^{r+1}s_0(\mathbf{x}):\cdots:x_0^{r+1}s_{n-1}(\mathbf{x})\,:x_0^rx_ns_0(\mathbf{x})\big)\\&=\big(x_0s_0(\mathbf{x}):\cdots:x_0s_{n-1}(\mathbf{x}):x_ns_0(\mathbf{x})\,\big)\\&=\sigma(\mathfrak{t}\circ\mathfrak{t}').\end{aligned}$$

By definition of the map $\rho\colon\mathrm{J}_o(1;\mathbb{P}^n)\to\mathrm{Cr}(\mathbb{P}^{n-1})$ it is clear that $\rho\circ\sigma$ is the identity map of the group $\mathrm{Cr}(\mathbb{P}^{n-1})$. ■

We proceed to elucidate further the geometric behavior. For this, recall that a rational map $\mathfrak{F}\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ *contracts* a subvariety $V \subset \mathbb{P}^n$ provided the restriction of $\mathfrak{F}$ to an open dense subset of V is well defined and its image has dimension strictly less than $\dim V$. It is known that if $\mathfrak{F}$ contracts an irreducible hypersurface $V \subset \mathbb{P}^n$, then the defining equation of V is a factor of the Jacobian determinant of the forms defining $\mathfrak{F}$.

If $V \subset \mathbb{P}^r$ is an irreducible projective variety of dimension m, the *geometric genus* $p_g(V)$ of V is the maximal number of linearly independent global differential m-forms on some (then all) desingularization of V; if V is a smooth hypersurface, then $p_g(V) = \binom{\ell-1}{r-1}$. In general, writing $\ell - 1 = s(r-m)+e$, where $0 \le e \le r-m-1$, one has the so-called Castelnuovo–Harris bound for the geometric genus of V (see [9]):

$$p_g(V) \le \binom{s}{m+1}(r-m) + \binom{s}{m}e.$$

Proposition 2.2 *Any $\mathfrak{F} \in \mathrm{J}_o(1;\mathbb{P}^n)$ contracts a finite number of irreducible hypersurfaces, each of which has geometric genus bounded by a number depending on* $\deg(\mathfrak{F})$.

Proof Let $\mathfrak{F} \in \mathrm{J}_o(1;\mathbb{P}^n)$ be a de Jonquières map of type 1. By Proposition 1.2(i) there exists $\mathfrak{G} = (g_0 : \cdots : g_{n-1}) \in \mathrm{Cr}(\mathbb{P}^{n-1})$ such that

$$\mathfrak{F} = (qg_0 : \cdots : qg_{n-1} : f), \tag{2.1}$$

for x_n-monoids $q = cx_n + d$, $f = ax_n + b$, for suitable forms $a, b, c, d \in k[x_0, \dots, x_{n-1}]$, with either $c \neq 0$ or $a \neq 0$, and $\gcd(q, f) = 1$ (equivalently, $ad - bc \neq 0$); set $\deg \mathfrak{G} = \deg \mathfrak{F} - t$, where $t = \deg(q)$ with $1 \le t \le \deg \mathfrak{F}$.

Note that $\mathfrak{F}$ maps the hyperplane $x_n = 0$ birationally onto a hypersurface. We deduce that $\mathfrak{F}$ is a local isomorphism at a point $p \in \mathbb{P}^n$ if and only if p is not a zero of $(ad - x_n^{\deg \mathfrak{F} - t} bc) q \operatorname{Jac}(\mathfrak{G})$, where $\operatorname{Jac}(\mathfrak{G})$ is the Jacobian determinant of the set $\{g_0, \dots, g_{n-1}\}$. Indeed, for p belonging to the open set $\{x_n = 1\}$, if $q(p) \neq 0$, then that map is not a local isomorphism at p if and only if it is a zero of the Jacobian determinant of the set $\{g_0, \dots, g_{n-1}, f/q\}$, *i.e.*, a zero of $(ad - bc)\operatorname{Jac}(\mathfrak{G})$. Then the reduced hypersurface $J(\mathfrak{F})$ of equation $(ad - x_n^{\deg \mathfrak{F} - t} bc) q \operatorname{Jac}(\mathfrak{G}) = 0$ has degree at most $t^2(\deg \mathfrak{F} - 1) n(\deg \mathfrak{G} - 1) \le n(\deg \mathfrak{F})^2(\deg \mathfrak{F} - 1)^2$.

Now let $V \subset \mathbb{P}^n$ denote an irreducible hypersurface contracted by $\mathfrak{F}$. One knows that $V \subset J(\mathfrak{F})$. Then the assertion follows by using the Castelnuovo–Harris bound for V. ■

Theorem 2.3 *Assume that k is uncountable* e.g., $k = \mathbb{C}$. *Let $\mathcal{G}$ be a set of generators for $\mathrm{J}_o(1;\mathbb{P}^n)$ and let $d \ge 2$. Then any subset $\mathcal{G}_0 \subset \mathcal{G}$ such that* $\mathrm{PGL}(2, k(\mathbb{P}^{n-1}))$ *is generated by elements of $\mathcal{G}_0$ contains uncountably many elements of degrees $\ge d$.*

Proof First note that for every ℓ, and every irreducible smooth hypersurface $\Gamma \subset \mathbb{P}^{n-1}$ of degree ℓ, we can construct an element of $\ker(\rho) \subset \mathrm{J}_o(1;\mathbb{P}^n)$ of degree $\ell + 1$ following the recipe in (2.1). Take $\mathfrak{G}$ to be the identity map, $q \in k[x_0, \dots, x_{n-1}]$ to be a defining equation of Γ, and $f = ax_n + b$, with $a \neq 0$.

