THE DUALITY THEOREM FOR CURVES OF ORDER n IN n-SPACE

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LET C_n be a curve in real projective n-space which is a continuous $1-1$ image of either the projective line or one of its closed segments. Consequently its points depend continuously on a real variable s for which $0 \leqslant s \leqslant 1$, with the understanding that $s = 0$ and $s = 1$ represent the same curve point in the case that C_n is the image of the complete projective line. The points of C_n will be described by their corresponding real numbers s.

We assume

(1) No $(n-1)$-dimensional hyperplane H cuts C_n in more than n points. An immediate consequence of the above is that any $k+1$ distinct curve points generate a linear k-subspace.

We assume

(2) The linear k-subspace L generated by $k+1$ curve points always converges to a linear k-subspace designated by (k, s) as the $k+1$ points all con verge to s, $0 \leqslant k < n$.

The subspaces (k, s) enable us to count multiple intersection points of a linear subspace L with C_n. A point s is said to be within L k-fold if $(k-1, s) \subset L$, $(k, s) \not\subset L$. We now assume that (1) and (2) are both true when the multiple intersection points of both H and L are counted by the above convention.

In 1936 Scherk[1] gave the first proof that the dual of C_n has properties (1) and (2). His proof first derives the result for the case where C_n is the map of the whole projective line and then derives the general result by showing that every C_n is part of such a curve. In the following an alternative proof is given which applies directly to any C_n. The methods are elementary. Use is made of the easily established fact that the projection of a C_n from one of its points s' is a C_{n-1} and each (k, s) of C_n projects either into a (k, s), $0 \leqslant k \leqslant n-2$, or into a $(k-1, s)$, $1 \leqslant k \leqslant n-1$, for the projected curve according as either $s' \neq s$ or $s' = s$.

THEOREM 1. *Where $\bar{s}$ is an interior point of C_n let $s^{\mu}{}_1$, $s^{\mu}{}_2$ be two sequences of real numbers which approach $\bar{s}$ and for which $s^{\mu}{}_1 \neq s^{\mu}{}_2$. If P^{μ} be a convergent sequence of space points selected from the intersection of $(n-1, s^{\mu}{}_1)$ and $(n-1, s^{\mu}{}_2)$ then it converges to a point P of $(n-2, \bar{s})$.*

For the proof of this result we shall use

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[1]P. Scherk. *Über differenzierbare Kurven und Bögen* II. Casopis pro pěstování, matematiky a fysiky 66 (1937), 172-191.

LEMMA 1. *If $\bar{s}$ is an interior point of C_n and $P \in (n-1, \bar{s})$ but P* non $\in$ $(n-2, \bar{s})$ *then for every sufficiently small curve neighborhood $I(\bar{s})$ a curve neighborhood $J(\bar{s})$, $J(\bar{s}) \subset I(\bar{s})$, together with a space neighborhood $N(P)$ of P exists with the following properties*:

(1) Curve points $s, s_1, s_2, \ldots, s_{n-2}$ from $J(\bar{s})$ and a point P' of $N(P)$ build a hyperplane which cuts $I(\bar{s})$ in exactly one additional point $q(s)$. (Some or all of $s_1, s_2, \ldots, s_{n-2}$ may coincide.)

(2) As s moves continuously in one direction in $J(\bar{s})$, $q(s)$ moves continuously in the opposite direction so that $q(s') \neq q(s'')$ if $s' \neq s''$.

