THE FRÉCHET VARIATION, SECTOR LIMITS, AND LEFT DECOMPOSITIONS

MARSTON MORSE AND WILLIAM TRANSUE

1. Introduction. The Fréchet variation of a function g defined over a 2-interval I^2 was introduced by Fréchet to enable him to generalize Riesz's theorem on the representation of functionals linear over the space C [7]. Recently the authors have found this variation fundamental in the study of functionals bilinear over the Cartesian product $A \times B$ of two normed linear spaces with certain characteristic properties, and in the further use of this theory in spectral and variational analysis. The recent discovery by the authors of several new properties of the Fréchet variation has made it possible to to give new and natural tests for the convergence of multiple Fourier series generalizing the classical Jordan, de la Vallée Poussin, Dini, Young and Lebesgue tests under considerably less restrictive hypotheses than those now accepted.

Many of the tests which generalize the classical tests make use of the Vitali variation $V(g)$. The theory so developed depends in essential ways on the decomposition of g into the difference of two monotone functions $P - N$, following the model of Jordan. No such decomposition is possible or needed when the Fréchet variation is used and $\mu > 1$. The classical second law of the mean has a counterpart in a fundamental inequality (see §9) governing multiple integrals, such as the Dirichlet integral. The use of the Fréchet variation makes this inequality possible and relieves a tendency to overuse absolute values.

As a result of the theorems of this paper the Fréchet variation now parallels the Jordan variation in striking fashion. The assumptions $\hat{F}$ on g are that the variation $P^\mu(g)$ is finite and that there is at least one r-section ($r = 1, \ldots, \mu$) of the interval I^μ parallel to each coordinate r-plane, on which $P^r(g)$ is finite. The condition $P^\mu(g) < \infty$ is much weaker than the condition $V(g) < \infty$ [12]. We enumerate the points of similarity of the Fréchet variation with the Jordan variation.

(1) If g satisfies $\hat{F}$ over I^μ, g is bounded and L-measurable [9] over I^μ. Its points of discontinuity lie on at most a countable number of $(\mu - 1)$-planes parallel to the coordinate $(\mu - 1)$-planes (see Theorem 8.4).

(2) With each point a in μ-space let 2^μ sectors S_a be associated, being the respective open regions into which the μ-space R^μ is divided by the $(\mu - 1)$-planes intersecting a parallel to the coordinate $(\mu - 1)$-planes. Extending

Received August 6, 1949. The contribution of Dr. Transue to this paper is in partial fulfilment of a contract between the Office of Naval Research and the Institute for Advanced Study.

the notion of the left and right limits for a function of bounded Jordan variation, we here prove that when g satisfies conditions $\hat{F}$, $g(s)$ has a limit as $s \to a$ from any one of the open sectors S_a. These 2^μ limits may be different at a.

(3) Generalizing left and right continuity, a function g defined over I^μ may be termed S-continuous ($\bar{S}$-continuous) of orientation type S_a, if $g(s) \to g(a)$ as $s \to a$ from S_a ($\bar{S}_a$) for each point a of I^μ. Given a g which satisfies conditions $\hat{F}$ over an open u-interval I^μ, and given a sector S_a invariant in orientation as a is varied, there exists a unique function g^S equal to g at the points of continuity of g and S-continuous ($\bar{S}$-continuous) of type S_a ($\bar{S}_a$). Certain additional properties of $P^\mu(g^S)$ as a minimum modulus of multilinear functionals cannot be described in the space of this introduction (See [8] and §8).

(4) When $\mu = 1$, the Jordan variation of g on a subinterval Q^1 of I^1 tends to zero as the vertices of Q^1 approach any point s_0 of I^1 from the open right or open left of s_0. If a is any point of I^μ the Fréchet variation of g over an r-interval Q^r in a fixed open sector S_a with vertex at a tends to zero as the vertices of Q^r tend to a, provided g satisfies $\hat{F}$ over I^μ (see Corollary 6.2).

The theorem [13] which extends the Jordan test in 1-dimension is as follows:

THEOREM 1.1. *Let g satisfy $\hat{F}$ over a closed μ-interval $I^\mu[\mathbf{0}, 2\pi\mathbf{i}]$ and have the period 2π in each coordinate. Then the multiple Fourier series for g converges in the sense of Pringsheim to the mean of the 2^μ sector limits of g at a. If g is continuous in addition, this convergence is uniform. (See* [13].)

For a statement of the more restrictive Hardy Krause test generalizing the Jordan test, see [6], where other tests are compared and developed. For the more recent use of spherical means in Fourier theory see [2] and [3]. For a theorem on convergence almost everywhere see [4]. Theorem 1.1 will be proved in paper [13]. The left decomposition of g obtained in §7 and the existence of a variation modulus (Theorem 7.3) are essential for the μ-dimensional proof.

2. Definitions and notations. Let R^μ denote a Cartesian space of points s with coordinates $s = [s^1, \ldots, s^\mu]$. Let $a = [a^1, \ldots, a^\mu]$ and $b = [b^1, \ldots, b^\mu]$ be two points in R^μ with $a^r < b^r$ ($r = 1, \ldots, \mu$). Let J_r represent an interval for s^r chosen from the intervals

$$(a^r, b^r), \ (a^r, b^r], \ [a^r, b^r), \ [a^r, b^r]. \tag{2.0}$$

By a μ-interval in R^μ determined by a and b we mean a Cartesian product

$$I^\mu = J_1 \times J_2 \times \ldots \times J_\mu. \tag{2.1}$$

When $J_r = (a^r, b^r)$ for each r, we shall denote I^μ by $I^\mu\,(a, b)$. The intervals $I^\mu\,(a, b]$, $I^\mu\,[a, b)$ and $I^\mu\,[a, b]$ are similarly defined. Thus $I^\mu\,(a, b)$ is open and $I^\mu\,[a, b]$ closed. By the left *r-closure* C_rJ_r of J_r is meant the union of J_r and the point $s^r = a^r$. The *left r-closure* C_rI^μ is then defined to be

$$C_rI^\mu = J_1 \times \ldots \times J_{r-1} \times C_rJ_r \times J_{r+1} \ldots \times J_\mu. \tag{2.2}$$

Note that for $r, m = 1, \ldots, \mu$,

$$C_r C_m I^\mu = C_m C_r I^\mu; \qquad C_r C_r I^\mu = C_r I^\mu. \tag{2.3}$$

By an *r-segment* Q^r in R^μ is meant a Cartesian product of the form (2.1) in which $\mu - r$ of the J_i's are points and the remaining r of the J_i's are intervals as above ($r = 1, \ldots, \mu$). If J_m is an interval, the *left m-closure* $C_m Q^r$ is defined as above. The orthogonal projection on a coordinate $(\mu - 1)$-plane $[s^m = 0]$ of an r-segment Q^r will be denoted by $X_m Q^r$. Observe that for m, $n = 1, \ldots, \mu$,

$$X_m X_n Q^r = X_n X_m Q^r; \qquad X_n X_n Q^r = X_n Q^r. \tag{2.4}$$

Let an r-segment Q^r in R^μ be given in the form

$$Q^r = J_1 \times \ldots \times J_\mu \qquad [r = 1, \ldots, \mu]. \tag{2.5}$$

By an *n-face* of Q^r will be meant an n-segment F^n ($0 < n \leqslant r$) of the form

$$F^n = J'_1 \times \ldots \times J'_\mu, \tag{2.6}$$

in which $J'_i = J_i$ if J_i is a point, while J'_i is an end point of J_i, or one of the intervals (2.0) in case J_i is one of these intervals. In particular Q^r is included as one of its own faces as is its closure $\bar{Q}^r$. It will also be convenient to refer to the vertices of Q^r as 0-faces of Q^r.

An r-segment Q^r of the form (2.5) will be said to be *left-closed* or *left-open* according as all of the 1-intervals J_i in the product (2.5) are closed, or open at the left. The terms right-closed and right-open are similarly defined.

Let g be a function mapping a closed μ-interval I^μ $[a, b]$ in R^μ into R^1.

A PARTITION π. A partition π of I^μ $[a, b]$ will be defined by giving partitions π^r of the respective intervals $[a^r, b^r]$ ($r = 1, \ldots, \mu$). The points of partition of $[a^r, b^r]$ in π^r shall satisfy the condition ($r = 1, \ldots, \mu$),

$$a^r = t^r_0 < t^r_1 < \ldots < t^r_m = b^r \qquad [\text{where } m = n^r(\pi)].$$

Corresponding to π^r let i be an integer on the range $1, \ldots, n^r(\pi)$ and set

$$\begin{aligned} &\Delta^r_i g(s^1, \ldots, s^{r-1}, \,.\, , s^{r+1}, \ldots, s^\mu) \\ &= g(s^1, \ldots, s^{r-1}, t_i^r, s^{r+1} \ldots, s^\mu) \\ &- g(s^1, \ldots, s^{r-1}, t^r_{i-1}, s^{r+1}, \ldots, s^\mu). \end{aligned} \tag{2.7}$$

A dot is used in place of an argument s^r to indicate that a function over the range of s^r is represented with the remaining displayed variables constants.

For t^r_i and t^r_{i-1} fixed the right member of (2.7) gives the *values* of a function to be denoted by $\Delta^r_i\, g$, mapping the (μ-1)-segment $X_r I^\mu$ into R^1. We understand that the differencing operator Δ^r_i may be applied not only to g but to any function defined on an m-segment in R^μ whose orthogonal projection on the rth coordinate axis is $[a^r, b^r]$.

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

With this understood suppose that $\mu \geqslant 2$. For a fixed partition π of $I^\mu[a,b]$,

$$\Delta^1{}_m \, \Delta^2{}_n \, g \qquad [m=1, \ldots, n^1(\pi); n=1, \ldots, n^2(\pi)]$$

is a function mapping the $(\mu - 2)$-segment $X_1X_2I^\mu$ into R^1. In general for $1 \leqslant m \leqslant \mu$, for fixed π, and for

$$n^r = 1, 2, \ldots, n^r(\pi) \qquad [1 \leqslant r \leqslant \mu], \tag{2.8}$$

the function

$$\Delta^1{}_{n^1} \, \Delta^2{}_{n^2} \ldots \Delta^m{}_{n^m} \, g \tag{2.9}$$

is defined over $X_1X_2 \ldots X_mI^\mu$.

Corresponding to the partition π we introduce μ sets of constants

$$\begin{aligned} e^1 &= [e^1{}_1, \ldots, e^1{}_{n^1(\pi)}], \\ &\ldots \qquad \ldots \\ e^\mu &= [e^\mu{}_1, \ldots, e^\mu{}_{n^\mu(\pi)}], \end{aligned}$$

associating $e^r{}_i$ with the ith interval of π^r. It will be convenient to set $[e^1, \ldots, e^r] = \mathbf{e}$.

We admit only those constants $e^r{}_i$ *whose absolute values are at most* 1.

THE FRÉCHET VARIATION $P^\mu[g, I^\mu]$. In the following definition and subsequent applications the summation convention of tensor algebra will be used. It will not be used in other connections. For $\mu = 1$, set

$$P^1[k, I^1] = \sup_{\pi, \mathbf{e}} e^1{}_n \, \Delta^1{}_n g \qquad [n = 1, \ldots, n^1(\pi)]$$

taking the sup over all admissible partitions π^1 of $[a^1, b^1]$ and associated sets e^1. Observe that when $\mu = 1$ this is the ordinary total Jordan variation $T[g, I^1]$ of g over I^1. For the case $\mu = 2$ see [5] and [7]. For a general μ we introduce the preliminary sum

$$\sigma^\mu[g, I^\mu, \pi, \mathbf{e}] = e^1{}_{n^1} \, \Delta^1{}_{n^1} \ldots e^\mu{}_{n^\mu} \, \Delta^\mu{}_{n^\mu} \, g, \tag{2.10}$$

where n^r has the range (2.8). The Fréchet variation of g over I^μ is then defined by setting

$$P^\mu[g, I^\mu] = \sup_{\pi, \mathbf{e}} \sigma^\mu[g, I^\mu, \pi, \mathbf{e}].$$

This variation may be finite or infinite. On the right of (2.10) the order of the differencing operators $\Delta^r{}_{n^r}$ is immaterial.

The following lemma is fundamental. There is just one relation when $\mu = 2$, established by Fréchet.