Recall that if $\ell > n$, then two smooth hypersurfaces $\Gamma_1, \Gamma_2 \subset \mathbb{P}^{n-1}$ of degree ℓ, which are birationally equivalent, are necessarily biregularly so. This is because the canonical class of such a hypersurface is ample (*cf.* [4, Thm. 0.2.1]). In particular, if Γ_1 and Γ_2 are not isomorphic, then $\Gamma_1 \times \mathbb{P}^1$ and $\Gamma_2 \times \mathbb{P}^1$ are not birationally equivalent by Lüroth's Theorem.

Since the moduli space of smooth hypersurfaces of degree ℓ in $\mathbb{P}^{n-1}$ has positive dimension, there are uncountably many such hypersurfaces that are pairwise nonisomorphic. We deduce that for any $\ell > n$ there exists a family $\mathcal{A}_\ell \subset \mathrm{J}_o(1, \mathbb{P}^n)$ with the following properties:

- $\mathcal{A}_\ell$ contains uncountably many elements.
- If $\mathfrak{F} \in \mathcal{A}_\ell$ then $\deg(\mathfrak{F}) = \ell + 1$ and $\mathfrak{F} \in \ker(\sigma)$.
- If $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{A}_\ell$, then there are irreducible components $V_1 \subset J(\mathfrak{F}_1)$ and $V_2 \subset J(\mathfrak{F}_2)$, which are not birationally equivalent. Indeed, we can choose $V_i \subset \mathbb{P}^n$ to be a cone, with vertex o, over a smooth hypersurface $\Gamma_i \subset \{x_0 = 0\} = \mathbb{P}^{n-1}$ of degree ℓ, then V_i is birationally equivalent to $\Gamma_i \times \mathbb{P}^1$.

It follows from [16, Lemma 4] that if $\mathcal{G}_1$ is a subset of $\mathrm{Cr}(\mathbb{P}^n)$ and if $\mathfrak{F} \in \mathrm{Cr}(\mathbb{P}^n)$ is written as a product of elements in $\mathcal{G}_1$, then every hypersurface contracted by $\mathfrak{F}$ is birationally equivalent to a hypersurface contracted by some element in $\mathcal{G}_1$.

Suppose that there exist an integer $d \geq 2$ and a subset $\mathcal{G}_0 \subset \mathcal{G}$ such that $\mathcal{G}_0$ contains at most countably many elements of degree $\geq d$. Now every member of such a countable sequence of elements of degrees $d_e := d+e$, $e = 0, 1, \ldots$, contracts a finite number of hypersurfaces. We then deduce that for any $\ell > n$, an uncountable subset $\mathcal{A}'_\ell \subset \mathcal{A}_\ell$ is generated by elements of degree $\leq d$ belonging to $\mathcal{G}_0$. This contradicts Proposition 2.2, since it suffices to take ℓ large enough in order to obtain elements in $\mathcal{A}'_\ell$ that contract a hypersurface with geometric genus larger than the stated bound. ∎

We now turn to the question of giving explicit families of maps that together generate $\mathrm{J}_o(1; \mathbb{P}^n)$. Since $\mathrm{J}_o(1, \mathbb{P}^n)$ is isomorphic to $\mathrm{PGL}(2, k(\mathbb{P}^{n-1})) \rtimes \mathrm{Cr}(\mathbb{P}^{n-1})$, it suffices to find generators for the two factors. In the case where $n > 3$, however, this approach does not help, since we do not know a workable set of generators of $\mathrm{Cr}(\mathbb{P}^{n-1})$. On the other hand, since $\mathrm{Cr}(\mathbb{P}^2)$ is sufficiently familiar, we can make these generators explicit for $n = 3$. For this, recall the generation of the Möbius group $\mathrm{PGL}(2, K)$, where K is an arbitrary field, by the elements defined by matrices of the following types

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\alpha, \beta \in K$, $\alpha \neq 0$. These three types of matrices are often called *elementary Möbius maps* over K, generating, respectively, the torus K^*, the additive group K, and the order 2 cyclic group defined by the "inversion" $t \mapsto 1/t$.

We denote by $\mathfrak{s} = (x_1x_2 : x_0x_2 : x_0x_1)$ the so-called *standard quadratic plane Cremona map*.

Proposition 2.4 *The group* $J_o(1;\mathbb{P}^3) = \mathrm{PGL}(2, k(\mathbb{P}^2)) \rtimes \mathrm{Cr}(\mathbb{P}^2)$ *is generated by the elements* $1 \rtimes \mathfrak{s}$, $1 \rtimes \mathfrak{t}$, *and* $\mathfrak{f} \rtimes 1$, *where* $\mathfrak{s}$ *is the standard quadratic plane Cremona map,* $\mathfrak{t} \in \mathrm{PGL}(3, k)$, *and* $\mathfrak{f} \in \mathrm{PGL}(2, k(\mathbb{P}^2))$ *is an elementary Möbius map.*

Proof The result is immediate from the above prolegomena on the generation of the Möbius group and the Noether–Castelnuovo Theorem on the generation of $\mathrm{Cr}(\mathbb{P}^2)$ by the standard quadratic transformation and by $\mathrm{PGL}(3, k)$. ■

An internal description of a set of generators of $J_o(1;\mathbb{P}^3)$ as a subgroup of $\mathrm{Cr}(\mathbb{P}^3)$ will be a consequence of Proposition 2.4, through the required interpretation. We can restate the proposition in the following compact form.