Proof of Lemma. As the lemma deals with local properties of C_n it is sufficient to prove it within an affine n-subspace of the projective space which contains P and $\bar{s}$. By hypothesis the linear $n-2$-subspace generated by any $n-1$ curve points will approach $(n-2, \bar{s})$ as these points all approach $\bar{s}$. Therefore and because P non $\in (n-2, \bar{s})$ a curve neighborhood $I(\bar{s})$, i.e. a set of points s containing $\bar{s}$ for which $s_a < s < s_b$, together with a point P' sufficiently close to P will always generate a hyperplane H. H converges to $(n-1, \bar{s})$ as $P' \to P$ and $s, s_1, s_2, \ldots, s_{n-2}$ converge to $\bar{s}$. The endpoints s_a, s_b of $I(\bar{s})$ will be on the same or opposite sides of H according as they are on the same or opposite sides of $(n-1, \bar{s})$ provided $s, s_1, s_2, \ldots, s_{n-2}$ are in a sufficiently small neighborhood $I'(\bar{s})$ and P' in a sufficiently small neighborhood N' of P. In this event the number of intersection points of H and $I(\bar{s})$ will be odd or even according as n is odd or even. Therefore H cuts $I(\bar{s})$ in a point $q(s)$ in addition to the points $s, s_1, \ldots, s_{n-2}$ and in no further points because of the order of C_n by (1). For fixed $s_1, s_2, \ldots, s_{n-2}$, $q(s)$ moves continuously with s because H moves continuously with s. As $q(s), s_1, \ldots, s_{n-2}$ and P' define H completely, two different positions of s cannot define the same $q(s)$ because the order of the curve would exceed n in this case. For the same reason $q(s)$ cannot experience a reversal as s moves continuously in a fixed direction. As $H \to (n-1, \bar{s}), q(s) \to \bar{s}$. Hence neighborhoods $J(\bar{s})$, $N(P)$ with $J(\bar{s}) \subset I'(\bar{s})$, $N(P) \subset N'$ exist so that if $s, s_1, s_2, \ldots, s_{n-2} \in J(\bar{s})$, $P' \in N(P)$ then $q(s) \in I'(\bar{s})$ Consequently $q(q(s))$ is defined and must be equal to s as $q(s), s_1, s_2, \ldots, s_{n-2}$ and P' define a unique hyperplane. If we project from $s_1, s_2, \ldots, s_{n-2}, P'$ then C_n will be projected into a curve of order two on the affine line. Points for which $s = q(s)$ will be projected into the reversal points of such a curve and as there are at most two such points we conclude $q(s) \neq s$ with at most two possible exceptions. Let $s' \in J(\bar{s})$, $q(s') \neq s'$. Then $q(s') \in I'(\bar{s})$. Let s move continuously in a fixed direction in $I'(\bar{s})$ from s' to $q(s')$. $q(s)$ will move from $q(s')$ to s' in a fixed direction and remain in $I(\bar{s})$. As $I(\bar{s})$ is not the whole curve C_n this can only happen if $q(s)$ moves in the direction opposite to that of s. The lemma is now completely proved.

We write $q(s)$ as $q(s, s_1, s_2, \ldots, s_{n-2})$ because it is a function of the $n-1$ variables $s, s_1, s_2, \ldots, s_{n-2}$. If any one of these variables moves in a fixed direction in $J(\bar{s})$ while all the others remain fixed, $q(s, s_1, \ldots, s_{n-2})$ will move

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in the opposite direction. To prove the theorem we note that, as P is the limit of P^μ, $P \in (n-1, \bar{s})$. We assume P non $\in (n-2, \bar{s})$, construct neighborhoods $I(\bar{s})$, $J(\bar{s})$, $N(P)$, satisfying the conditions of the lemma and select $s^\mu{}_1, s^\mu{}_2 \in J(\bar{s}), P^\mu \in N(P)$. Because $P^\mu \in (n-1, s^\mu{}_1)$, $q(s^\mu{}_1, s^\mu{}_1, \ldots, s^\mu{}_1) = s^\mu{}_1$. Now if we move each of the variables successively from $s^\mu{}_1$ to $s^\mu{}_2$ the point q will move in the opposite direction and remain on $I(\bar{s})$ in accordance with the lemma. But as $I(\bar{s})$ is not the whole curve C_n and $q(s^\mu{}_2, s^\mu{}_2, \ldots, s^\mu{}_2) = s^\mu{}_2$, this is impossible. Hence $P \in (n-2, \bar{s})$ and the theorem is proved.

THEOREM 2. *If s belongs to an arc $s_1 < s < s_2$ then not all of $(n-1, s)$ can pass through a single point.*

Proof. The result is true for a C_1 as by definition two different values of s define different curve points $(0, s)$. We assume the result true for C_{n-1} and proceed by induction. Should an arc $s_1 < s < s_2$ of C_n exist together with a point P so that all $(n-1, s)$, $s_1 < s < s_2$, pass through P then by Theorem 1 all $(n-2, s)$, $s_1 < s < s_2$, must pass through the same point. If we project the curve C_n from one of its points the resulting curve is a C_{n-1} for which all $(n-2, s)$, $s_1 < s < s_2$ pass through the projection of P. This contradicts the induction assumption and thus the theorem is proved.