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

LEMMA 2.1. *The variation $P^{\mu}[k, I^{\mu}]$ has the values*

$$\begin{aligned}&\sup_{\pi,\,\mathbf{e}} P^{\mu-1}[e^1{}_{n^1}\,\Delta^1{}_{n^1}\,g,\ X_1 I^{\mu}]\\ &= \sup_{\pi,\,\mathbf{e}} P^{\mu-2}[e^1{}_{n^1}\,\Delta^1{}_{n^1}\,e^2{}_{n^2}\,\Delta^2{}_{n^2}\,g,\ X_1X_2I^{\mu}]\\ &\ldots \qquad\qquad\qquad \ldots\\ &= \sup_{\pi,\,\mathbf{e}} P^{1}[e^1{}_{n^1}\,\Delta^1{}_{n^1}\ldots e^{\mu-1}{}_{n^{\mu-1}}\,\Delta^{\mu-1}{}_{n^{\mu-1}}\,g,\ X_1X_2\ldots X_{\mu-1}\,I^{\mu}],\end{aligned}$$

where the range of n^r is given in (2.8).

To establish these relations recall that

$$P^{\mu}[g, I^{\mu}] = \sup_{\pi,\,\mathbf{e}} \sigma^{\mu}[g, I^{\mu}, \pi, \mathbf{e}]. \tag{2.11}$$

We shall obtain the sup in (2.11) in two steps. Taking the sup of $\sigma^{\mu}[g, I^{\mu}, \pi, \mathbf{e}]$ over all admissible partitions $\pi^{r+1}, \ldots, \pi^{\mu}$ and associated sets $e^{r+1}, \ldots, e^{\mu}$, keeping $\pi^1, \ldots, \pi^r$ and $e^1, \ldots, e^r$ fixed, one obtains the relation

$$\begin{aligned}&\sigma^{\mu}[g, I^{\mu}, \pi, \mathbf{e}]\\ &\qquad \leqslant P^{\mu-r}[e^1{}_{n^1}\,\Delta^1{}_{n^1}\ldots, e^r{}_{n^r}\,\Delta^r{}_{n^r}\,g,\ X_1X_2\ldots X_rI^{\mu}]\\ &\qquad \leqslant P^{\mu}[g, I^{\mu}].\end{aligned} \tag{2.12}$$

Completing the process indicated in (2.11) we now take the sup of the first two members of (2.12). The definition of $P^{\mu}[g, I^{\mu}]$ then implies that

$$P^{\mu}[g, I^{\mu}] \leqslant \sup_{\pi,\,\mathbf{e}} \Phi \leqslant P^{\mu}[g, I^{\mu}],$$

where Φ is the middle term in (2.12). The lemma follows.

THE GENERAL VARIATION $P^r[g, Q^r]$. The preceding definition has been given for the case of a closed μ-interval $I^{\mu}[a, b]$. If Q^{μ} is a general μ-interval, we give the definition

$$P^{\mu}[g, Q^{\mu}] = \sup_{I^{\mu}} P^{\mu}[g, I^{\mu}],$$

taking the sup over all closed μ-intervals $I^{\mu} \subset Q^{\mu}$. We shall also define $P^r[g, Q^r]$ in case Q^r is an r-segment in I^{μ}. Such an r-segment lies in an r-plane θ^r. Referring θ^r to any coordinate system with axes parallel to coordinate axes in R^{μ} and corresponding coordinates $t = [t^1, \ldots, t^r]$, the function g given over I^{μ} defines a function h over Q^r with values $h(t)$. In θ^r, $P^r[h, Q^r]$ is well defined and we set

$$P^r[g, Q^r] = P^r[h, Q^r].$$

It is evident that when $Q_1{}^r \subset Q^r$,

$$P^r[g, Q_1{}^r] \leqslant P^r[g, Q^r]. \tag{2.13}$$

If $Q^r = Q_1{}^r \cup Q_2{}^r$, where $Q_1{}^r$ and $Q_2{}^r$ are two non-overlapping closed r-segments, it is clear that

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

$$P^r[g, Q^r] \leqslant P^r[g, Q_1^r] + P^r[g, Q_2^r].$$

Finally, as shown in [7, Lemma 3.4] in the case $\mu = 2$, for a general μ one can restrict the constants $e^r{}_i$ to values ± 1 in defining $P^\mu[g, I^\mu]$ without changing the variation thereby defined.

3. Conditions F and $\hat{\mathbf{F}}$. Let g be defined on a general μ-interval I^μ of R^μ, and let Q^r be any r-segment in I^μ. Condition F on g over Q^r requires that $P^r[g, Q^r]$ be finite. Conditions $\hat{F}$ on g over Q^r also require that $P^r[g, Q^r]$ be finite and that there shall be an m-section H^m of Q^r by an m-plane parallel to an arbitrary m-face of Q^r, $0 < m \leqslant r$, such that $P^m[g, H^m]$ is finite.

THEOREM 3.1. *If g satisfies $\hat{F}$ over I^μ, then $P^r[g, Q^r]$ is bounded over all r-segments Q^r of I^μ.*

The theorem is clearly true when $r = \mu$ since $P^\mu[g, I^\mu]$ is finite. Taking account of the definition of the Fréchet variation for non-closed intervals Q^r it is sufficient to prove the theorem for closed r-segments Q^r.

We begin with the case in which $r = \mu - 1$. Without loss of generality we can suppose that $Q^{\mu-1}$ lies in a $(\mu - 1)$-plane $[s^1 = a^1]$. By hypothesis there is at least one $(\mu - 1)$-section $H^{\mu-1}$ of I^μ by a $(\mu - 1)$-plane $s^1 = b^1$ such that

$$P^{\mu-1}[g, H^{\mu-1}] < \infty.$$

We shall show that

$$P^{\mu-1}[g, Q^{\mu-1}] \leqslant P^\mu[g, I^\mu] + P^{\mu-1}[g, H^{\mu-1}], \tag{3.0}$$

thereby establishing the theorem when $r = \mu - 1$. When $a^1 = b^1$, $Q^{\mu-1} \subset H^{\mu-1}$ and (3.0) holds trivially. For definiteness suppose that $a^1 < b^1$. The case $a^1 > b^1$ will be seen to be similar.

Let $Q_0^{\mu-1}$ be the orthogonal projection of $Q^{\mu-1}$ into $H^{\mu-1}$ and let J^μ be the μ-interval with the faces $Q^{\mu-1}$ and $Q_0^{\mu-1}$. Let π be an arbitrary partition of J^μ specialized in that the only vertices in the partition π^1 of $[a^1, b^1]$ are to be a^1 and b^1. In the set $\mathbf{e}$ of constants associated with π we suppose that $e^1{}_1 = 1$. With J^μ represented as an interval $I^\mu[a, b]$, let π' be the partition of the $(\mu - 1)$ interval

$$[a^2, b^2] \times \ldots \times [a^\mu, b^\mu],$$

with π' defined by $\pi^2, \ldots, \pi^\mu$, and let $\mathbf{e}'$ be the subset of the constants $\mathbf{e}$ associated in π with the intervals defined by π'. Then

$$\sigma^\mu[g, J^\mu, \pi, \mathbf{e}] = [e^2{}_{n^2}\, \Delta^2{}_{n^2} \ldots e^\mu{}_{n^\mu}\, \Delta^\mu{}_{n^\mu}\, g]_{s^1=a^1}^{s^1=b^1},$$

or otherwise written

$$\sigma^\mu[g, J^\mu, \pi, \mathbf{e}] = \sigma^{\mu-1}[g, Q_0^{\mu-1}, \pi', \mathbf{e}'] - \sigma^{\mu-1}[g, Q^{\mu-1}, \pi', \mathbf{e}'].$$

Hence

$$\sigma^{\mu-1}[g, Q^{\mu-1}, \pi', \mathbf{e}'] \leqslant P^{\mu}[g, J^{\mu}] + P^{\mu-1}[g, Q_0^{\mu-1}]. \tag{3.1}$$

On taking the sup of the left member of (3.1) over all admissible π', $\mathbf{e}'$ defined as above, one has

$$\begin{aligned} P^{\mu-1}[g, Q^{\mu-1}] &\leqslant P^{\mu}[g, J^{\mu}] + P^{\mu-1}[g, Q_0^{\mu-1}] \\ &\leqslant P^{\mu}[g, I^{\mu}] + P^{\mu-1}[g, H^{\mu-1}]. \end{aligned} \tag{3.2}$$

The lemma follows for $(\mu - 1)$-segments in I^{μ}.

If Q^r is an arbitrary closed r-segment of I^{μ}, we shall refer to the condition

$$P^r[g, Q^r] < B^r, \tag{3.3r}$$

in which B^r is a constant independent of Q^r. Condition (3.3μ) is satisfied for proper choice of B^{μ}, since g satisfies F Proceeding inductively we shall assume that [3.3$(r + 1)$] holds for proper choice of B^{r+1} for closed $(r + 1)$-segments of I^{μ} $(1 < r + 1 \leqslant \mu)$, and show that (3.3$r$) holds for proper choice of B^r. By hypothesis there exists an r-section H^r of I^{μ} parallel to Q^r for which $P^r[g, H^r] < \infty$. As a matter of elementary geometry, there then exists a sequence,

$$Q_0^r, Q_1^r, \ldots, Q_{\nu}^r = Q^r, \qquad [0 \leqslant \nu \leqslant \mu - r]$$

of closed r-segments of I^{μ} parallel to the given closed r-segment Q^r, such that Q_0^r is in H^r and, if $\nu > 0$, Q_{i-1}^r and Q_i^r are r-faces of a closed $(r + 1)$-segment Q_i^{r+1} of I^{μ}. As in the proof of the preceding paragraph, it follows here that

$$P^r[g, Q_i^r] \leqslant B^{r+1} + P^r[g, Q_{i-1}^r] \qquad [i = 1, \ldots, \nu].$$

We infer that

$$\begin{aligned} P^r[g, Q^r] &\leqslant \nu B^{r+1} + P^r[g, Q_0^r] \\ &\leqslant \nu B^{r+1} + P^r[g, H^r] = B_1^r, \end{aligned}$$

introducing B_1^r. The constant B_1^r is a bound for the left member of (3.3r) whenever Q^r is parallel to H^r. It is clear that there are a finite number of sections H_i^r (such as H^r) such that each r-segment G^r is parallel to H_i^r for some i, and $P^r[g, H_i^r]$ is finite for each i, and we accordingly infer the existence of a bound B^r for the left member of (3.3r) as required.

The theorem follows.

COROLLARY 3.1. *Under conditions $\hat{F}$ on k over a general interval I^{μ}, $|k(s)|$ is bounded for $s \in I^{\mu}$.*

If $\mu = 1$, the corollary is true. Proceeding inductively we can assume that $|k(s)|$ has a bound B over a $(\mu - 1)$-section $H^{\mu-1}$ of I^{μ}. According to the theorem, $P^1[k, Q^1]$ taken over all 1-sections Q^1 of I^{μ} orthogonal to $H^{\mu-1}$ has a bound B_1. Hence $|k(s)| \leqslant B_1 + B$ for $s \in I^{\mu}$.

COROLLARY 3.2. *If k satisfies $\hat{F}$ over I^{μ}, the function $k|Q^r$ defined by k over any r-segment Q^r in I^{μ} satisfies $\hat{F}$ over Q^r.*

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

4. The Fréchet variation in the small. Let Q^r be an r-segment in R^μ. By the *vertex distance* of Q^r from a point $s \in R^\mu$ is meant the maximum of the distances of the vertices of Q^r from s.

We write $Q^r \to s$ in case the vertex distance of a variable Q^r from s tends to zero.

In the following sections it will be convenient to refer to a μ-interval determined by the origin O and the point i all of whose coordinates are 1. For this purpose we shall set[1]

$$\mathbf{0} = [0, \ldots, 0], \qquad \mathbf{i} = [1, \ldots, 1],$$

understanding that $\mathbf{0}$ and $\mathbf{i}$ are vectorial representations of O and i in the space R^μ unless the context indicates otherwise. There will be no loss of generality in replacing an interval I^μ (a, b) by I^μ $(\mathbf{0}, \mathbf{i})$ in the study of the Fréchet variation, since these intervals are images one of the other under 1-1 affine linear mappings of R^μ onto itself.