Theorem 2.5 *The group* $J_o(1;\mathbb{P}^3)$ *is generated by a set of generators of* $J_o(2;\mathbb{P}^3)$ *and the cubic Cremona map*

$$\mathfrak{T}_3 = (x_0x_1x_2 : x_0^2x_2 : x_0^2x_1 : x_1x_2x_3). \tag{2.2}$$

Proof We have $\mathfrak{T}_3 = \sigma(\mathfrak{s}) = 1 \rtimes \mathfrak{s}$, where $\mathfrak{s}$ is the standard quadratic map of $\mathrm{Cr}(\mathbb{P}^2)$.

On the other hand, any $\mathfrak{t} \in \mathrm{PGL}(3, k) = \mathrm{Aut}(\mathbb{P}^2)$ is defined by three independent linear forms $\ell_0, \ell_1, \ell_2 \in k[x_0, x_1, x_2]$. Its image by σ is

$$1 \rtimes \mathfrak{t} = (x_0\ell_0 : x_0\ell_1 : x_0\ell_2 : x_3\ell_0),$$

which has degree at most 2 (actually, equal to 2, provided $\ell_0 \neq \alpha x_0$ for $\alpha \in k$). Note that $1 \rtimes \mathfrak{t}$ belongs to $J_o(2;\mathbb{P}^3)$.

Finally, we know that the $\mathrm{PGL}(2, k(\mathbb{P}^2))$ is identified with $\ker(\rho) \subset J_o(2;\mathbb{P}^3)$ (Proposition 1.3). ■

To close the section, we state yet another result individualizing further the set of generators into some explicit families.

For this, we consider the set $\mathbf{T_{22}}$ of Cremona maps of $\mathbb{P}^3$ of degree 2 with inverse of degree 2. We consider the usual action of $\mathrm{PGL}(4, k) \times \mathrm{PGL}(4, k)$ on $\mathbf{T_{22}}$,

$$(\mathfrak{T}_1, \mathfrak{T}_2) \cdot \mathfrak{F} = \mathfrak{T}_1\mathfrak{F}\mathfrak{T}_2^{-1}. \tag{2.3}$$

One knows from [18, Prop. 2.4.1, Thm. 3.1.1] that $\mathbf{T_{22}}$ is an irreducible variety of dimension 26 with 7 orbits under the action (2.3). It can be seen that any $\mathfrak{F} \in \mathbf{T_{22}}$ admits points $o_1, o_2 \in \mathbb{P}^3$ such that $\mathfrak{F}$ transforms a general plane going through o_1 in a plane going through o_2. Up to projective transformations, every orbit meets $J_o(2;\mathbb{P}^3)$, and also $\ker(\rho) \subset J_o(2;\mathbb{P}^3)$ by Proposition 1.3.

Considering the induced action of $\mathrm{PGL}(4, k)_o \times \mathrm{PGL}(4, k)_o$ on $\ker(\rho) \cap \mathbf{T_{22}}$, one has the following lemma.

Lemma 2.6 *Any orbit of* $\ker(\rho) \cap \mathbf{T_{22}}$ *under the action of* $\mathrm{PGL}(4, k)_o \times \mathrm{PGL}(4, k)_o$ *is the restriction to* $\ker(\rho)$ *of an orbit of* $\mathbf{T_{22}}$ *under the action of* $\mathrm{PGL}(4, k) \times \mathrm{PGL}(4, k)$. *In addition, every such orbit of* $\ker(\rho) \cap \mathbf{T_{22}}$ *contains involutions.*

Proof Let $O \subset \mathbf{T_{22}}$ be an orbit under the action of $\mathrm{PGL}(4, k) \times \mathrm{PGL}(4, k)$. We know that $O \cap \ker(\rho) \neq \varnothing$. To prove the first assertion it suffices to show that $\mathfrak{F}, \mathfrak{G} \in \ker(\rho)$ implies $\mathfrak{T}, \mathfrak{T}' \in \mathrm{PGL}(4, k)_o$ for $\mathfrak{F}, \mathfrak{G} \in O$ and $\mathfrak{T}, \mathfrak{T}' \in \mathrm{PGL}(4, k)$ such that $\mathfrak{T}\mathfrak{F}\mathfrak{T}' = \mathfrak{G}$.

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The linear system associated with an element in $\mathbf{T}_{22}$ is defined by smooth quadrics containing a conic C of rank r, where $r \in \{1,2,3\}$, and going through a unique "special" point p that either does not belong to the conic plane or $p \in C$ and the tangent plane at p of a general member in that linear system is constant and does not contain C. In particular, such an element in $\mathbf{T}_{22}$ is not defined (only) along $C \cup \{p\}$, and if it belongs to $\mathrm{PGL}(4,k)\mathrm{J}_o(1;\mathbb{P}^3)$, then $p = o$.

Let $L \simeq \mathbb{P}^1$ be a line in $\mathbb{P}^3$. If L intersects the open set on which $\mathfrak{F}$ is injective, the restriction of $\mathfrak{F} \in \ker(\rho) \cap \mathbf{T}_{22}$ to L induces a biregular map onto the image $\nu\colon \mathbb{P}^1 \to \mathbb{P}^3$. Since $\mathfrak{F}$ is defined by quadratic polynomials, then $\nu^*\mathcal{O}_{\mathbb{P}^3}(1) = \mathcal{O}_{\mathbb{P}^1}(n)$ with $n \in \{1,2\}$. We have $n = 1$ if and only if all these polynomials vanish at a point $p' \in L$; in this case the point p' is the special point associated with $\mathfrak{F}$ as element in $\mathbf{T}_{22}$, that is, $p' = o$.