DEFINITION. A system of linear subspaces $S^\mu{}_r$ is defined to converge to a subspace S_r if a basis $\mathbf{a}^\mu{}_1, \mathbf{a}^\mu{}_2, \ldots, \mathbf{a}^\mu{}_{r+1}$ exists for each $S^\mu{}_r$, with $\mu \geqslant \mu_0$, such that $\mathbf{a}^\mu{}_k$, $1 \leqslant k \leqslant r+1$, converges to $\mathbf{a}_k$ where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{r+1}$ is a basis of S_r.

LEMMA 2. *$S^\mu{}_r$ is a set of linear subspaces of dimension $\geqslant r$, $0 \leqslant r < n$, defined for positive integers μ. The limit points of any point set P^μ, $P^\mu \in S^\mu{}_r$, are all within a linear r-subspace S_r. Then $S^\mu{}_r$ converges to S_r as μ approaches infinity.*

Proof. Let T_{n-r-1} be any linear $(n-r-1)$-subspace such that the projective n-space is the direct sum of T_{n-r-1} and S_r. We choose μ_0 so large that $S^\mu{}_r$ contains no elements of T_{n-r-1} for $\mu \geqslant \mu_0$. This is possible as T_{n-r-1} is a closed compact set which contains no elements of S_r. If vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{r+1}$ form a basis of S_r each $S^\mu{}_r$, $\mu \geqslant \mu_0$ will have a basis $\mathbf{a}_1 + \mathbf{p}_1$, $\mathbf{a}_2 + \mathbf{p}_2, \ldots, \mathbf{a}_{r+1} + \mathbf{p}_{r+1}$ where the vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{r+1}$ define points of T_{n-r-1}. Hence all these $S^\mu{}_r$ will have dimension r. All the vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{r+1}$ must approach the null vector as μ approaches infinity otherwise we could construct a subsequence which would contradict the hypothesis. Thus the lemma is proved.

We introduce the following multiplicity convention:

A point P is said to be within the space $(n-1, s)$ exactly k-fold if $P \in (n-k, s)$, P non $\in (n-k-1, s)$, $0 < k < n$, and n-fold if $P = s$.

LEMMA 3. *For $n > 1$, $k > 1$ an arc A of C_n contains points $s_1, s_2, \ldots, s_k$ with $s_1 \leqslant s_2 \leqslant \ldots \leqslant s_k$ and all different from one of its endpoints s_a. P is a space* *point for which $P \neq s_a$ and $P \in (n-1, s_i)$, $1 \leqslant i \leqslant k$. Then the projection of P from s_a will be included within at least $k-1$ spaces $(n-2, s)$ of the projection C_{n-1} of C_n for which $s_1 \leqslant s \leqslant s_k$. Multiple inclusions are to be interpreted in accordance with the multiplicity convention.*

Proof. For s on the given arc A of C_n let $Q(s)$ be the intersection of $(n-1, s)$ and the line s_aP; $Q(s)$ is uniquely defined except possibly for $s = s_a$. It is continuous as $(n-1, s)$ is continuous by (2). It cannot cover the full projective line $s_a P$ as $Q(s) \neq s_a$, $s \neq s_a$, for all s in A including the second endpoint. For $i < k$ let $s_i < s_{i+1}$; $Q(s_i) = Q(s_{i+1}) = P$ but $Q(s)$ cannot be equal to P for all s with $s_i < s < s_{i+1}$ by Theorem 2. Hence $Q(s)$ must attain an extremum at a point s'_i for which $s_i < s'_i < s_{i+1}$. Within every curve neighborhood of s'_i two points separated by s'_i must exist for which $Q(s)$ attains the same value. Then by Theorem 1 and the continuity of $Q(s)$, $Q(s) \in (n-2, s'_i)$.

Let m be the number of different values of s_i and let s_j run through each of these different values exactly once. Let n_j be the number of s_i which assume the value s_j. By hypothesis $\sum_j n_j = k$. Let $\overline{P}$ be the projection of P from s_a and C_{n-1} that of C_n. As the space $(n-2, s'_i)$ of C_n projects into the space $(n-2, s'_i)$ of C_{n-1}, $\overline{P} \in (n-1-1, s'_i)$. Similarly, if $P \in (n-n_j, s_j)$ of C_n then $\overline{P} \in (n-1-(n_j-1), s_j)$ of C_{n-1}. Hence $\overline{P}$ is contained in at least $m - 1 + \sum_j (n_j - 1) = k - 1$ spaces $(n-2, s)$ of C_{n-1} for which $s_1 \leqslant s \leqslant s_k$. Thus the lemma is proved.