The following notation will aid in the proof of Theorem 4.1. Referring to (2.10), let it be understood that the typical term on the right of (2.10) corresponds to an *elementary μ-interval* whose Cartesian product representation is

$$[t^1{}_{n^1-1}, t^1{}_{n^1}] \times \ldots \times [t^\mu{}_{n^\mu-1}, t^\mu{}_{n^\mu}]. \tag{4.1}$$

It will be convenient to break I^μ up into the union of several non-overlapping intervals I_i^μ $(i = 1, \ldots, \omega)$ and to give a corresponding decomposition of the sum $\sigma^\mu [g, I^\mu, \pi, \mathbf{e}]$. We suppose that the vertices of I_i^μ $(i = 1, \ldots, \omega)$ are among the vertices of π. In this case, π defines a partition of I_i^μ which will be denoted by $\pi|I_i^\mu$. Corresponding to the set $\mathbf{e}$ of constants associated with the 1-intervals of π [typified by the factors of (4.1)], let $\mathbf{e}|I_i^\mu$ denote the subset of constants thereby associated with respective 1-intervals of $\pi|I_i^\mu$. We extend this notation to closed n-faces θ_i^n of I_i^μ so that $\pi|\theta_i^n$ is well defined and the associated set of constants is $\mathbf{e}|\theta_i^n$. With this understood we have

$$\sigma^\mu [g, I^\mu, \pi, \mathbf{e}] = \sum_i \sigma^\mu [g, I_i^\mu \, \pi|I_i^\mu, \mathbf{e}|I_i^\mu]. \tag{4.2}$$

The principal theorem of this section follows.

THEOREM 4.1μ. *If g satisfies $\hat{F}$ over $U^\mu = I^\mu$ $(\mathbf{0}, \mathbf{i})$, then $P^\mu [g, Q^\mu] \to 0$ as an arbitrary μ-interval $Q^\mu \to O$ in U^μ.*

Theorem 4.1μ is true when $\mu = 1$, as is well known. Proceeding inductively we shall assume $\mu > 1$ and that Theorem 4.1m is true for $m = 1, 2, \ldots, \mu - 1$. We shall then establish Theorem 4.1μ. To that end we shall assume Theorem 4.1μ false and arrive at a contradiction with the fact that $P^\mu [g, U^\mu]$ is finite. The assumption that Theorem 4.1μ is false can be equivalently stated as follows For a suitably chosen positive constant η there exist μ-intervals $I^\mu [u, v] \subset U^\mu$ with arbitrary small vertex distances from O and such that

[1]We are confining our use of boldface symbols for points to 0 and i to avoid the ambiguity which arises in the representation of these points, although one should logically extend the use of boldface to the general point s.

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

$$P^\mu [g, I^\mu] > \eta \tag{4.3}$$

We continue the proof with the following lemma.

LEMMA 4.1. *Let $K^\mu [a, b]$ be an arbitrary closed μ-interval in U^μ. Under the assumption that Theorem 4.1m is true for $m = 1, \ldots, \mu - 1$, and that (4.3) holds as stated, there exists an interval $H^\mu [u, b] \subset U^\mu$ with $0 < u^p < a^p$ $(p = 1, \ldots, \mu)$ such that*

$$P^\mu [g, H^\mu] > P^\mu [g, K^\mu] + \eta/2. \tag{4.4}$$

The interval $K^\mu [a, b]$ is given and fixed. We shall choose an interval $I^\mu [u, v] \subset U^\mu$ such that

$$0 < u^p < v^p < a^p \qquad [p = 1, \ldots, \mu], \tag{4.5}$$

subjecting I^μ to other conditions which we now describe. Let π be a partition of the μ-interval $H^\mu [u, b]$ such that the vertices of I^μ and K^μ are vertices of π. We are concerned with 1-intervals

$$\begin{aligned} [u^p, v^p] &= I_p, & [a^p, b^p] &= K_p, \\ [v^p, a^p] &= J_p, & [u^p, b^p] &= H_p. \end{aligned} \qquad [p = 1, \ldots, \mu].$$

Note that

$$H_p = I_p \cup J_p \cup K_p \qquad [p = 1, \ldots, \mu].$$

If Π indicates a Cartesian product,

$$H^\mu = \prod_p [I_p \cup J_p \cup K_p] \qquad [p = 1, \ldots, \mu]. \tag{4.6}$$

The set of constants $\mathbf{e}$ associated with π will be determined in three steps. We first require that the constants in $\mathbf{e}$ associated with the subintervals of J_p $(p = 1, \ldots, \mu)$ be zero. We next require that the partition $\pi|K^\mu$ and associated constants $\mathbf{e}|K^\mu$ be such that

$$\sigma^\mu [g, K^\mu, \pi|K^\mu, \mathbf{e}|K^\mu] > P^\mu [g, K^\mu] - \eta/4. \tag{4.7}$$

So chosen $\pi|K^\mu$ and $\mathbf{e}|K^\mu$ will be fixed. There remains the choice of I^μ, $\pi|I^\mu$, $\mathbf{e}|I^\mu$, subject to (4.5).

With $p = 1, \ldots, \mu$,

$$H^\mu \supset (\prod_p I_p) \cup (\prod_p K_p) \cup (\text{Union } Q^\mu) = I^\mu \cup K^\mu \cup (\text{Union } Q^\mu), \tag{4.8}$$

where Q^μ is any μ-interval of the form

$$Q^\mu = I_{r_1} \times \ldots \times I_{r_n} \times K_{m_1} \times \ldots \times K_{m_p} \qquad [n + p = \mu], \tag{4.9}$$

in which $r_1 \ldots r_n\, m_1 \ldots m_p$ is a permutation of the integers $1, \ldots, \mu$ with $0 < n < \mu$. The μ-intervals on the right of (4.8) include all those in the expansion (4.6) which contain elementary intervals in the partition of H^μ by π making a non-zero contribution to $\sigma^\mu [g, H^\mu, \pi, \mathbf{e}]$. That is

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

$$\sigma^\mu [g, H^\mu, \pi, \mathbf{e}] = \sigma^\mu [g, I^\mu, \pi|I^\mu, \mathbf{e}|I^\mu] + \sigma^\mu [g, K^\mu, \pi|K^\mu, \mathbf{e}|K^\mu] + \sum_{Q^\mu} \sigma^\mu [g, Q^\mu, \pi|Q^\mu, \mathbf{e}|Q^\mu]. \tag{4.10}$$

Relative to the choice of $I^\mu(u, v)$ the following will be established.

(A) *Let DI^μ be the vertex distance of I^μ from O. With $\pi|K^\mu$ and $\mathbf{e}|K^\mu$ fixed as above there exists a positive constant δ so small that when $DI^\mu < \delta$, then*

$$|\sum_{Q^\mu} \sigma^\mu [g, Q^\mu, \pi|Q^\mu\ \mathbf{e}|Q^\mu]| < \eta/4, \tag{4.11}$$

regardless of the choice of $\pi|I^\mu$ and $\mathbf{e}|I^\mu$.

The term in the sum σ in (4.11) contributed by the general elementary interval of the partition $\pi|Q^\mu$ has the form (without summing),

$$e^{r_1}{}_{\alpha_1} \Delta^{r_1}{}_{\alpha_1} \dots e^{r_n}{}_{\alpha_n} \Delta^{r_n}{}_{\alpha_n} \varphi,$$

where φ has the form (without summing),

$$\varphi = e^{m_1}{}_{\beta_1} \Delta^{m_1}{}_{\beta_1} \dots e^{m_p}{}_{\beta_p} \Delta^{m_p}{}_{\beta_p} g, \tag{4.12}$$

and φ is defined over the n-interval $\theta^n = I_{r_1} \times \dots \times I_{r_n}$ in a space R^n with coordinates $[s^{r_1}, \dots, s^{r_n}]$. Hence

$$\begin{aligned} &|\sigma^\mu [g, Q^\mu, \pi|Q^\mu, \mathbf{e}|Q^\mu]| \\ &\leqslant \sum_\phi |e^{r_1}{}_{\alpha_1} \Delta^{r_1}{}_{\alpha_1} \dots e^{r_n}{}_{\alpha_n} \Delta^{r_n}{}_{\alpha_n} \varphi| \quad [\alpha_1, \dots, \alpha_n \text{ summed}] \\ &\leqslant \sum_\phi |P^n [\varphi, \theta^n]|, \end{aligned} \tag{4.13}$$

summing over all φ given by (4.12). The number of such φ is at most the number N of elementary intervals in $\pi|K^\mu$. The integer α_i is the index of the α_ith interval of $\pi|I_{r_i}$, and β_j the index of the β_jth interval of $\pi|K_{r_j}$. In (4.13) $(\alpha_1, \dots, \alpha_n)$ determines the general interval of $\pi|\theta^n$, and the sum as to $(\alpha_1, \dots, \alpha_n)$ is over $\pi|\theta^n$.

We are assuming Theorem 4.1m true when $m = 1, \dots, \mu - 1$. In particular, Theorem 4.1n is assumed true. Hence there exists a positive constant δ so small that when $DI^\mu < \delta$,

$$|P^n [\varphi, \theta^n]| < \frac{\eta}{4N2^\mu} \tag{4.14}$$

for each φ given by (4.12). Account has here been taken of the fact that $\theta^n \to$ (the origin in R^n) as $I^\mu \to O$, and that for a fixed partition $\pi|K^\mu$, φ satisfies $\hat{F}$ over the n-interval $(0 < s^{r_1} < 1) \dots (0 < s^{r_n} < 1)$. Hence (4.14) and (4.13) imply that

$$|\sigma^\mu [g, Q^\mu, \pi|Q^\mu, \mathbf{e}|Q^\mu]| \leqslant \frac{\eta}{4 \cdot 2^\mu}. \tag{4.15}$$

Finally there are $2^\mu - 2 < 2^\mu$ different intervals Q^μ so that (4.15) yields (4.11), thus establishing (A).

Proof of Lemma 4.1. Returning to (4.10), we choose I^μ subject to (4.5) and to the condition $DI^\mu < \delta$, together with $\pi|I^\mu$ and $\mathbf{e}|I^\mu$ so that the first sum on the right of (4.10) exceeds η. This is possible since (4.3) holds as stated, by hypothesis. From (4.10), making use of (4.7 and (A),

$$\begin{aligned}\sigma^\mu[g, H^\mu, \pi, \mathbf{e}] &> \eta + [P^\mu[g, K^\mu] - \eta/4] - \eta/4 \\ &> P^\mu[g, K^\mu] + \eta/2.\end{aligned}$$

Relation (4.4) follows.

This completes the proof of Lemma 4.1.

The theorem follows at once from Lemma 4.1, since Lemma 4.1 implies the existence of μ-intervals in U^μ with arbitrarily large Fréchet variations. Hence the hypothesis that Theorem 4.1μ is false is untenable, and the proof is complete.

5. A sector limit. The problem of the existence of sector limits mentioned in §1 is a part of the more general problem of the boundary values of a function g, satisfying $\hat{F}$ over the open interval $U^\mu = I^\mu(\mathbf{0}, \mathbf{i})$. We are not concerned with boundary values in the classical sense but in limits of $g(s)$ as $s \in U^\mu$ approaches an open $(\mu - 1)$-face $I^{\mu-1}$ of U^μ on a straight line orthogonal to $I^{\mu-1}$. We shall extend g over $I^{\mu-1}$ and then be concerned with the limits of the extension of g as $s \in I^{\mu-1}$ approaches an open $(\mu - 2)$-face $I^{\mu-2}$ of $I^{\mu-1}$ on a straight line orthogonal to $I^{\mu-2}$, and so on down through the open faces of U^μ of every dimension. It is convenient to regard the problem of extending g, originally defined only on U^μ over the boundary of U^μ, as a special case of *extending* or *transforming* the boundary values of a function g defined over a more general μ-interval V^μ such that

$$I^\mu(\mathbf{0}, \mathbf{i}) \subset V^\mu \subset I^\mu[\mathbf{0}, \mathbf{i}]. \tag{5.1}$$

We begin with a general definition.

THE FUNCTIONS g^r. Suppose that the domain of definition in R^μ of a function g includes a line segement of points with coordinates

$$(s^1, \ldots, s^{r-1}, s^r + t, s^{r+1}, \ldots, s^\mu),$$

where $(s^1, \ldots, s^\mu)$ is fixed and t ranges over an open interval $0 < t < c$. We then set

$$g(s^1, \ldots, s^{r-1}, s^r + t, s^{r+1}, \ldots, s^\mu) = g^r(s), \tag{5.2}$$

provided the implied limit exists and regardless of whether $g(s)$ is defined or not.

It is not practical to transform or extend the boundary values of g in one step. The faces $X\ V^\mu$ which are incident with O will be successively considered. Observe the inclusion relations

$$V^\mu \subset C_1 V^\mu \subset C_2 C_1 V^\mu \subset \ldots \subset C_\mu \ldots C_1 V^\mu. \tag{5.3}$$

Corresponding to this sequence of intervals, we shall define a sequence of operations on g, progressively extending or transforming the boundary values of g. We suppose that g satisfies $\hat{F}$ over the μ-interval V^μ of (5.1). Let r be any one of the integers $1, \ldots, \mu$. The operator E_r which we shall now define extends or modifies the boundary values of g over $X_r V^\mu$. Observe first that g^r exists at each point $s \in X_r V^\mu$ regardless of whether $g(s)$ is there defined or not.