By applying this argument to lines of the form $L = \mathfrak{T}'(L_o)$, where L_o is a general line passing through o, and taking into account that $\mathfrak{G}$ maps L_o birationally onto a line of the same type, we deduce that the special point of $\mathfrak{T}\mathfrak{F}$ is o, hence $\mathfrak{T}'(o) = o$. By symmetry the same holds for $\mathfrak{T}^{-1}(L_o)$ and $(\mathfrak{T}')^{-1}\mathfrak{F}^{-1}$, hence $\mathfrak{T}(o) = o$.

This proves the first assertion. For the second one we may use the normal forms obtained in [18, Thm. 3.1.1] or draw upon the content of [19, Cor. 5.3] (also [19, Thm. 5.11]). ■

We now deduce the following theorem.

Theorem 2.7 *The group* $\mathrm{J}_o(1;\mathbb{P}^3)$ *is generated by the cubic Cremona involution* $\mathfrak{T}_3$, *seven involutions of degree* 2, *generators for* $\mathrm{PGL}(4,k)_o$, *and the Cremona maps of degree* ≥ 3 *coming from elementary Möbius maps in* $\mathrm{PGL}(2,k(\mathbb{P}^2))$.

3 Generalized de Jonquières Ideals

In previous sections we focused on the group theoretic and geometric properties of certain Cremona maps. As we have seen, these maps admit a very special form in terms of their defining coordinates. In this section we take a more abstract view of a map of de Jonquières type, with an emphasis on the ideal theoretic and homological properties of the base ideal thereof.

Quite generally, let there be given a rational map

$$\mathfrak{G} = (g_0 : \cdots : g_{n-1})\colon \mathbb{P}^{n-1}_k \dashrightarrow \mathbb{P}^{n-1}_k,$$

where g_i's are forms of degree $d \geq 1$ in the polynomial ring

$$R := k[\mathbf{x}] = k[x_0, \dots, x_{n-1}].$$

Write $I := (g_0, \dots, g_{n-1}) \subset R$ for the corresponding base ideal. Consider the flat extension $S := k[\mathbf{x}, x_n] = R[x_n] = k[x_0, \dots, x_{n-1}, x_n]$, where x_n is a new indeterminate. Let $q, f \in S$ be additional forms of degrees $\mathfrak{d} \geq 1$ and $D := d + \mathfrak{d}$, respectively, where $\mathfrak{d}$ is arbitrary. We assume throughout that q and f are relatively prime.

Definition 3.1 The rational map $\mathfrak{F} := (qg_0 : \cdots : qg_{n-1} : f)\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ will be called a *map of* (q, f)*-type* with *underlying map* $\mathfrak{G}$.

The corresponding base ideal $J := (qI, f) \subset S$ will also be called a (q, f)-*ideal* (with *underlying ideal* I).

These rational maps admit a fairly structured homological behavior.

3.1 The Homology of a Map of (q, f)-type

We refer to [3] or [8] for the basic homological notions used in this and subsequent parts.

Let S be as above and let $J \subset S$ stand for a homogeneous ideal. The S-module S/J has a *graded free resolution* of finite length ([8, Section 1.10]). Such a resolution of minimal length is essentially unique, and its length is the *homological dimension* of S/J, denoted $\mathrm{hd}(S/J)$. This number is an upper bound for the codimension of J. The ring S/J is said to be *Cohen–Macaulay* if the two numbers coincide. In this case the ideal J is said to be *perfect*. (We observe that these notions are definable in more general contexts, but the present ones suffice for our purpose.)

We write a minimal graded free resolution as an exact sequence

$$0 \to F_r \xrightarrow{\psi_r} F_{r-1} \cdots \xrightarrow{\psi_2} F_1 \xrightarrow{\psi_1} F_0 \longrightarrow S/J \to 0,$$

where F_i is a free graded module. The maps ψ_i are homogeneous of degree 0. Typically, $F_i \simeq \bigoplus_{j=1}^{s_i} S(-C_{ij})$, where s_i is the rank of F_i and C_{ij} are the *shifts* needed to make of degree 0 the corresponding map of the resolution.

A set of minimal generators of the submodule $\mathrm{Im}(\psi_1)$ is called a set of minimal *syzygies* of S/J. Writing ψ_1 in matrix form, the column vectors corresponding to these generators are the (minimal, generating) relations of a set of minimal generators of J.

Throughout we will freely draw on these notions without further ado.

We keep the notation as in the beginning of the section.

Let $I \subset R = k[x_0, \ldots, x_{n-1}] \subset S = R[x_n]$ be an ideal generated by forms $\mathbf{g} = \{g_0, \ldots, g_{n-1}\}$ of degree $d \geq 1$. Let

$$\cdots \to \bigoplus_{j=1}^{m_1} R(-a_{1j}) \xrightarrow{\varphi} \bigoplus_{i=0}^{n} R(-d) \xrightarrow{\mathbf{g}} R \longrightarrow R/I \to 0,$$

$$\cdots \to \bigoplus_{j=1}^{s_1} S(-C_{1j}) \xrightarrow{\psi} \bigoplus_{j=1}^{s} S(-C_j) \xrightarrow{\pi} S \longrightarrow S/IS\colon f \to 0$$

stand for minimal graded free resolutions of R/I and $R/IS\colon f$, over R and S respectively, from which we trivially derive minimal graded free resolutions of S/qIS and $R/q(IS\colon f)$ over S:

$$\cdots \to \bigoplus_{j=1}^{m_1} S(-a_{1j} - \deg(q)) \xrightarrow{\varphi_1 = \varphi} S(-(d + \deg(q)))^n \xrightarrow{q\,\mathbf{g}} S \to S/qIS \to 0,$$

$$\cdots \to \bigoplus_{j=1}^{s_1} S(-C_{1j} - \deg(q)) \xrightarrow{\psi_1 = \psi} \bigoplus_{j=1}^{s} S(-C_j - \deg(q)) \xrightarrow{q\,\pi} S \to S/q(IS\colon f) \to 0.$$

