ON CERTAIN REMARKABLE TRIANGLES

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1. Introduction. In his fine and stimulating book *Introduction to geometry* **(1)**, Coxeter considers three mutually tangent circles C_1, C_2, and C_3 (Fig. 1) and calculates the radii of the two circles E_1 and E_2 that touch all

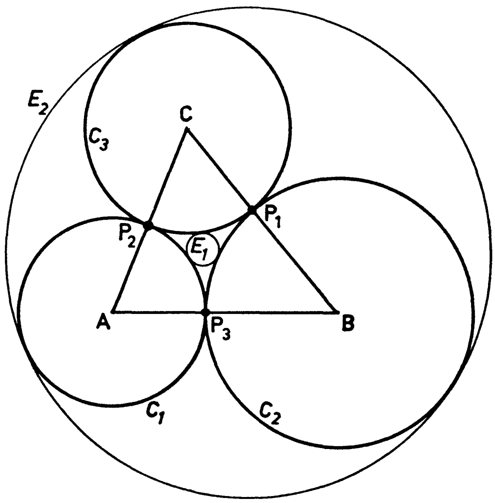

FIGURE 1

these three. If the centres of C_i are the vertices of a triangle ABC their radii are $s - a$, $s - b$, and $s - c$; α, β, and γ are defined by

$$\alpha(s - a) = \beta(s - b) = \gamma(s - c) = 1.$$

The radii R_1 and R_2 of E_1 and E_2 are then found to be

$$R_{1,2} = [\alpha + \beta + \gamma \pm 2(\beta\gamma + \gamma\alpha + \alpha\beta)^{\frac{1}{2}}]^{-1}. \tag{1}$$

R_1, the radius of the smaller circle, is obviously always positive; the other radius, R_2, is said to be "usually" negative (which means that E_2 encloses C_i as in our figure), but there are "very obtuse" triangles for which $R_2 > 0$ so that the circles C_i are all outside E_2. In the border-case E_2 is a straight line and the condition for this to happen is

$$(\alpha + \beta + \gamma)^2 = 4(\beta\gamma + \gamma\alpha + \alpha\beta) \tag{2}$$

or, what is shown to be the same thing,

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(3) $$4R + r - 2s = 0,$$

where R and r are the radii of the circumscribed and inscribed circles of ABC.

We determine here the triangles for which (2) holds; in fact, we express the sides a, b, c and angles A, B, C of such a triangle explicitly as functions of a parameter; in this manner the description "very obtuse" may be given a more precise meaning.

2. Alternative methods. A proof of (1) is given by Coxeter in a purely elementary way. He remarks that the well-known chemist Soddy considered the problem and that the formulae for R_i may be found in Hobson's *Trigonometry* **(2)**. We add to this that (1) was already known to Steiner, who published a proof, by means of Heron's formula for the height of a triangle, in his "Einige geometrische Betrachtungen" which appeared in the first volume of *Crelle's Journal* in 1826 (pp. 161–184, 252–288). The paper is republished in his collected works **(4)** and the formulae are given with a clarifying note by Weierstrass on the signs of the radii concerned.

We remark that (1) may also easily be shown by *inversion*. If the contact-point P_3 of C_1 and C_2 is taken as the centre of inversion and its modulus is 2, then the images C_1^1 and C_2^1 are two parallel lines (Fig. 2), that of the inscribed

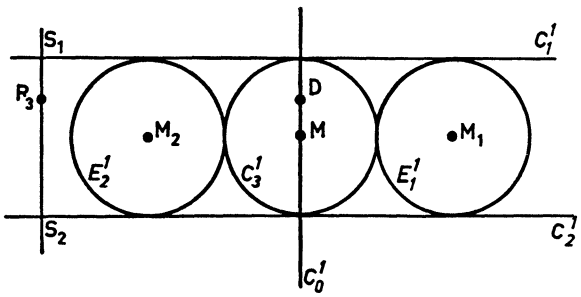

FIGURE 2

circle C_0 is the line C_0^1, C_3^1 has M as its centre, $P_3 S_1 = \alpha$, $P_3 S_2 = \beta$, $P_3 D = \rho = r^{-1}$. Obviously E_1^1 and E_2^1 are circles congruent to C_3^1 and touching it at the right- and the left-hand side. From

$$P_3 M_1^2 = (\rho + \alpha + \beta)^2 + \tfrac{1}{4}(\alpha - \beta)^2$$

and

$$P_3 M_2^2 = (\rho - \alpha - \beta)^2 + \tfrac{1}{4}(\alpha - \beta)^2$$

it follows that

(4) $$R_1 = (\alpha + \beta)[(\rho + \alpha + \beta)^2 - \alpha\beta]^{-1}, \quad R_2 = (\alpha + \beta)[(\rho - \alpha - \beta)^2 - \alpha\beta]^{-1}.$$