THE TRANSFORM $E_r g$ OF g. With g satisfying $\hat{F}$ over V^μ, $E_r g$ will be defined over $C_r V^\mu$ by setting

$$E_r g(s) = g(s), \qquad [\text{for } s \in V^\mu - X_r V^\mu] \tag{5.4$'$}$$

$$E_r g(s) = g^r(s), \qquad [\text{for } s \in X_r V^\mu]. \tag{5.4$''$}$$

LEMMA 5.1. *If g satisfies $\hat{F}$ over V^μ, then for any m-segment $Q^m \subset C_r V^\mu$ with $Q^m \neq X_r Q^m$,*

$$P^m [E_r g, Q^m] = P^m [g, Q^m - X_r Q^m] \qquad [r, m = 1, \ldots, \mu]. \tag{5.5}$$

If $(X_r Q^m) \cap Q^m$ is the null set, (5.5) is trivially true, since $E_r g = g$ over Q^m in this case.

To proceed we first suppose Q^m closed. In this case and with $X_r Q^m \cap Q^m$ non-empty, the orthogonal projection of Q^m on the s^r-axis is a closed interval $[0, b^r]$. Let π be a partition of Q^m. Referring to the partition π let t be a point on the s^r-axis between $s_0^r = 0$ and s_1^r. Let Q_t^m be the m-segment obtained from Q^m on removing from Q^m the subinterval of Q^m on which $s^r < t$, and let π_t be the partition of Q_t^m obtained from π by replacing $s_0^r = 0$ by $s_0^r = t$.

We associate the same set $\mathbf{e}$ with π_t as is given with π. It follows from the definition of $E_r g$ that for fixed π, $\mathbf{e}$,

$$\sigma^m [g, Q_t^m, \pi_t, \mathbf{e}] \to \sigma^m [E_r g, Q^m, \pi, \mathbf{e}] \tag{5.6}$$

as $t \to 0+$. Since $Q^m - X_r Q^m \supset Q_t^m$,

$$P^m [g, Q^m - X_r Q^m] \geqslant \sigma^m [g, Q_t^m, \pi_t, \mathbf{e}]. \tag{5.7}$$

From (5.6) and (5.7) we infer that

$$P^m [g, Q^m - X_r Q^m] \geqslant \sigma^m [E_r g, Q^m, \pi, \mathbf{e}]. \tag{5.8}$$

On taking the sup of the right member of (5.8) over admissible π, $\mathbf{e}$, we conclude that

$$P^m [g, Q^m - X_r Q^m] \geqslant P^m [E_r g, Q^m] \tag{5.9}$$

in case Q^m is closed.

Relation (5.9) holds even when Q^m is not closed. In this case, let K^m be an arbitrary closed m-segment in Q^m. Then $K^m \neq X_r K^m$ and

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

$$P^m[g, Q^m - X_rQ^m] \geqslant P^m[g, K^m - X_rK^m] \qquad \text{[by (2.13)]}$$
$$\geqslant P^m[E_rg, K^m] \qquad \text{[using (5.9)]}.$$

By virtue of this relation and the definition of P^m over a non-closed interval Q^m, (5.9) holds without exception. But (5.9) also holds with the inequality reversed since

$$P^m[g, Q^m - X_rQ^m] = P^m[E_rg, Q^m - X_rQ^m] \qquad \text{[cf. (5.4)']}$$
$$\leqslant P^m[E_rg, Q^m] \qquad \text{[by (2.13)]}.$$

Thus (5.5) holds as stated.

We note the following consequence of Lemma 5.1.

LEMMA 5.2. *If g satisfies $\hat{F}$ over V^μ, E_rg satisfies $\hat{F}$ over C_rV^μ.*

It follows from Lemma 5.2 that for $n, r = 1, \ldots, \mu$, E_nE_rg and E_rE_ng are well defined over the interval

$$C_nC_rV^\mu = C_rC_nV^\mu, \tag{5.10}$$

but there is no simple *a priori* reason why

$$E_nE_rg = E_rE_ng \tag{5.11}$$

over the interval (5.10). In fact if g did not satisfy $\hat{F}$ over V^μ, simple examples would show the falsity of (5.11) even when the limits necessary to define the two members of (5.11) existed. Until this difficulty is resolved in §6 the order in which the operators $E_1, \ldots, E_\mu$ are applied is very material. Bearing this in mind, note that for $r = 1, \ldots, \mu$,

$$E_rE_{r-1}\ldots E_1g \tag{5.12}$$

is well defined and satisfies $\hat{F}$ over

$$C_rC_{r-1}\ldots C_1V^\mu$$

whenever g satisfies $\hat{F}$ over V^μ.

LEMMA 5.3. *If g satisfies $\hat{F}$ over V^μ, then*

$$P^m[E_rE_{r-1}\ldots E_1g, Q^m] = P^m[g, Q^m - X_1Q^m - \ldots - X_rQ^m]$$

for $r, m = 1, \ldots, \mu$ and for any m-segment such that

$$Q^m \subset C_rC_{r-1}\ldots C_1V^\mu, \qquad Q^m \not\subset X_1Q^m \cup \ldots \cup X_rQ^m.$$

As a consequence of Lemma 5.2, $E_rE_{r-1}\ldots E_1g$ satisfies $\hat{F}$ over $C_rC_{r-1}\ldots C_1V^\mu$. The application of Lemma 5.1 then gives the successive relations

$$
\begin{aligned}
P^m [E_r E_{r-1} \ldots E_1 g, Q^m] \\
&= P^m [E_{r-1} E_{r-2} \ldots E_1 g, Q^m - X_r Q^m] \\
&\quad \ldots \qquad\qquad \ldots \\
&= P^m [E_1 g, Q^m - X_r Q^m - X_{r-1} Q^m - \ldots - X_1 Q^m] \\
&= P^m [g, Q^m - X_r Q^m - X_{r-1} Q^m - \ldots - X_1 Q^m],
\end{aligned}
$$

thus establishing Lemma 5.3.

It will be convenient to set

$$C_\mu C_{\mu-1} \ldots C_1 V^\mu = Cl_O{}^\mu V^\mu,$$

and refer to $Cl_O{}^\mu$ as the *left closure* of V^μ. When g satisfies $\hat{F}$ over V^μ we also set

$$E_\mu E_{\mu-1} \ldots E_1 g = \mathfrak{E}_O{}^\mu g.$$

Lemma 5.3 gives the following:

Lemma 5.4. *If g satisfies $\hat{F}$ over V^μ, then*

$$P^m [\mathfrak{E}_O{}^\mu g, Q^m] = P^m [g, Q^m \cap I^\mu (\mathbf{0}, \mathbf{i}]]$$

for each m-segment $Q^m \subset Cl_O{}^\mu V^\mu$ intersecting I^μ $(\mathbf{0}, \mathbf{i}]$. As a consequence $\mathfrak{E}_O{}^\mu g$ satisfies $\hat{F}$ over $Cl_O{}^\mu V^\mu$.

We come to a major theorem:

Theorem 5.1μ. *If g satisfies $\hat{F}$ over a μ-interval V^μ of type* (5.1), *then $g(s)$ has a unique limit $g(O+)$ as $s \to O$ in $U^\mu = I^\mu$ $(\mathbf{0}, \mathbf{i})$.*

Theorem 5.1μ is true when $\mu = 1$. We shall suppose $\mu > 1$ and fixed. Proceeding inductively, let m be any integer such that $1 < m \leqslant \mu$. We shall assume that Theorem 5.1r is true for $0 < r < m$, and establish Theorem 5.1m.

With $s \in U^m$ set I^m $[\mathbf{0}, s] = H_s{}^m$. Let g and V^m be the function and interval in Theorem 5.1m. Then $\mathfrak{E}_O{}^m g$ satisfies $\hat{F}$ over $Cl_O{}^m V^m$ by Lemma 5.4, and

$$P^m [\mathfrak{E}_O{}^m g, H_s{}^m] = P^m [g, H_s{}^m \cap U^m]. \tag{5.13}$$

The right member of (5.13) tends to 0 as $s \to O$ in U^m by Theorem 4.1 so that

$$P^m [\mathfrak{E}_O{}^m g, H_s{}^m] \to 0 \qquad [\text{as } s \to O \text{ in } U^m]. \tag{5.14}$$

Use will be made of a partition π of $H_s{}^m$ in which the only vertices are those of $H_s{}^m$. The differencing operator Δ^p ($p = 1, \ldots, m$) will then correspond to the interval $[0, s^p]$, and (5.14) implies that, for $g' = \mathfrak{E}_O{}^m g$,

$$\Delta^1 \Delta^2 \ldots \Delta^m g'(s) \to 0 \tag{5.15}$$

as $s \to O$ in U^m. Two of the terms in the sum (5.15) when expanded are $g'(s)$ and $(-1)^m g'(O)$. Each of the other terms is of the form $\pm g'(s_\lambda)$, where

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

s_λ is the orthogonal projection of s on one of the open r-faces λ of U^m incident with O $(r = 1, \ldots, m-1)$. Let the coordinates of s_λ which are variable over λ, taken in the order of their superscripts, be denoted by $t^1, \ldots, t^r$. Then $t = (t^1, \ldots, t^r)$ ranges over U^r as s ranges over U^m with $s_\lambda = (t^1, \ldots, t^r)$. For $s \in U^m$ set $g'(s_\lambda) = \varphi(t)$. Since $\mathfrak{E}_O{}^m g$ satisfies $\hat{F}$ over $Clo^m U^m$ by Lemma 5.4, φ satisfies $\hat{F}$ over $U^r \subset Clo^m U^m$ by Corollary 3.2. By our inductive hypothesis $\varphi(t)$ tends to a limit as $t \to O$ in U^r. It follows from (5.15) that $g'(s)$ tends to a limit as $s \to O$ in U^m. But $g'(s) = g(s)$ for $s \in U^m$, and the theorem follows.

The limit $g(\mathbf{0}+)$ *proved to exist in Theorem* 5.1μ *is the sector limit corresponding to the sector* S_O *on which* $s^i > 0$ $(i = 1, \ldots, \mu)$.

6. Canonical left boundary values.

The transform

$$\mathfrak{E}_O{}^\mu g = E_\mu E_{\mu-1} \ldots E_1 g$$

of g defined in §5 does not in reality depend upon the order of application of the operator E_r $(r = 1, \ldots, \mu)$ when g satisfies $\hat{F}$ over V^μ. To establish this it will be sufficient to show that $\mathfrak{E}_O{}^\mu g$ has canonical left boundary values in the sense of the following definition.

DEFINITION. *If an interval* V^μ *of type* (5.1) *is left closed* (cf. §2) *a function* g *defined over* V^μ *will be said to have canonical left boundary values if for* $s \in X_r V^\mu$,

$$E_r g(s) = g(s) \qquad [r = 1, \ldots, \mu]. \tag{6.1}$$

We term the *union* $X_r V^\mu$ $(r = 1, \ldots, \mu)$ the *left boundary* of V^μ and note that V^μ is left closed if and only if it contains its left boundary. Condition (6.1) is automatically fulfilled if V^μ is left closed and g continuous over V^μ. In general (6.1) will not be satisfied if g is discontinuous. At a point $a \in V^\mu$ for which

$$a^{r_1} = a^{r_2} = \ldots = a^{r_n} = 0, \qquad [0 < n \leqslant \mu]$$

and for which the remaining coordinates $a^i \neq 0$, (6.1) imposes the conditions

$$g^{r_1}(a) = g^{r_2}(a) = \ldots = g^{r_n}(a) = g(a),$$

not in general satisfied. We shall, however, show that when g satisfies $\hat{F}$ over V^μ the transform $\mathfrak{E}_O{}^\mu g$ satisfies (6.1), and so has canonical left boundary values.

For this purpose it is useful to generalize the limits g^r previously defined by defining the limits $g^{r_1 \ldots r_n}$. We suppose g defined over a μ-interval in R^μ and that r_i $(i = 1, \ldots, n \leqslant \mu)$ is an integer in the set $1, \ldots, \mu$. Proceeding inductively we suppose that $g^{r_1 \ldots r_{n-1}}$ has well defined values at points with coordinates

$$s^1, \ldots, s^{r_n - 1}, s^{r_n} + t, s^{r_n + 1}, \ldots, s^\mu, \tag{6.2}$$

where $(s^1, \ldots, s^\mu)$ is fixed and t variable on an interval $(0, c)$. We then set

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

$$(g^{r_1 \cdots r_{n-1}})^{r_n}(s) = g^{r_1 \cdots r_n}(s), \tag{6.3}$$

whenever the implied limit exists.