Shifting the second of these resolutions by $-(d+\deg(q))$, one obtains a map of complexes, where the vertical homomorphisms are also homogeneous of degree 0 induced by multiplication by f on the rightmost modules:

$$
\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_{j=1}^{m_i} S(-a_{ij}-\mathfrak{d}) \arrow[r] & \cdots \arrow[r, "\varphi"] & S(-(d+\mathfrak{d}))^n \arrow[r] & S \arrow[r] & S/qIS \arrow[r] & 0 \\
\cdots \arrow[r] & \bigoplus_{j=1}^{s_i} S(-C_{ij}-(d+2\mathfrak{d})) \arrow[r] \arrow[u, "c_i(f)"] & \cdots \arrow[r, "\psi"] & \bigoplus_{j=1}^{s} S(-C_j-(d+2\mathfrak{d})) \arrow[r] \arrow[u, "c(f)"] & S(-(d+\mathfrak{d})) \arrow[r] \arrow[u, "f"] & \frac{S}{q(IS:f)}(-(d+\mathfrak{d})) \arrow[r] \arrow[u, "f"] & 0
\end{tikzcd}
$$

where we have written $\mathfrak{d} := \deg(q)$ for editing purposes.

Proposition 3.2 *The mapping cone of the above map of complexes is a graded free resolution of the ideal (qI, f) of (q, f)-type with underlying ideal I. Moreover, if*

$$\mathrm{hd}(S/q(IS\colon f)) \le \mathrm{hd}(R/I) - 1$$

*(*e.g., *if $f \in IS$ and I has codimension ≥ 2), then* $\mathrm{hd}(S/(qI, f)) \le \mathrm{hd}(R/I)$.

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Proof This result was essentially proved in [11]. This case only requires minor changes. ■

We draw attention to the syzygy matrix of the generators of (qI, f), which has the form

$$\Psi = \begin{pmatrix} \varphi & c(f) \\ \mathbf{0} & -q\pi \end{pmatrix}.$$

Here φ denotes a syzygy matrix of the given set of generators of I, while $\pi\colon S^s \twoheadrightarrow IS\colon(f)$ stands for a surjective S-module homomorphism based on the given homogeneous generators $\{c_1, \dots, c_s\} \subset S$ of $IS\colon(f)$, and $c(f)\colon S^s \longrightarrow S^{n+1}$ is the induced *content map* whose j-th column vector gives the coefficients of fc_j as a combination over S of the generators of I.

We will write $D := \deg(f) = d + \mathfrak{d}$.

Example 3.3 Suppose that $f \in IS$. Then π is the identity map of S and the content map $c(f)\colon S \to S^{n+1}$ is represented by one single column. A graded free resolution of (qI, f) has the form

$$0 \to F_r \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \oplus S(-(D+\mathfrak{d})) \longrightarrow S^{n+1}(-D) \longrightarrow S,$$

where $0 \to F_r \to \cdots \to F_2 \to F_1 \to S^n(-D) \to S$ is a graded free resolution of qIS over S with suitable self-understood shifts.

Two important cases emerge as follows.

3.1.1 Hilbert–Burch ideal of (q, f)-type

If I is a codimension 2 perfect ideal, then so is (qI, f). Namely, a graded free resolution of (qI, f) is

$$0 \to \Big(\bigoplus_j S\big(-(\mathfrak{d} + d_j)\big)\Big) \oplus S(-(D+\mathfrak{d})) \longrightarrow S^{n+1}(-D) \longrightarrow S,$$

where $0 \to \bigoplus_j S(-d_j) \to S(-d)^n \to S$ is a graded free resolution of IS.

We also call these ideals *Hilbert–Burch* ideals because they satisfy the Hilbert–Burch theorem, by which they are generated by the maximal minors of the corresponding matrix of syzygies ([8, Theorem 20.15]).

For convenience we introduce the following definition.

Definition 3.4 We will say that a Cremona map is *Cohen–Macaulay* if its base ideal $J \subset S$ is perfect (*i.e.,* S/J is Cohen–Macaulay).

We often say that a Cremona map is Hilbert–Burch if it is Cohen–Macaulay of codimension 2. The Cremona map of degree 3 in (2.2) falls within this class: it is based on the standard plane quadratic Cremona map, which is Cohen–Macaulay of codimension 2.

Remark 3.5 The classification of codimension 2 Cohen–Macaulay Cremona maps seems to be largely unknown. If $n - 1 = 2$, *i.e.,* for plane Cremona maps, this

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property is equivalent to requiring that the base ideal be saturated. It has been proved in [10, Theorem 1.5] that a plane Cremona map of codimension 2 and degree at most 4 is Cohen–Macaulay. This is false for degree ≥ 5, and the classification gets harder and harder as the degree increases.

3.1.2 Almost Koszul Ideal of (q, f)-type

This is the case where I is generated by a regular sequence (necessarily of degree 1), which is resolved by the Koszul complex on the elements of the regular sequence. The graded free resolution of (qI, f) has the form

$$0 \to S(-(n+\mathfrak{d})) \to \cdots \to S(-(3+\mathfrak{d}))^{\binom{n}{3}} \to S(-(2+\mathfrak{d}))^{\binom{n}{2}} \oplus S(-(D+\mathfrak{d})) \to S^{n+1}(-D) \to S.$$

The class of these maps and the class of Hilbert–Burch ones are quite apart. Indeed, the only Cremona maps of de Jonquières type that are both Cohen–Macaulay of codimension 2 and based on a regular sequence are the classical plane de Jonquières maps (thus forcing $n = 2$).