E_2 is a straight line if P_3 is on E_2^1, or what is the same thing, if

(5) $$\rho^2 - 2(\alpha + \beta)\rho + \alpha^2 + \alpha\beta + \beta^2 = 0.$$

In the same way we obtain

(6) $$\rho^2 - 2(\beta + \gamma)\rho + \beta^2 + \beta\gamma + \gamma^2 = 0.$$

The third analogous equation is seen to be linearly dependent on (5) and (6), which also imply that

(7) $$\rho^2 = \beta\gamma + \gamma\alpha + \alpha\beta, \qquad 2\rho = \alpha + \beta + \gamma.$$

We thus have again the condition (2), of which both sides are now seen to be equal to $4r^{-2}$.

3. The deltoid. If we multiply both sides of (2) by $(s-a)^2(s-b)^2(s-c)^2$ it becomes

(8) $$(bc + ca + ab - s^2)^2 = 4s(s-a)(s-b)(s-c).$$

This is therefore the relation between the sides of a triangle for which E_2 is a straight line. We regard a, b, c as the barycentric co-ordinates of a point in a Euclidean plane with respect to an equilateral triangle $O_1 O_2 O_3$ (Fig. 3).

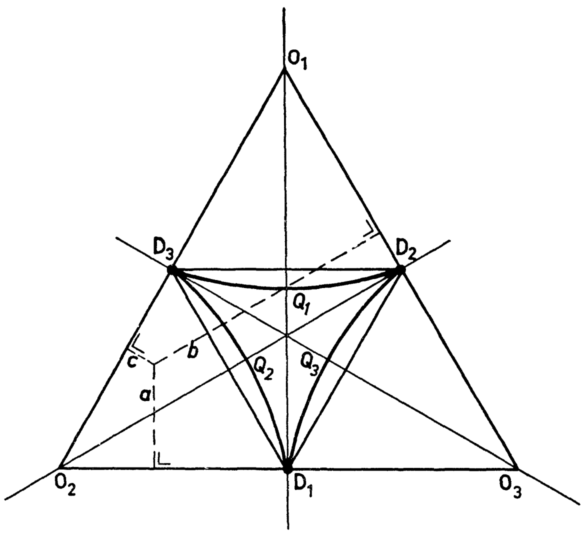

FIGURE 3

For real triangles the image points are inside the triangle $D_1 D_2 D_3$. Points on the line-segments $D_2 D_3$, $D_3 D_1$, $D_1 D_2$ correspond to improper triangles. The line l at infinity is given by $s = 0$; $bc + ca + ab = 0$ is the equation of the circumcircle Ω of $O_1 O_2 O_3$; $s - a = 0$, $s - b = 0$, $s - c = 0$ are the lines $D_2 D_3$, $D_3 D_1$, and $D_1 D_2$ respectively.

We see that (8) is the equation of a quartic curve K, which has the three lines $O_i D_i$ as axes of symmetry. For $s = 0$ we obtain the isotropic points each

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counted twice, but these points are one-fold intersections of K and Ω. Hence l touches K in the isotropic points. For $s - a = 0$ we obtain the points D_2 and D_3 each counted twice and so on; therefore D_1, D_2, and D_3 are double points of K. The axis of symmetry $b = c$ meets K three times at D_1 (the fourth point of intersection being $Q_1 = (8, 5, 5)$), which means that D_1 (and D_2 and D_3) is a cusp. From these properties it follows, in view of a theorem of Cremona-Clebsch **(3)**, that K is *Steiner's hypocycloid.* It is drawn in Figure 3 and we see that no points of K are outside $D_1 D_2 D_3$. Therefore if a, b, c satisfy (8) they are the sides of a real triangle, which is improper only if $a = 0$, $b = c$, etc. K is generated by a point of a circle of radius $\frac{1}{3}$ which rolls within the inscribed circle of $O_1 O_2 O_3$, which is given the radius 1 (Fig. 4). If g is the

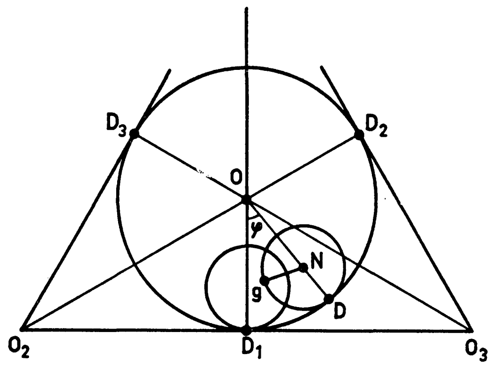

FIGURE 4

moving point, $\angle D_1 OD = \phi$, then $\angle g\, ND = 3\phi$ and by projecting the broken line $D_1 ONg$ on $O_1 D_1$, $O_2 D_2$, and $O_3 D_3$ the co-ordinates of g are easily found. If we write $\alpha = \pi/3$ the result is