The following lemma is a corollary of Theorem 5.1.

LEMMA 6.1. *If g satisfies $\hat{F}$ over an interval of V^μ of type* (5.1), *then*

$$\mathfrak{E}_O{}^\mu g\,(O) = g^{r_1 \cdots r_\mu}\,(O) \tag{6.4}$$

where $r_1 \ldots r_n$ is any permutation of the integers $1, \ldots, \mu$.

The right member of (6.4) is a limit of values of $g(s)$ for $s \in I^\mu(\mathbf{0}, e\mathbf{i})$ where e is an arbitrarily small positive constant. It follows from Theorem 5.1 that the right member of (6.4) equals $g(O+)$ and is accordingly independent of the permutation of the integers $1, \ldots, \mu$. On the other hand it follows from the definition of $\mathfrak{E}_O{}^\mu g$ that

$$\mathfrak{E}_O{}^\mu g(O) = g^{\mu(\mu-1) \cdots 1}(O),$$

so that (6.4) follows.

We can now obtain a representation of $\mathfrak{E}_O{}^\mu g$ which shows its independence of the order of application of the operator E_r.

THEOREM 6.1. *Suppose that g satisfies $\hat{F}$ over an interval V^μ of type* (5.1). *Let a be any point on the left boundary of V^μ at which*

$$a^{r_1} = a^{r_2} = \ldots = a^{r_n} = 0 \qquad [n > 0], \tag{6.5}$$

while the remaining $\mu - n$ coordinates $a^i \neq 0$. Let $m_1 \ldots m_\mu$ be any permutation of the integers $1, \ldots, \mu$. Then

$$E_{m_1} \ldots E_{m_\mu}\, g(a) = g^{r_1 \cdots r_n}\,(a), \tag{6.6}$$

where the ordering of the integers $r_1, \ldots, r_n$ or $m_1, \ldots, m_\mu$ is immaterial.

In the ordered set $m_1, \ldots, m_\mu$, suppose that $r_1, \ldots, r_n$ appear in the order $\rho_1, \ldots, \rho_n$. Let $\Lambda_a{}^n$ be the n-section of V^μ by the n-plane on which

$$(s^{\alpha_1}, \ldots, s^{\alpha_p}) = (a^{\alpha_1}, \ldots, a^{\alpha_p}) \qquad [p = \mu - n], \tag{6.7}$$

where the a's on the right of (6.7) are the coordinates of a which are not zero. On $\Lambda_a{}^n$ the coordinates $s^{\rho_1}, \ldots, s^{\rho_n}$ are variable, and it follows from the definition (5.4) of E_r ($r = 1, \ldots, \mu$) that

$$E_{m_1} \ldots E_{m_\mu}\, g(a) = E_{\rho_1} \ldots E_{\rho_n}\, g(a). \tag{6.8}$$

On the other hand one has successively

$$\begin{aligned}
E_{r_1}\, g(s) &= g^{r_1}(s) && [s \in X_{r_1}\Lambda_a{}^n], \\
E_{r_2} E_{r_1}\, g(s) &= g^{r_1 r_2}(s) && [s \in X_{r_2} X_{r_1}\Lambda_a{}^n], \\
\ldots &\qquad \ldots && \ldots \\
E_{r_n} \ldots E_{r_2} E_{r_1}\, g(s) &= g^{r_1 \cdots r_n}\,(s) && [s \in X_{r_n} \ldots X_{r_1}\Lambda_a{}^n = a].
\end{aligned} \tag{6.9}$$

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

The existence of $g^{r_1 \cdots r_i}$ in (6.9) has been inferred from Lemma 5.2 $(i = 1, \ldots, n)$.

To show that the right member of (6.6) is independent of the order of $r_1, \ldots, r_n$, observe that the function $g|\Lambda_a{}^n$ satisfies $\hat{F}$ over $\Lambda_a{}^n$ in accordance with Corollary 3.2. It then follows from Theorem 5.1 that as $s \to a$ in $\Lambda_a{}^n$ with $s \neq a$, $g(s) \to c$, a constant. We conclude as in the proof of Lemma 6.1 that

$$g^{r_1 \ldots r_n}(a) = c \tag{6.10}$$

regardless of the ordering of the integers $r_1, \ldots, r_n$. In accordance with (6.9), (6.8) can be written in the form

$$E_{m_1} \ldots E_{m_\mu} g(a) = g^{\rho_n \cdots \rho_1}(a).$$

Equation (6.6) then follows from (6.10), since $\rho_1, \ldots, \rho_n$ is an ordering of the integers $r_1, \ldots, r_n$.

Corollary 6.1. *For any ordering $m_1, \ldots, m_\mu$ of the integers $1, \ldots, \mu$,*

$$\mathfrak{E}_O{}^\mu g = E_{m_1} \ldots E_{m_\mu} g.$$

Theorem 6.2. *If g satisfies $\hat{F}$ over a left closed interval V^μ of type* (5.1), *then a necessary and sufficient condition that g have canonical left boundary values is that $\mathfrak{E}_O{}^\mu g = g$.*

The condition $\mathfrak{E}_O{}^\mu g = g$ is sufficient. We have

$$\begin{aligned} g = \quad \mathfrak{E}_O{}^\mu g &= E_r(E_1E_2 \ldots E_{r-1}E_{r+1} \ldots E_\mu)g && \text{[by (6.6)]}, \\ E_r g = E_r \mathfrak{E}_O{}^\mu g &= E_rE_r(E_1E_2 \ldots E_{r-1}E_{r+1} \ldots E_\mu)g && [r = 1, \ldots, \mu] \\ &= E_r(E_1E_2 \ldots E_{r-1}E_{r+1} \ldots E_\mu)g \\ &= \mathfrak{E}_O{}^\mu g = g. \end{aligned}$$

The condition is necessary. For $E_r g = g$ by hypothesis, so that

$$g = E_1 g = E_2E_1 g = \ldots = E_\mu E_{\mu-1} \ldots E_1 g = \mathfrak{E}_O{}^\mu g.$$

The proof of Theorem 6.3 requires the following lemma.

Lemma 6.2. *If g satisfies $\hat{F}$ over a left closed interval V^μ of type* (5.1) *and has canonical left boundary values, and if σ is a left closed n-face of V^μ incident with O, then $g|\sigma$ has canonical left boundary values.*

Let $s^{r_1}, \ldots, s^{r_n}$ be the coordinates which are variable over σ. To show that $g|\sigma$ has canonical left boundary values, it is sufficient to show that

$$(g|\sigma)^{r_i} = g|\sigma \qquad [\text{over } X_{r_i}\sigma]. \tag{6.11}$$

We have

$$(g|\sigma)^{r_i} = g^{r_i}|\sigma \qquad [\text{over } X_{r_i}\sigma]. \tag{6.12}$$

Since $g^{r_i}(s) = g(s)$ for $s \in X_{r_i}\sigma$ by hypothesis, (6.11) follows from (6.12). This completes the proof of the lemma.

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

Theorem 4.1 concerns an interval Q^μ such that $Q^\mu \to O$ in $I^\mu(\mathbf{0}, \mathbf{i})$. In the following extension of Theorem 4.1, Q^r is a segment of *any* dimension r between 1 and μ inclusive, and $Q^r \to O$ in $I^\mu[\mathbf{0}, \mathbf{i})$.

THEOREM 6.3. *If g satisfies $\hat{F}$ over $V^\mu = I^\mu\ [\mathbf{0}, \mathbf{i})$ and has canonical left boundary values, then*

$$P^r\,[g, Q^r] \to 0 \qquad [r = 1, \ldots, \mu]$$

as an arbitrary r-segment $Q^r \to O$ in $I^\mu\ [\mathbf{0}, \mathbf{i})$.

We shall refer to the theorem as Th (μ, r), and first establish Th (μ, μ). Observe that $\mathfrak{E}_O{}^\mu g = g$ by Theorem 6.2, so that

$$\begin{aligned} P^\mu\,[g, Q^\mu] &= P^\mu\,[\mathfrak{E}_O{}^\mu\, g, Q^\mu] \\ &= P^\mu\,[g, Q^\mu \cap I^\mu\,(\mathbf{0}, \mathbf{i})] \qquad \text{[by Lemma 5.4].} \end{aligned}$$

This final variation tends to zero as $Q^\mu \to O$ in accordance with Theorem 4.1. Thus Th (μ, μ) holds.

To prove Th(μ, r) we shall use a double induction. Let m be a fixed integer with $1 < m \leqslant \mu$. We shall assume the truth of

$$\text{(6.13)} \qquad \text{Th}\,(m-1, 1), \text{Th}\,(m-1, 2), \ldots, \text{Th}\,(m-1, m-1),$$

and establish

$$\text{(6.14)} \qquad \text{Th}\,(m, 1), \text{Th}\,(m, 2), \ldots, \text{Th}\,(m, m-1).$$

For this purpose we assume the truth of the theorems listed in (6.13) and of

$$\text{(6.15)} \qquad \text{Th}\,(m, m), \text{Th}\,(m, m-1), \ldots, \text{Th}\,(m, r+1), \qquad [0<r<m]$$

and prove the truth of Th (m, r).

When $m = 2$, the theorems in (6.13) reduce to Th $(1, 1)$, which holds as proved above. In (6.15) the theorems listed reduce to Th (m, m) when $r = m - 1$, and Th (m, m) holds. We are concerned with increasing m and decreasing r. We proceed to establish Th (m, r) assuming theorems listed in (6.13) and (6.15).

CASE I. In this case Q^r is assumed to lie in a left closed $(m-1)$-face σ of V^μ. Then

$$P^r\,[g, Q^r] = P^r\,[g|\sigma, Q^r].$$

But $g|\sigma$ has canonical left boundary values in accordance with the preceding lemma. Here Th $(m-1, r)$ is in the list (6.13), and implies that

$$P^r\,[g|\sigma, Q^r] \to 0, \qquad [\text{as } Q^r \to O \text{ in } \sigma]$$

so that $P^r\,[g, Q^r] \to 0$ as $Q^r \to O$ in σ

CASE II. (Not Case I.) In this case, there is an $(m-1)$-face σ of V^μ parallel to Q^r and not intersecting Q^r. We can suppose σ closed relative to V^μ. Let $Q_0{}^r$ be the orthogonal projection of Q^r into σ, and let Q^{r+1} be the

$(r+1)$-segment with faces Q_r and $Q_0{}^r$. Without loss of generality we can suppose Q^r closed, taking into account the definition of the Fréchet variation when Q^r is not closed. Then by a proof similar to that of (3.2), we infer that

$$P^r[g, Q^r] \leqslant P^{r+1}[g, Q^{r+1}] + P^r[g, Q_0{}^r]. \tag{6.16}$$

The last term in (6.16) tends to zero as $Q_0{}^r \to O$ in σ (or in any of the other $(m-1)$-faces of V^μ) by virtue of the conclusion under Case I. Moreover,

$$P^{r+1}[g, Q^{r+1}] \to 0$$

as Q^r, and hence $Q^{r+1} \to O$, since Th $(m, r+1)$ is assumed true. Hence $P^r[g, Q^r] \to 0$ as $Q^r \to O$ in V^μ.

As $Q^r \to O$ in the general case Q^r may come under either Case I or Case II as it varies, but it is clear that $P^r[g, Q^r] \to 0$ in this case as well, as a consequence of the conclusions under Case I and Case II. Thus Th (m, r) is true for $0 < r \leqslant m$, and the proof of the theorem is complete.

COROLLARY 6.2. *If g satisfies $\hat{F}$ over $I^\mu(\mathbf{0}, \mathbf{i})$, then*

$$P^r[g, Q^r] \to 0 \qquad [r = 1, \ldots, \mu] \tag{6.17}$$

as an arbitrary r-segment $Q^r \to O$ in $I^\mu(\mathbf{0}, \mathbf{i})$.

The function $\mathfrak{E}_O{}^\mu g$ has canonical left boundary values by Theorem 6.2. Hence Theorem 6.3 implies that

$$P^r[\mathfrak{E}_O{}^\mu g, Q^r] \to 0 \tag{6.18}$$

as $Q^r \to O$ in $I^\mu(\mathbf{0}, \mathbf{i})$. But $\mathfrak{E}_O{}^\mu g(s) = g(s)$ for $s \in Q^r$ so that (6.17) follows from (6.18).

The following theorem will enable us to extend properties of a function g satisfying $\hat{F}$ over a *closed* interval I^μ to a function satisfying $\hat{F}$ over an open interval.