Infringing our notation for a minute, writing $d := \deg(q) + 1 = \mathfrak{d} + 1$, the graded free resolution of an almost Koszul Cremona map (*i.e.,* based on a complete intersection) has the form

$$0 \to S(-(d+n-1)) \to \cdots \to S(-(d+2))^{\binom{n}{3}} \\ \to S(-(d+1))^{\binom{n}{2}} \oplus S(-(2d-1)) \xrightarrow{\varphi} S^{n+1}(-d) \to S.$$

3.2 Inverse Results

From the previous subsection we transcribe the relevance of the two discussed special classes of ideals of (q, f)-type in the following condensed result.

Proposition 3.6 *The de Jonquières group $\mathrm{J}_o(1; \mathbb{P}^3)$ is generated by de Jonquières maps of Hilbert–Burch type and of almost Koszul type.*

In this part we will reverse our considerations by assuming a resolution format is given from which we wish to deduce the nature of the ideal. Unfortunately, due to the difficulty in classifying the plane Cremona maps with Cohen–Macaulay base ideals, it becomes hard to recover which elements of $\mathrm{J}_o(1; \mathbb{P}^3)$ are determined by the minimal free resolution of the respective perfect base ideals. In the subsequent part we will find a satisfactory result for the case of almost Koszul maps.

3.2.1 Homological Characterization

We will continue to write d for the degree of the generating forms.

Thus, let $S = k[x_0, \dots, x_n]$ be a standard graded polynomial ring over a field k, where $n \geq 3$, and let $J \subset S$ denote a homogeneous ideal of codimension at least 2, having a minimal free graded resolution of the form

$$(3.1)\ \ 0 \to S(-(d+2)) \xrightarrow{\psi} S^n(-(d+1)) \oplus S(-(2d-1)) \xrightarrow{\varphi} S^{n+1}(-d) \longrightarrow J \to 0.$$

Clearly, for such homological dimension, J is saturated (*i.e.*, $J^{\mathrm{sat}} := J : (\mathbf{x})^\infty = J$) and $d \geq 2$. Moreover, if $n = 3$, then J has codimension at most 2. In fact, otherwise J would have codimension at least 3, hence of codimension exactly 3, because the projective dimension is 3. Then S/J would be Cohen–Macaulay, hence Gorenstein, because the Cohen–Macaulay type is 1 ([3, Theorem 3.2.10]). But this is nonsense since any (homogeneous or local) Gorenstein ideal of codimension $c \geq 3$ that is not a complete intersection is minimally generated by at least $c + 2 \geq 5$ elements.

Thus, J has codimension 2 if $n = 3$. We assume throughout that $n = 3$. Recall that the *unmixed part* J^{un} of an ideal $J \subset S$ is the intersection of its primary components of minimal codimension and that J is said to be *unmixed* if $J^{\mathrm{un}} = J$.

Theorem 3.7 *Let $n = 3$ and let $J \subset S$ denote an ideal of codimension 2 having a minimal free resolution as in (3.1). If $d = 2$, we assume, moreover, that J is not unmixed.*

(i) *S/J has a unique associated prime of codimension 3 and, moreover, this prime is generated by the entries of ψ.*

(ii) *Up to a linear change of variables and elementary row operations one has*

$$\psi = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ 0 \end{pmatrix}, \ \varphi = \left(\begin{array}{ccc|c} & & & -q_0 \\ & \mathcal{K} & & -q_1 \\ & & & -q_2 \\ \hline 0 & 0 & 0 & q \end{array}\right), \ J = (qx_0, qx_1, qx_2, q_0x_0 + q_1x_1 + q_2x_2),$$

for suitable forms $q_0, q_1, q_2, a \in S$ of degree $d - 1$, where $\mathcal{K}$ is the Koszul syzygy matrix of the regular sequence x_0, x_1, x_2; in particular, $J^{\mathrm{un}} = (q, q_0x_0 + q_1x_1 + q_2x_2)$, a complete intersection of degree $(d - 1)d$.

Proof Let us first assume that $d \geq 3$. Then $2d - 1 \geq d + 2$, hence we can assume that the last entry in ψ vanishes. Since the remaining entries are linear and generate an ideal of codimension ≥ 3 by the acyclicity criterion of Buchsbaum–Eisenbud ([8, Theorem 20.9]), then they form a regular sequence of linear forms. By a change of variables, we may assume that the entries of ψ are $x_0, x_1, x_2, 0$. Consider the 4×3 linear submatrix $\mathcal{L}$ of φ. Its rows are Koszul relations of x_0, x_1, x_2, therefore it has rank at most 2. Then clearly its rank is exactly 2, since φ has rank 3. Therefore, we can write, up to elementary row operations,

$$\varphi = \left(\begin{array}{ccc|c} & & & -q_0 \\ & \mathcal{K} & & -q_1 \\ & & & -q_2 \\ \hline 0 & 0 & 0 & q \end{array}\right),$$

where $\mathcal{K}$ denotes the transposed Koszul syzygy matrix of x_0, x_1, x_2. Note that all the operations so far have changed the entries on the right most column of φ, but not their degrees ($= d - 1$).

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But this immediately implies that the columns of $\mathcal{K}$ are syzygies of say, the first three minimal generators of J. This implies that the latter have the form px_0, px_1, px_2, for some form $p \in S$ of degree $d-1$. This already proves the assertion in (i), namely, since $p \notin J$ by degree consideration, the prime ideal $P = (x_0, x_1, x_2)$ is an associated prime of S/J. Moreover, by a well-known fact (see, *e.g.,* [8, Corollary 20.14(a)]) this is necessarily the only associated prime thereof of codimension 3.