$$\begin{aligned} \lambda a &= 3 - 2\cos\phi - \cos 2\phi, \\ \lambda b &= 3 + 2\cos(\phi + \alpha) - \cos 2(\phi + \alpha), \\ \lambda c &= 3 + 2\cos(\phi - \alpha) - \cos 2(\phi - \alpha), \end{aligned} \tag{9}$$

where λ is a proportionality factor. By (9) the sides of our remarkable triangles are given explicitly as functions of the parameter ϕ with $0 \leqslant \phi < 2\pi$. For $\phi = 0$, 2α, and 4α we have the cusps D_1, D_2, and D_3; for $\phi = \alpha$, 3α, and 5α the points Q_3, Q_1, and Q_2. If ϕ increases with 2α the sides a, b, c are cyclically interchanged.

We remark that (9) may be written:

$$\begin{aligned} \lambda a &= 2(1 - \cos\phi)(2 + \cos\phi), \\ \lambda b &= 2\{1 + \cos(\phi + \alpha)\}\{2 - \cos(\phi + \alpha)\}, \\ \lambda c &= 2\{1 + \cos(\phi - \alpha)\}\{2 - \cos(\phi - \alpha)\}. \end{aligned} \tag{10}$$

Furthermore from (9) it follows that

$$
(11)\quad
\begin{aligned}
2\lambda s &= 9,\\
2\lambda(s-a) &= (2\cos\phi+1)^2 = 16\cos^2\frac{\phi+\alpha}{2}\cos^2\frac{\phi+5\alpha}{2},\\
2\lambda(s-b) &= \{2\cos(\phi+\alpha)-1\}^2 = 16\cos^2\frac{\phi+5\alpha}{2}\cos^2\frac{\phi+3\alpha}{2},\\
2\lambda(s-c) &= \{2\cos(\phi-\alpha)-1\}^2 = 16\cos^2\frac{\phi+3\alpha}{2}\cos^2\frac{\phi+\alpha}{2},
\end{aligned}
$$

and therefore, if Δ stands for the area of triangle ABC:

$$(12)\qquad \lambda^2\Delta = 48\cos^2\frac{\phi+\alpha}{2}\cos^2\frac{\phi+3\alpha}{2}\cos^2\frac{\phi+5\alpha}{2} = 3\sin^2\frac{3\phi}{2},$$

$$(13)\qquad \lambda r = \tfrac{2}{3}\sin^2\tfrac{3}{2}\phi,$$

$$(14)\qquad \lambda r_a = 6\sin^2\tfrac{1}{2}\phi,\qquad \lambda r_b = 6\sin^2(\tfrac{1}{2}\phi-\alpha),\qquad \lambda r_c = 6\sin^2(\tfrac{1}{2}\phi+\alpha),$$

$$(15)\qquad \tan\tfrac{1}{2}A = \tfrac{4}{3}\sin^2\tfrac{1}{2}\phi,\quad \tan\tfrac{1}{2}B = \tfrac{4}{3}\sin^2(\tfrac{1}{2}\phi-\alpha),\quad \tan\tfrac{1}{2}C = \tfrac{4}{3}\sin^2(\tfrac{1}{2}\phi+\alpha),$$

$$(16)\qquad 12\lambda R = 27 - 2\sin^2\tfrac{3}{2}\phi.$$

4. Bounds for the angles. The formulae (9) and (12) to (16) give some characteristic values for the triangles ABC. Obviously all triangles of the set are essentially determined if ϕ is chosen in the interval $0 < \phi \leqslant \alpha$, the image-point then lying on the arc D_1Q_3 of K. We have then $A \leqslant B \leqslant C$. From (15) we draw the conclusion that $1 < \tan\frac{1}{2}C \leqslant \frac{4}{3}$, from which it follows that ABC is always an obtuse-angled triangle, but the largest angle is at most $2\arcsin\frac{4}{5}$ (that is, about 106°); this maximum is found for the isosceles triangle of the set, namely $a:b:c = 5:5:8$. Furthermore $0 < \tan\frac{1}{2}A \leqslant \frac{1}{3}$ and therefore $A \leqslant \arcsin\frac{3}{5}$ (that is, about 37°); for the angle B we find $\arcsin\frac{3}{5} \leqslant B < \frac{1}{2}\pi$.

5. Centres of similitude. G. R. Veldkamp has noticed that the centres of similitude of the circles E_1 and E_2 are the incentre I and the Gergonne point M (which lies on the lines joining the vertices to the points of contact of the opposite sides with the incircle). Therefore, in a "remarkable" triangle (such that E_2 is a straight line l), the circle E_1 has IM as a diameter; moreover, l is the polar line of M with respect to the incircle.

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