THEOREM 6.4. *A function k which satisfies $\hat{F}$ over $U^\mu = I^\mu(\mathbf{0}, \mathbf{i})$ admits an extension g over $\bar{U}^\mu$ which satisfies $\hat{F}$ over $\bar{U}^\mu$, has canonical left boundary values and satisfies the relation*

$$P^r[k, Q^r \cap U^\mu] = P^r[g, Q^r], \tag{6.19}$$

where Q^r is any r-segment in $\bar{U}^\mu$ which intersects U^μ.

Let $s = \varphi(t)$ be a (1-1)-linear transformation of $\bar{U}^\mu$ onto itself which interchanges the vertices $\mathbf{0}$ and $\mathbf{i}$. Set

$$\begin{aligned} k[\varphi(t)] &= g'(t) && [t \in U^\mu], \\ \mathfrak{E}_O{}^\mu g'(t) &= g''(t) && [t \in I^\mu[\mathbf{0}, \mathbf{i})], \\ g''[\varphi^{-1}(s)] &= g'''(s) && [s \in I^\mu(\mathbf{0}, \mathbf{i}]], \\ \mathfrak{E}_O{}^\mu g'''(s) &= g(s) && [s \in \bar{U}^\mu]. \end{aligned} \tag{6.20}$$

We shall see that g satisfies the theorem.

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

It is first clear that g' satisfies F over U^μ, and hence by Lemma 5.4 g'' satisfies $\hat{F}$ over $I^\mu\,[\mathbf{0}, \mathbf{i})$. Hence g''' satisfies $\hat{F}$ over $I^\mu\,(\mathbf{0}, \mathbf{i}]$, and again by Lemma 5.4 g satisfies $\hat{F}$ over $\bar{U}^\mu$. Set

$$H^r = Q \cap I^\mu\,(\mathbf{0}, \mathbf{i}].$$

Two applications of Lemma 5.4, first to k and its extension g''', and second to g''' and its extension g give

$$P^r\,[k, K^r] = P^r\,[g''', H^r] = P^r\,[g, Q^r],$$

where

$$K^r = H^r \cap U^\mu = Q^r \cap U^\mu.$$

Relation (6.19) follows.

7. A left decomposition of g. Suppose that g satisfies $\hat{F}$ over a left closed interval V^μ of type (5.1) and has canonical left boundary values. Let σ be any face of V^μ incident with O and closed relative to V^μ. Each such σ is left closed since V^μ is left closed by hypothesis. Special faces σ are the origin O and V^μ. On σ, g defines a function $g|\sigma$ which has canonical left boundary values (Lemma 6.2). In case $\sigma = O$ this condition is vacuous. Let s_σ be the orthogonal projection of $s \in V^\mu$ into σ. We understand that $s_\sigma = O$ when $\sigma = O$, and that $s_\sigma = s$ when $\sigma = V^\mu$.

DEFINITION. *Let σ be a face of V^μ incident with O and closed relative to V^μ. A function g which has canonical left boundary values will be said to be a left σ-function if*

$$g(s) = g(s_\sigma) \qquad [\text{for } s \in V^\mu] \tag{7.1}$$

and if $g(s) = 0$ when s is on the left boundary of σ.

When σ is the 0-face O, a left σ-function g has but one value $g(O)$. This value need not be zero.

The following theorems are useful in the theory of multiple Fourier series, and in particular in the treatment of the Dirichlet integral.

THEOREM 7.1. *When g satisfies $\hat{F}$ over a μ-interval V^μ of type* (5.1) *and has canonical left boundary values, then g is a sum*

$$g = \sum_\sigma g_\sigma \tag{7.2}$$

of left σ-functions belonging to the faces σ of V^μ which are incident with O and closed relative to V^μ.

We term (7.2) a *left decomposition of* g.

The determination of g_σ when $\sigma = V^\mu$. Let Δ^i ($i = 1, \ldots, \mu$) be a differencing operator corresponding to any interval $[0, s^i]$ of the s^i-axis with $s^i \geqslant 0$. For $\sigma = V^\mu$ and $s \in V^\mu$ set

$$g_\sigma = \Delta^\mu \Delta^{\mu-1} \ldots \Delta^1 g. \tag{7.3}$$

We must show that g_σ satisfies $\hat{F}$ over V^μ, has canonical left boundary values, and is a left σ-function. That g_σ satisfies $\hat{F}$ over V^μ follows immediately from its definition (7.3). To prove that g_σ has canonical boundary values we must show that $E_r g_\sigma = g_\sigma$ for $r = 1, \ldots, \mu$. Since $E_r g = g$ by hypothesis,

$$E_r g_\sigma = \Delta^\mu \Delta^{\mu-1} \ldots \Delta^1 E_r g = \Delta^\mu \Delta^{\mu-1} \ldots \Delta^1 g = g_\sigma$$

as required. To show that g_σ is a left σ-function requires a verification of the relation $g_\sigma(s) = g_\sigma(s_\sigma)$ for $s \in V^\mu$. This relation is trivial when $\sigma = V^\mu$, since $s_\sigma = s$ when $\sigma = V^\mu$. Finally $g_\sigma(s) = 0$ on the left boundary of V^μ as a consequence of the definition (7.3) of g_σ.

The determination of g_σ when $\sigma \neq V^\mu$. The right member of (7.3) is the sum of g and functions defined over V^μ with values obtained from $g(s)$ by setting some of the coordinates $s^i = 0$ in $g(s)$. It is clear that each of these derived functions satisfies $\hat{F}$ over V^μ and has canonical left boundary values (Lemma 6.2). For $r = 1, \ldots, \mu$, let $\lambda(r)$ be the $(\mu - 1)$-face of V^μ on which $s^r = 0$, taken closed relative to V^μ. When $\sigma = V^\mu$, denote g_σ by g_1. It follows from (7.3) that g admits the form

$$g = g_1 + \sum_r f_r \qquad [\text{with } f_r(s) = f_r(s_{\lambda(r)})] \tag{7.4}$$

where f_r satisfies $\hat{F}$ over V^μ and has canonical left boundary values. It follows from Lemma 6.2 that the function φ_r defined over $\lambda(r)$ by the values $f_r(s_{\lambda(r)})$ satisfies $\hat{F}$ over $\lambda(r)$ and has canonical left boundary values.

Proceeding inductively we assume that the theorem holds when μ is replaced by $\mu - 1$ (supposed positive), noting that the theorem holds when $\mu = 1$. We seek to prove the theorem for the general μ. Under our inductive hypothesis, φ_r admits a left decomposition over $\lambda(r)$ of the form

$$\varphi_r = \sum_{\sigma \subset \lambda(r)} \Phi_{r\sigma}, \tag{7.5}$$

taken over those faces σ of V^μ incident with O which are also faces of $\lambda(r)$. From (7.5) we obtain a left decomposition of f_r of the form

$$f_r(s) = f_r(s_{\lambda(r)}) = \sum_{\sigma \subset \lambda(r)} f_{r\sigma}(s). \tag{7.6}$$

For a given r $f_{r\sigma} = 0$ if $\sigma \not\subset \lambda(r)$. Decomposition (7.6), taken for $r = 1, \ldots, \mu$, and substituted in (7.4), gives a left decomposition of g in which $g_\sigma = \sum_r f_{r\sigma}$.

This completes the proof of the theorem.

THEOREM 7.2. *In the left decomposition of g given by* (7.2), *g_σ is uniquely determined by g and σ. Specifically if $s^{r_1}, \ldots, s^{r_n}$, are the coordinates which are variable over σ and if s_σ is the orthogonal projection of $s \in V^\mu$ into σ, then in case $n > 0$,*

$$g_\sigma|\sigma = \Delta^{r_1} \ldots \Delta^{r_n}(g|\sigma) \qquad [s \in V^\mu], \tag{7.7}$$

while $g_\sigma(s) = g(O)$ *when* $n = 0$.

Let ω be an arbitrary one of the respective faces of V^μ incident with O and closed relative to V^μ. There are 2^μ such faces ω of which σ is one. Let $\beta\omega$ be the left boundary of ω. According to (7.2),

$$g = \sum_\omega g_\omega. \tag{7.8}$$

We shall prove the following. For the given σ, for $s \in V^\mu$, and s_σ the orthogonal projection of s into σ,

$$g(s_\sigma) = g_\sigma(s_\sigma) + \sum_{\omega \subset \beta\sigma} g_\omega(s_\sigma). \tag{7.9}$$

Proof of (7.9). It follows from (7.8) that for $s \in V^\mu$,

$$g(s) = \sum_\omega g_\omega(s_\omega), \tag{7.10}$$

since g_ω is a left ω-function. Then (7.10) implies that

$$g(s_\sigma) = g_\sigma(s_\sigma) + \sum_{\omega \neq \sigma} g_\omega(s_{\sigma\omega}) \tag{7.11}$$

since $s_{\sigma\sigma} = s_\sigma$. Moreover for $\omega \neq \sigma$,

$$s_{\sigma\omega} \subset \sigma \cap \omega \subset (\sigma \cap \beta\omega) \cup (\omega \cap \beta\sigma). \tag{7.12}$$

In case $\omega \not\subset \beta\sigma$,

$$\omega \cap \beta\sigma = \beta\omega \cap \beta\sigma, \tag{7.13}$$

and (7.12) with (7.13) implies that $s_{\sigma\omega} \subset \beta\omega$. In this case $g_\omega(s_{\sigma\omega}) = 0$ since ω is a left ω-function. Hence (7.11) reduces to (7.9).

Proof of (7.7). The form of (7.7) suggests that (7.9) be written as a relation between functions rather than between values. Thus (7.9) implies that

$$g|\sigma = g_\sigma|\sigma + \sum_{\omega \subset \beta\sigma} (g_\omega|\sigma). \tag{7.14}$$

We apply the differencing operator $\Delta^{r_1} \ldots \Delta^{r_n}$ to the respective functions in (7.14). In particular

$$\Delta^{r_1} \ldots \Delta^{r_n}(g_\sigma|\sigma) = g_\sigma|\sigma,$$

since $g_\sigma(s)$ vanishes when s is on the left boundary of σ. Moreover when $\omega \subset \beta\sigma$,

$$\Delta^{r_1} \ldots \Delta^{r_n} (g_\omega|\sigma) = 0.$$

For $g_\omega(s_\sigma) = g(s_{\sigma\omega})$ and at least one of the coordinates s^{r_i} of $s_{\sigma\omega}$ vanishes when $\omega \subset \beta\sigma$, so that $\Delta^{r_i}(g_\omega|\sigma)$ yields a null difference. Hence (7.14) implies (7.7) when $n > 0$. The relation $g_\sigma(s) = g(O)$ when $n = 0$ requires no comment.

This completes the proof of the theorem.

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

We have stated that the Fourier series of a function g which satisfies $\hat{F}$, is continuous over $I^\mu[0, 2\pi\mathbf{i}]$ and has the period 2π in each of its arguments, converges uniformly in the sense of Pringsheim. In establishing this, the concept of the *variation modulus* of g, as used in Theorem 7.3, will play a fundamental role.

DEFINITION. *The variation modulus of g.* Let g satisfy $\hat{F}$ over an interval I^μ. A function ρ with values $\rho(\eta) > 0$ defined for $0 < \eta < 1$ is called a variation modulus of g over I^μ if $\rho(\eta) \to 0$ as $\eta \to 0$ and if for any r-segment $Q^r \subset I^\mu$,

$$P^r[g, Q^r] < \rho(\eta) \qquad [r = 1, \ldots, \mu]$$

whenever the diameter of Q^r is less than η.

THEOREM 7.3. (α). *A function g which satisfies $\hat{F}$ and is continuous over $I^\mu[\mathbf{0}, \mathbf{i}]$ admits a variation modulus ρ.* (β). *Moreover $2^\mu\rho$ is a variation modulus common to the functions g_σ in a left decomposition of g.*

Proof of (α). If statement (α) of the theorem were false, there would exist some point $a \in I^\mu[\mathbf{0}, \mathbf{i}]$ together with an integer m $(0 < m \leqslant \mu)$ and a positive constant $e > 0$ with the following property. There exist m-segments $Q^m \subset I^\mu[\mathbf{0}, \mathbf{i}]$ with arbitrarily small vertex distances from a and with

$$P^m[g, Q^m] > e. \tag{7.15}$$

But any such Q^m is the sum of at most 2^μ non-overlapping m-segments Q_i^m, each of which lies in one of the 2^μ closed sectors $\overline{S_a}$ with vertex at a. Taking account of the continuity of g one may infer from Theorem 6.3 that $P^m[g, Q_i^m] \to 0$ as $Q_i^m \to a$, provided Q_i^m remains in one of the sectors $\overline{S_a}$. Since

$$P^m[g, Q^m] \leqslant \sum_i P^m[g, Q_i^m], \tag{7.16}$$

a contradiction with (7.15) is evident. We conclude that g possesses a variation modulus.