To complete the proof of (ii) it remains to show that we can assume $p = q$, and the fourth minimal generator of J is of the stated form. Let $\mathcal{N}$ denote, say, the submatrix of φ consisting of the three rightmost columns, which we may clearly assume has rank 3. Recall that, up to order and signs, the set of 3×3 minors of $\mathcal{N}$ divided by their gcd coincide with the given set of minimal generators of J on which (3.1) is based. This fact is well known and follows by dualizing (3.1) into S to get

$$0 \to J^* \simeq S \longrightarrow S^4(d) \xrightarrow{\varphi^t} S^3((d+1)) \oplus S((2d-1)),$$

from where follows that the entries of the vector generating the image of $S \to S^4(d)$ are the maximal minors of the transpose of $\mathcal{N}$ divided by their gcd.

In this case, by the explicitness of $\mathcal{N}$, we immediately see that this gcd is x_0, so the determinant of

$$\begin{pmatrix} -x_0 & 0 & -x_2 \\ 0 & -x_0 & x_1 \\ q_2 & q_1 & q_0 \end{pmatrix},$$

further divided by x_0 gives the required expression up to signs adjustment. To verify that p can be taken to be q, one inspects easily the other 3-minors.

The last statement in (ii) is obvious, since localizing at any minimal prime of codimension 2, a variable among x_0, x_1, x_2 becomes invertible.

We now consider the case $d = 2$. Our starter fails right at the outset, since there are in fact unmixed ideals with the given resolution shape, *e.g.,* the classical $J = (x_0, x_1) \cap (x_2, x_3)$. Thus, we must assume that J is not unmixed. But then S/J has some associated prime of codimension 3. As remarked earlier, any such prime must contain the ideal generated by the entries of ψ. Since these are all linear, we conclude as before that there is only one associated prime of codimension 3, necessarily generated by a regular sequence of 3 linear forms. This shows (i) for $d = 2$, and we can pick up from here by repeating the argument used in the case where $d \geq 3$. ∎

We say that two rational maps $\mathcal{F}, \mathcal{G} \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ are *linearly equivalent* if they belong to the same orbit of the action of $\mathrm{PGL}(n+1, k) \times \mathrm{PGL}(n+1, k)$ as in (2.3). We note that under this action the base ideal $J = (f_0(\mathbf{x}), \dots, f_n(\mathbf{x})) \subset S = k[\mathbf{x}] = k[x_0, \dots, x_n]$ of a map $\mathcal{F} \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ changes to the ideal generated by the forms $f_0(\mathbf{x} \cdot (P')^{-1}), \dots, f_n(\mathbf{x} \cdot (P')^{-1})$ where $(P, P') \in \mathrm{PGL}(n+1, k) \times \mathrm{PGL}(n+1, k)$ (considered as matrices). Clearly then, the base ideals of two linearly equivalent maps have the same typical algebraic invariants, such as equivalent minimal graded free resolutions.

As a consequence of the above results, we obtain a homological characterization of the elements of the subgroup $\mathrm{J}_o(n-1; \mathbb{P}^3)$ for $n = 3$.

Theorem 3.8 *Let $\mathcal{F}$ denote a Cremona map of $\mathbb{P}^3$ of degree $d \geq 2$. The following conditions are equivalent.*

(i) *$\mathcal{F}$ is linearly equivalent to an element of $\mathrm{J}_o(2;\mathbb{P}^3)$.*

(ii) *The base ideal $J \subset S = k[x_0, x_1, x_2, x_3]$ of $\mathcal{F}$ admits a minimal graded free resolution of the form*

$$0 \longrightarrow S\big(-(d+2)\big) \longrightarrow S^3\big(-(d+1)\big) \oplus S\big(-(2d-1)\big) \longrightarrow S^4(-d) \longrightarrow S/J,$$

and is not unmixed if $d = 2$.

We list some further properties of the above free resolution of the base ideal of a map $\mathfrak{F}$ in $\mathrm{J}_o(2;\mathbb{P}^3)$. By Theorem 3.7, we may assume that $J = (x_0q, x_1q, x_2q, f)$ for some relatively prime forms $q, f \in S$ of respective degrees $d-1, d$, with $f \in (x_0, x_1, x_2)S$. Moreover, $\mathfrak{F}$ being birational implies that q, and hence f as well, is an x_3-monoid. Let us write $f = \alpha x_3 + \beta, q = \gamma x_3 + \delta$, with $\alpha, \beta, \gamma, \delta \in k[x_0, x_1, x_2]$ forms of degrees $d-1, d, d-2, d-1$, respectively, such that $\alpha\delta - \beta\gamma \neq 0$.

Proposition 3.9 *With the above notation, set $P = (x_0, x_1, x_2)$.*

(i) *The following conditions are equivalent:*
   - (a) *P is an embedded prime of S/J;*
   - (b) *either $\mathfrak{F}$ has degree ≥ 3 (*i.e., *$\gamma \neq 0$ and* $\deg(\gamma) \geq 1$*) or else $\gamma = 0$;*
   - (c) *$(\alpha x_3 + \beta, \gamma x_3 + \delta) \subset P$.*

(ii) *If the ideal $(c, \beta, \gamma, \delta)$ containing $(\alpha x_3 + \beta, \gamma x_3 + \delta)$ is a proper ideal of codimension at least* 3*, then the minimal primes of $S/(\alpha x_3 + \beta, \gamma x_3 + \delta)$ are those that contract in $k[x_0, x_1, x_2]$ to an irreducible factor of the determinant $\alpha\delta - \beta\gamma$; otherwise (*i.e., *if $(\alpha, \beta, \gamma, \delta)$ has codimension* 2*), the additional minimal primes are the defining primes of straight lines through o and a point of $V(\alpha, \beta, \gamma, \delta) \cap V(x_3)$.*

Proof (i) Set $q := \gamma x_3 + \delta$.