THEOREM 3. *No space point P is within more than n spaces $(n-1, s)$ of C_n.*

Proof. This theorem is the statement that the dual of C_n has property (1). As C_1 is self-dual it is true for C_1. We assume the result for curves C_{n-1} and proceed by induction. If the result is false for a curve C_n then an arc of this curve exists with distinct endpoints s_a, s_b together with $n+1$ points $s_1, s_2, \ldots, s_{n+1}$ with $s_a \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_{n+1} \leqslant s_b$ so that $P \in (n-1, s_i)$, $1 \leqslant i \leqslant n+1$. Multiple inclusions are interpreted in accordance with the multiplicity convention. P cannot be the point s_a for in this case P would be included in $(n-1, s_a)$ n-fold and by (1) (with the added multiplicity convention) in no other spaces $(n-1, s)$. Let P be included in $(n-1, s_a)$ k-fold, $0 \leqslant k < n$ where $k = 0$ is to be interpreted as P non $\in (n-1, s_a)$. Then P is contained in $n - k + 1$ spaces $(n-1, s)$ with $s \neq s_a$. If we project from s_a then the projection $\overline{P}$ of P will, by Lemma 3, be contained in at least $n - k$ spaces $(n-2, s)$ of the projected curve C_{n-1} in addition to being contained in $(n-2, s_a)$ k-fold. In all, $\overline{P}$ is contained in at least n spaces $(n-2, s)$ of C_{n-1} in contradiction to the induction assumption. Hence P can be contained in at most n spaces $(n-1, s)$ and the theorem is proved.

THEOREM 4. *Points $s^{\mu}_1, s^{\mu}_2, \ldots, s^{\mu}_{k+1}$ are defined for $\mu = 0, 1, 2, 3, \ldots,$ and all converge to $\bar{s}$ as μ approaches infinity. Then the intersection S^{μ} of the* *spaces* $(n-1, s^{\mu}_1)$, $(n-1, s^{\mu}_2), \ldots, (n-1, s^{\mu}_{k+1})$, $0 \leqslant k < n$, *converges to* $(n-k-1, \bar{s})$. *The points of* S^{μ} *are to be included h-fold within any hyperplane which occurs h times in this set.*

Proof. The theorem is the statement that the dual of (2) is true for C_n. For $k = 0$ the result is a statement of the continuity of $(n-1, s)$ which we assume by (2). In particular the result is true for C_1. Therefore let $k > 0$. We assume the result for C_{n-1} and proceed by induction. We select a point P^{μ} from each S^{μ}. As the dimension of $S^{\mu} \geqslant n-k-1$ the truth of the theorem will result from Lemma 2 if we prove that every convergent subsequence P^{ν} of P^{μ} has its limit P within $(n-k-1, \bar{s})$. We may assume $s^{\mu}_1 \leqslant s^{\mu}_2 \leqslant \ldots \leqslant s^{\mu}_{k+1}$. With the help of Theorem 2 we select an arc A containing $\bar{s}$ for one of the endpoints s_a of which $\bar{s} \neq s_a$ and P non $\in (n-1, s_a)$. If we choose P^{ν} sufficiently close to P, we may assume P^{ν} non $\in (n-1, s_a)$ and also, if $s^{\nu}_1, s^{\nu}_2, \ldots, s^{\nu}_{k+1}$ are sufficiently close to $\bar{s}$, that these points will be within A and different from s_a. Let $\bar{P}$ be the projection of P from s_a, C_{n-1} that of C_n and $\bar{P}^{\nu}$ that of P^{ν}. By Lemma 3, $\bar{P}$ will be contained in k spaces $(n-2, s)$ of C_{n-1} with $s^{\nu}_1 \leqslant s \leqslant s^{\nu}_{k+1}$. $\bar{P}^{\nu}$ will converge to $\bar{P}$ and, by the induction assumption applied to C_{n-1}, $\bar{P} \in (n-1-k, \bar{s})$. Therefore P is contained in the space generated by s_a and $(n-k-1, \bar{s})$ of C_n. If P non $\in (n-k-1, \bar{s})$. then s_a will be in the space generated by P and $(n-k-1, \bar{s})$. As s_a may be chosen in infinitely many ways this would contradict the assumption (1). Hence $P \in (n-k-1, \bar{s})$. The theorem is then completely proved.

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