Proof of (β). Statement (β) follows from the existence of the variation modulus ρ, as affirmed in (α) and the specific formula (7.7) for g_σ. In the sum on the right side of (7.7) there are at most 2^μ different functions defined over σ and obtained from $g|\sigma$, by setting some or none of the coordinates of s_σ equal to zero. Each of these functions has the variation modulus ρ of g so that $g_\sigma(s)$ has $2^\mu\rho$ as a variation modulus.

8. S-continuity. At each point a in R^μ the $(\mu - 1)$-planes on which $s^r = a^r$ $(r = 1, \ldots, \mu)$ divide R^μ into 2^μ open regions or sectors. Let S_a be any one of these regions. As a varies over R^μ, we suppose that S_a remains invariant in orientation, and refer to S as a *sector type*. There are 2^μ sector types.

Let g be defined over a general μ-interval I^μ. We say that g is S-continuous ($\bar{S}$-continuous) at a point a of I^μ if S_a ($\bar{S}_a$) intersects I^μ and if $g(s) \to g(a)$ as $s \to a$ in S_a ($\bar{S}_a$). The following theorem is useful.

THEOREM 8.1. *If g satisfies $\hat{F}$ over an open μ-interval I^μ and is S-continuous over I^μ relative to a sector type S of invariant orientation, then g is $\bar{S}$-continuous over I^μ.*

Let a be fixed in I^μ. For any $u \in \bar{S}_a \cap I^\mu$, $g(s) \to g(u)$ as $s \to u$ in $S_u \cap S_a$. Since $g(s) \to g(a)$ as $s \to a$ in S_a, it follows that $g(s) \to g(a)$ as $s \to a$ in $\bar{S}_a$.

THE FUNCTIONS g^S. Corresponding to a function f of bounded Jordan variation with values $f(s)$ over a 1-interval $[a, b]$, one commonly associates a function f^+ such that $f^+(a) = f(a)$, $f^+(b) = f(b)$, and $f^+(s) = f(s+)$ for $a < s < b$ Functions f^- are similarly defined. It is necessary to preserve the end values of f in defining f^+ or f^- in order that f^+ or f^- may be useful in applications such as the Riesz-Stieltjes representation of functionals linear over C. For the case $\mu = 2$ the generalizations are the functions $k^{\mu,\nu}$ of [7]. The generalizations of f^+ and f^- in the case of a general μ are as follows.

Let g satisfy $\hat{F}$ over I^μ $[\mathbf{0}, \mathbf{i}]$, and let S be a fixed sector type. Let F^r be any open r-face on the boundary of I^μ $(r = 1, \ldots, \mu - 1)$. For $t \in F^r$, let $S_t{}^r$ denote the orthogonal projection of S_t into the r-plane of F^r. Then $S_t{}^r$ is an open r-sector[2] in this r-plane with vertex t. With g we now associate a function g^S mapping I^μ into R^1 and defined as follows:

$$g^S(t) = \lim_{s \to t | (s \in S_t)} g(s) \qquad [t \in I^\mu\ (\mathbf{0}, \mathbf{i})], \tag{8.1}$$

$$g^S(t) = \lim_{s \to t | (s \in S_t{}^r)} g(s) \qquad [t \in F^r], \tag{8.2}$$

$$g^S(t) = g(t) \qquad [t \text{ a vertex of } I^\mu], \tag{8.3}$$

where (8.2) applies to all r-faces F^r of $I^\mu[\mathbf{0}, \mathbf{i}]$. That the limits implied in this definition exist follows from Theorem 5.1. We note the following:

$$g^S | F^r = [g | F^r]^{S^r}.$$

The proof of the following theorem depends upon the *lower semi-continuity* of P^r defined as follows. Let $[g_n]$ $(n = 0, 1, 2, \ldots)$ be a sequence of functions mapping an r-segment $Q^r \subset R^\mu$ into R^1 and such that for each $s \in Q^r$, $g_n(s) \to g_0(s)$ as $n \to \infty$. Then P^r is lower semi-continuous in the sense that

$$\liminf_{n \to \infty} P^r[g_n, Q^r] \geqslant P^r[g_0, Q^r].$$

The proof of this fact is essentially independent of the dimensions r and μ. In the case $\mu = r = 2$, the proof is given in [7, §3], and need not be repeated here.

The following theorem is fundamental in the theory of functionals multilinear over Cartesian products of C.

THEOREM 8.2. *Let g satisfy $\hat{F}$ over $I^\mu[\mathbf{0}, \mathbf{i}]$ and let S be a sector type in R^μ invariant in orientation. The function g^S is S-continuous over $I^\mu(\mathbf{0}, \mathbf{i})$ and S^r-continuous[2] over each open r-face F^r of I^μ. Moreover for the closed interval I^μ,*

[2] S^r is a sector type associated with F^r and derived from S.

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

(8.4) $$P^\mu[g^S, I^\mu] \leqslant P^\mu [g, I^\mu].$$

By virtue of its definition g^S is S-continuous over $I^\mu(\mathbf{0}, \mathbf{i})$. The S^r-continuity of $g^S|F^r$ follows similarly. In general g^S will not be S-continuous over $I^\mu[\mathbf{0}, \mathbf{i}]$ nor S^r-continuous over $\bar{F}^r$.

To continue, we suppose that S is of the "right" type in which the coordinates $s^i > a^i$ $(i = 1, \ldots, \mu)$ for points in S_a. The cases of other sector types can be reduced to the case of the "right" type on making a suitable affine mapping of $I^\mu[\mathbf{0}, \mathbf{i}]$ onto itself.

To establish (8.4) let φ_n be a homeomorphic mapping of the 1-interval [0, 1] onto [0, 1] leaving the end points of [0, 1] fixed and being such that

$$0 < \varphi_n(x) - x < 1/n \qquad [0 < x < 1].$$

Such a mapping exists. Set

$$g\,[\varphi_n(s^1), \ldots, \varphi_n(s^\mu)] = g_n(s) \qquad [s \in I^\mu\,[\mathbf{0}, \mathbf{i}]].$$

It is clear that $g_n(s) \to g^S(s)$ as $n \to \infty$ for each $s \in I^\mu\,[\mathbf{0}, \mathbf{i}]$, since $\varphi_n(x) \to x$ from the right. A direct consideration of the definition of the Fréchet variation makes it clear [7, Lemma 3.3] that for the closed interval I^μ,

(8.5) $$P^\mu\,[g, I^\mu] = P^\mu\,[g_n, I^\mu].$$

Because of the lower semi-continuity of the Fréchet variation,

(8.6) $$\liminf_{n\to\infty} P^\mu\,[g_n, I^\mu] \geqslant P^\mu\,[g^S, I^\mu].$$

Relation (8.4) follows from (8.5) and (8.6).

THEOREM 8.3. *If g satisfies $\hat{F}$ over $I^\mu\,[\mathbf{0}, \mathbf{i}]$, then*

$$P^r\,[g^S, I^r] \leqslant P^r\,[g, I^r],$$

where I^r is any closed r-face of I^μ $(r > 0)$.

The proof of Theorem 8.2 yields a proof of Theorem 8.3, replacing μ by r, I^μ by I^r.

COROLLARY 8.1. *If g satisfies $\hat{F}$ over $I^\mu[\mathbf{0}, \mathbf{i}]$, then g^S satisfies $\hat{F}$ over $I^\mu[\mathbf{0}, \mathbf{i}]$.*

THEOREM 8.4. *If g satisfies $\hat{F}$ over $I^\mu[\mathbf{0}, \mathbf{i}]$, the points of discontinuity of g lie on a countable set of $(\mu - 1)$-planes parallel to the coordinate $(\mu - 1)$-planes.*

The proof of this theorem is similar to the proof of the corresponding theorem when $\mu = 2$ as given in [7, §6], and need not be repeated. It depends upon the fact that whenever a is a point in $I^\mu\,[\mathbf{0}, \mathbf{i}]$ and S_a intersects $I^\mu[\mathbf{0}, \mathbf{i}]$ then $g(s)$ has a limit as $s \to a$ in S_a. Theorem 5.1 implies the existence of these sector limits.

The following theorem concerns a determination of the minimum modulus of a multilinear functional defined and continuous over the μ-fold Cartesian product of the Banach space C by itself [8, Theorem 12.1].

THEOREM 8.5. *Let g satisfy $\hat{F}$ over I^μ $[\mathbf{0}, \mathbf{i}]$ and let S and S' be two sector types in R^μ. Then*

$$P^\mu [g^S, I^\mu] = P^\mu [g^{S'}, I^\mu]. \tag{8.7}$$

Recall that $g^{S'}$ satisfies $\hat{F}$ over I^μ by Corollary 8.1. Hence $[g^{S'}]^S$ exists over I^μ. We shall write $[g^{S'}]^S = g^{S'S}$, and prove that

$$g^{S'S} = g^S. \tag{8.8}$$

The points of continuity of g in I^μ form a set A everywhere dense in I^μ in accordance with Theorem 8.4. From the definition of the sector limits it follows that

$$g(s) = g^S(s) = g^{S'}(s) \qquad [\text{for } s \in A]. \tag{8.9}$$

By virtue of (8.9), g^S and $[g^{S'}]^S$ may be evaluated at each point $t \in I^\mu$ $(\mathbf{0}, \mathbf{i})$ as the limit of a common sequence of values of $g(s)$ with $s \in A$. Hence (8.8) holds not only for $s \in A$ but also for $s \in I^\mu$ $(\mathbf{0}, \mathbf{i})$. That (8.8) holds at each point of an arbitrary open r-face F^r of I^μ is similarly established. Relation (8.8) holds at the vertices of I^μ by virtue of the definition of g^S and $g^{S'}$ at a vertex.

It follows from (8.8) that

$$\begin{aligned} P^\mu [g^S, I^\mu] &= P^\mu [g^{S'S}, I^\mu] \\ &\leqslant P^\mu [g^{S'}, I^\mu] \qquad [\text{by Theorem 8.2}]. \end{aligned}$$

Since S and S' can be interchanged in this relation, the equality must prevail and (8.7) is established.

If k satisfies $\hat{F}$ over the open interval $U^\mu = I^\mu$ $(\mathbf{0}, \mathbf{i})$, k^S can be defined over U^μ by using (8.1) alone, and the following theorem can be proved.

THEOREM 8.6. *If k satisfies $\hat{F}$ over U^μ and S is a sector type of fixed orientation, k^S satisfies $\hat{F}$ over U^μ and*

$$P^\mu [k^S, U^\mu] \leqslant P^\mu [k, U^\mu]. \tag{8.10}$$

By virtue of Theorem 6.4, k admits an extension g which satisfies $\hat{F}$ over $\bar{U}^\mu$ and for which (6.19) holds. We have

$$\begin{aligned} P^\mu [k, U^\mu] &= P^\mu [g, \bar{U}^\mu] && [\text{by (6.19)}] \\ &\geqslant P^\mu [g^S, \bar{U}^\mu] && [\text{by (8.4)}] \\ &\geqslant P^\mu [g^S, U^\mu] && [\text{by (2.13)}], \end{aligned}$$

thereby establishing (8.10). Moreover g^S satisfies $\hat{F}$ over $\bar{U}^\mu$, and in particular over U^μ. Hence k^S, which equals g^S over U^μ, satisfies $\hat{F}$ over U^μ.

The existence of a sector limit function k^S satisfying $\hat{F}$ over U^μ enables us, as in [9, §2], to prove the following theorem.

THEOREM 8.7. *If k satisfies $\hat{F}$ over U^μ, then k is L-measurable over U^μ.*

If g satisfies $\hat{F}$ over $\bar{U}^\mu$, g is then L-measurable over U^μ by Theorem 8.7, and hence L-measurable over $\bar{U}^\mu$ as well as over U^μ.

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

9. A replacement for the second law of the mean. Lemma 2.1 leads to a basic inequality whose one-dimensional form is as follows. Let f and g with values $f(s)$ and $g(s)$ map the open 1-interval $I^1 = (0, 1)$ into R^1. Recall that $P^1[g, I^1]$ is the total Jordan variation of g over I^1.

LEMMA 9.1. *If f is in L over the 1-interval $I^1 \doteq (0, 1)$, if g has a finite total variation over I^1, and if moreover $g(0+) = 0$, then*

$$|\textstyle\int_0^1 f(s)\, g(s)\, ds| \leqslant P^1[g, I^1] \sup_a |\int_a^1 f(s)\, ds|, \tag{9.1}$$

where $0 < a < 1$.