(a) $\Rightarrow$ (b) If $\mathfrak{F}$ has degree 2 and $\alpha \neq 0$, then $q \notin P$. But if P is embedded, then $P \supset Q$ for some minimal prime of S/J of codimension 2. Since $q \notin Q$, Q must contain the variables x_0, x_1, x_2, hence contains P, a contradiction.

(b) $\Rightarrow$ (c) This is obvious.

(c) $\Rightarrow$ (a) If P is a minimal prime of S/J and If $(\alpha x_3 + \beta, \gamma x_3 + \delta) \subset P$ then P contains a minimal prime of $S/(\alpha x_3 + \beta, \gamma x_3 + \delta)$; since the latter has codimension 2 it is also a minimal prime of S/J because clearly $J \subset (\alpha x_3 + \beta, \gamma x_3 + \delta)$. Therefore P is an embedded prime of S/J.

(ii) The ideal $(\alpha, \beta, \gamma, \delta)$ is proper if and only if $\deg(\gamma) > 0$. Therefore, according to item (i) this ideal is proper if and only if P is an embedded prime of S/J. If, moreover, $(\alpha, \beta, \gamma, \delta)$ has codimension ≥ 3 then the minimal primes of $S/(\alpha, \beta, \gamma, \delta)$ cannot be minimal primes of $S/(\alpha x_3 + \beta, \gamma x_3 + \delta)$. Then, locally at any minimal prime Q of S/J, $\alpha\delta - \beta\gamma$ is a generator of $(\alpha x_3 + \beta, \gamma x_3 + \delta)$. This implies that some irreducible factor of $\alpha\delta - \beta\gamma$ is a minimal generator of Q, hence generates its contraction to $k[x_0, x_1, x_2]$ since this contraction has codimension 1. The alternative

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statement follows immediately since $\alpha, \beta, \gamma, \delta \in k[x_0, x_1, x_2]$ now implies that every minimal prime of $S/(\alpha, \beta, \gamma, \delta)$ is generated by two linear forms in $k[x_0, x_1, x_2]$ defining the straight line through a point of $V(\alpha, \beta, \gamma, \delta) \cap V(x_3)$ and o. ■

3.2.2 Homologically Near Cases

The purpose of this piece is to convey examples of ideals whose minimal free resolution is obtained from (3.1) by a slight perturbation of its shifts. Such ideals will actually be base ideals of Cremona maps which, therefore, cannot be elements of $\mathrm{J}_o(2; \mathbb{P}^3)$.

The first example appears on M. Noether's original papers ([14]; also [20, Example after Remark 2.3]).

Example 3.10 $J = (x_0x_3, x_1x_3, x_0(x_1 - x_2), x_1(x_0 - x_1))$.

The minimal free resolution is of the form

$$0 \to S(-5) \longrightarrow S^3(-3) \oplus S(-4) \longrightarrow S^4(-2) \longrightarrow J \to 0.$$

Note that it fits the template

$$0 \to S(-(d+3)) \longrightarrow S^3(-(d+1)) \oplus S(-2d) \xrightarrow{\varphi} S^4(-d) \longrightarrow J \to 0.$$

Here the linear submatrix of φ has rank 3 (not 2 as in the de Jonquières case) and the coordinates of the tail map generate a radical ideal in degrees $1, 2$ whose minimal primes are

$$(x_0, x_1, x_3), \quad (x_1, x_2, x_3), \quad (x_0 - x_2, x_1 - x_2, x_3).$$

The unmixed radical of J is (x_0, x_1), hence J has one embedded associated prime and two minimal primes of codimension 3.

Moreover, a calculation with *Macaulay* ([2]) shows that the initial degree of $J{:}P$ is 2, where P is any of the above three associated primes. Therefore, there is no associated prime of codimension 3 driven inside J by a form of degree 1.

The next example defines the polar map of the determinant of a so-called 3×3 sub-Hankel matrix ([5]):

$$\varphi = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & 0 \end{pmatrix}.$$

Example 3.11 $J = (x_3^2, x_2x_3, x_2^2 - 2/3x_1x_3, x_1x_2 - x_0x_3)$.

The minimal free resolution is again of the form

$$0 \to S(-5) \longrightarrow S^3(-3) \oplus S(-4) \longrightarrow S^4(-2) \longrightarrow J \to 0.$$

This time around, however, S/J admits a unique associated prime of codimension 3, and this prime is an embedded prime. This makes up for a sensitive geometric distinction between this example and the previous one (although both have the same degree ($= 1$) as schemes). The first is the scheme-theoretic union of a straight line with an embedded point and two isolated points, while the second is a straight line with an embedded point. Clearly, any of these two is very distinct from de Jonquières

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ideal, whose scheme has multiplicity 2. Nevertheless, the second example is more akin to a de Jonquières, as it is in a sense an "iteration" of Cohen–Macaulay de Jonquières schemes [5, Remark 4.6 (b)].

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Centro de Matemática, Facultad de Ciencias, Universidad de la República, 11400 Montevideo, Uruguay
e-mail: ivan@cmat.edu.uy

Departamento de Matemática, CCEN, Universidade Federal de Pernambuco, 50740-560 Recife, PE, Brazil
and

Departamento de Matemática, CCEN, Universidade Federal da Paraíba, 58059-900 João Pessoa, PB, Brazil
e-mail: aron@dmat.ufpe.br