This lemma is an easy consequence of the second law of the mean, and for many purposes is an adequate replacement. We shall generalize Lemma 9.1 in Theorem 9.3.

Let f_r $(r = 1, \ldots, \mu)$ be in L with values $f_r(t)$ defined over the 1-interval $(0, 1)$. Set

$$M_r = \sup_a |\textstyle\int_a^1 f_r(t)\, dt| \qquad [r = 1, \ldots, \mu] \tag{9.2}$$

for $0 < a < 1$. Let g satisfy $\hat{F}$ over the interval $I^\mu[\mathbf{0}, \mathbf{i}]$. According to Corollary 3.1, $g(s)$ is bounded over $I^\mu[\mathbf{0}, \mathbf{i}]$ while Theorem 8.7 implies that g is L-measurable. The L-integral

$$J_r[s^1, \ldots, s^r] = \textstyle\int_0^1 \ldots \int_0^1 f_{r+1}(s^{r+1}) \ldots f_\mu(s^\mu)\, g(s^1, \ldots s^\mu)\, ds^{r+1} \ldots ds^\mu \tag{9.3}$$

is well defined for $[s^1, \ldots, s^r]$ in $I^r[\mathbf{0}, \mathbf{i}]$ and $0 < r < \mu$. For the domain of integration in (9.3) is a $(\mu - r)$-section of $I^\mu[\mathbf{0}, \mathbf{i}]$ on which g is again L-measurable by Theorem 8.7.

DEFINITION. *If g satisfies $\hat{F}$ over a left closed interval V^μ of type* (5.1), *we shall say* (1) *that g has null left boundary limits if for each $r = 1, \ldots, \mu$,*

$$g^r(s) = 0 \qquad [\text{for } s \in X_r V^\mu] \tag{9.4$'$}$$

and (2) *that g has null left boundary values if*

$$g^r(s) = g(s) = 0 \qquad [\text{for } s \in X_r V^\mu]. \tag{9.4$''$}$$

THEOREM 9.1. *If g satisfies $\hat{F}$ over $I^\mu[\mathbf{0}, \mathbf{i}]$ and has null left boundary values, then J_r satisfies $\hat{F}$ over $I^r[\mathbf{0}, \mathbf{i}]$ for $r = 1, \ldots, \mu - 1$ and has null left boundary values. Moreover,*

$$P^r[J_r, I^r] \leqslant P^\mu[g, I^\mu] M_{r+1} \ldots M^\mu \tag{9.5(μ, r)}$$

for $r = 1, \ldots, \mu - 1$.

Proof of (9.5)$(\mu, \mu - 1)$. Observe that

$$J_{\mu-1}[s^1, \ldots, s^{\mu-1}] = \textstyle\int_0^1 f_\mu(t)\, g[s^1, \ldots, s^{\mu-1}, t]\, dt. \tag{9.6}$$

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

For fixed $t \in [0, 1]$, let g_t be the function mapping $I^{\mu-1}[\mathbf{0}, \mathbf{i}]$ into R^1 with values $g[s^1, \ldots, s^{\mu-1}, t]$. For an arbitrary partition π of I^μ and associated set $\mathbf{e}$, let π', $\mathbf{e}'$ be the partition $\pi' = [\pi^1, \ldots, \pi^{\mu-1}]$ of $I^{\mu-1}$ and associated set $\mathbf{e}' = [e^1, \ldots, e^{\mu-1}]$. Then

$$\sigma^{\mu-1}[J_{\mu-1}, I^{\mu-1}, \pi', \mathbf{e}'] = \int_0^1 f_\mu(t)\, h(t)\, dt, \tag{9.7}$$

where for fixed π', $\mathbf{e}'$,

$$h(t) = \sigma^{\mu-1}[g_t, I^{\mu-1}, \pi', \mathbf{e}']. \tag{9.8}$$

It follows from the last formula in Lemma 2.1 that

$$P^\mu[g, I^\mu] \geqslant P^1[h, I^1] \qquad [I^1 = [0, 1]]. \tag{9.9}$$

Lemma 9.1 then implies that

$$\int_0^1 f_\mu(t)\, h(t)\, dt \leqslant P^1[h, I^1]\, M_\mu \leqslant P^\mu[g, I^\mu]\, M_\mu, \tag{9.10}$$

since $h(0+) = 0$ on account of the null left boundary values of g. On taking the sup of the left member of (9.10) over admissible π', $\mathbf{e}'$, (9.10) gives

$$P^{\mu-1}[J_{\mu-1}, I^{\mu-1}] \leqslant P^\mu[g, I^\mu]\, M_\mu. \tag{9.11}$$

Thus (9.5)$(\mu, \mu - 1)$ holds.

Proof that J_r has null left boundary values. The integrand $K(s)$ in the right member of (9.3) satisfies the relation

$$|K(s)| \leqslant |f_{r+1}(s^{r+1})| \ldots |f_\mu(s^\mu)|\, B,$$

where B is a bound for $|g(s)|$ over I^μ. We refer to the limits g^m of §6 $(m = 1, \ldots, \mu)$. A theorem of Lebesgue gives the result

$$J_r^m(s) = J_r(s) = 0 \qquad [\text{for } s \in X_m I^r]$$

$(m = 1, \ldots, r)$, taking account of the given relation

$$g^m(s) = g(s) = 0 \qquad [\text{for } s \in X_m I^\mu].$$

Thus J_r has null left boundary values.

Proof that J_r satisfies $\hat{F}$ over I^r and that (9.5)(μ, r) *holds.* That $J_{\mu-1}$ satisfies F over $I^{\mu-1}$ follows from (9.5)$(\mu, \mu - 1)$ already established. That $J_{\mu-1}$ satisfies $\hat{F}$ over $I^{\mu-1}$ then follows from the fact that $J_{\mu-1}$ has null left boundary values. Proceeding inductively, we shall assume that J_n satisfies $\hat{F}$ over I^n and that (9.5)(μ, n) holds with $1 < n < \mu$, and show that J_{n-1} satisfies $\hat{F}$ over I^{n-1} and that (9.5)$(\mu, n - 1)$ holds.

Observe that

$$J_{n-1}[s^1, \ldots, s^{n-1}] = \int_0^1 f_n(t)\, J_n[s^1, \ldots, s^{n-1}, t]\, dt. \tag{9.12}$$

We compare this relation with (9.6) and note that J_n satisfies the same conditions as g in (9.6), with μ replaced by n and I^μ by I^n. We infer, as in (9.11), that

$$P^{n-1}[J_{n-1}, I^{n-1}] \leqslant P^n[J_n, I^n]\, M_n. \tag{9.13}$$

Thus J_{n-1} satisfies $\hat{F}$ over I^{n-1}. By inductive hypothesis (9.5)(μ, n) holds so that

$$P^n[J_n, I^n] \leqslant P^\mu[g, I^\mu]\, M_{n+1} \ldots M_\mu. \tag{9.14}$$

Relations (9.13) and (9.14) imply (9.5)$(\mu, n-1)$. This completes the proof.

THEOREM 9.2. *Under the hypothesis of Theorem* 9.1, *the integral*

$$J_0 = \int_0^1 \ldots \int_0^1 f_1(s^1) \ldots f_\mu(s^\mu)\, g(s^1, \ldots, s^\mu)\, ds^1 \ldots ds^\mu \tag{9.15}$$

satisfies the relation

$$|J_0| \leqslant P^\mu[g, I^\mu]\, M_1 \ldots M_\mu. \tag{9.15$'$}$$

Observe that

$$J_0 = \int_0^1 f_1(t)\, J_1(t)\, dt.$$

Moreover $J_1(0+) = 0$ and J_1 has a finite total Jordan variation over $[0, 1]$ in accordance with Theorem 9.1. It follows from Lemma 9.1 that

$$|J_0| \leqslant M_1 P^1[J_1, I^1] \qquad [I^1 = [0, 1]], \tag{9.16}$$

so that (9.15)$'$ holds as a consequence of (9.16) and of the relation

$$P^1[J_1 I^1] \leqslant P^\mu[g, I^\mu]\, M_2 M_3 \ldots M_\mu$$

implied by Theorem 9.1. This completes the proof of Theorem 9.2.

Theorem 9.2 is not a precise extension of Lemma 9.1 since the interval I^μ in Theorem 9.1 is closed while the interval $(0, 1)$ in Lemma 9.1 is open. Theorem 9.2 leads, however, to a precise extension of Lemma 9.1 once Lemma 9.2 is established.

The proof of Lemma 9.2 requires the introduction of limits $g^{r-}(s)$ $(r = 1, \ldots, \mu)$, defined as were the limits $g^r(s)$, using a left limit instead of a right limit. More generally, if $r_1, \ldots, r_n$ are distinct arbitrary integers chosen from the set $1, \ldots, \mu$, the limits $g^{r-_1 \cdots r-_n}(s)$ may be supposed well defined. Suppose that k satisfies $\hat{F}$ over $U^\mu = I^\mu(\mathbf{0}, \mathbf{i})$. Let a be any point on the boundary of U^μ at which

$$\begin{gathered} a^{m_1} = a^{m_2} = \ldots = a^{m_p} = 0, \\ a^{r_1} = a^{r_2} = \ldots = a^{r_n} = 1, \end{gathered} \qquad [m_i \neq m_j \text{ if } i \neq j] \tag{9.17}$$

with the remaining coordinates (if any) neither 0 nor 1. By an obvious extension of Theorem 6.1, the limit

$$k^{r-_1 \ldots r-_n\, m_1 \ldots m_p}(a) \tag{9.18}$$

exists and is independent of the order of writing of the superscripts.

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

LEMMA 9.2. *A function k which satisfies* $\hat{F}$ *over* $U^\mu = I^\mu(\mathbf{0}, \mathbf{i})$ *and has null left boundary limits, admits an extension g over* $\bar{U}^\mu$ *which satisfies* $\hat{F}$ *over* $\bar{U}^\mu$, *has null left boundary values and satisfies the relation*

$$P^r[k, Q^r \cap U^\mu] = P^r[g, Q^r], \tag{9.19}$$

where Q^r *is an arbitrary r-segment in* $\bar{U}^\mu$ *intersecting* U^μ.

This lemma is identical with Theorem 6.4 except that k is here assumed to have null left boundary limits and g is then affirmed to have *null*, instead of canonical, left boundary values. We accordingly define g as in (6.20), and seek to prove that g has null left boundary values. As stated in Theorem 6.4, g has canonical left boundary values so that for $r = 1, \ldots, \mu$,

$$g^r(a) = g(a) \qquad [\text{for } a \in X_r I^\mu(\mathbf{0}, \mathbf{i}] = X_r \bar{U}^\mu]. \tag{9.20}$$

It remains to show that $g(a) = 0$ in (9.20). Suppose that a has coordinates given by (9.17) with the remaining coordinates (if any) neither zero nor one. Note that for a point a in (9.20), $p > 0$ in (9.17). The definition of $g(a)$ in (6.20) is such that

$$g(a) = k^{r_{-1} \cdots r_{-n}\, m_1 \cdots m_p}(a).$$

But as noted in writing (9.18), $g(a)$ also has the form

$$g(a) = k^{m_1 \cdots m_p\, r_{-1} \cdots r_{-n}}(a) \qquad [p > 0]. \tag{9.21}$$

But $k^{m_1 \ldots m_p}(s) = 0$ for $s \in X_{m_1} \ldots X_{m_p} U^\mu$, since k has null left boundary limits. It follows that $g(a) = 0$ in (9.21). Thus g has null left boundary values.

This establishes the lemma.

Although no use will be made of the following, the preceding analysis implies the following theorem of which Lemma 9.2 is a special consequence.

THEOREM 9.3. *A function k which satisfies* $\hat{F}$ *over* U^μ *admits an extension g over* $\bar{U}^\mu$ *with the following properties. The function g is continuous at each vertex of* U^μ. *At each point u on any open r-cell* σ_r *of the boundary of* $U^\mu (r = 1, \ldots, \mu - 1)$, $g(u)$ *is the limit of* $k(s)$ *as* $s \to u$ *with* $s \in \bar{U}^\mu \cap \pi_{\mu-r}$ *where* $\Pi_{\mu-r}$ *is the* $(\mu - r)$*-plane orthogonal to* σ_r *at* $s = u$. *Moreover,*

$$P^r[k, Q^r \cap U^\mu] = P^r[g, Q^r] \qquad [r = 1, \ldots, \mu]$$

for each r-segment Q^r *in* $\bar{U}^\mu$ *intersecting* U^μ.

The generalization of Lemma 9.1 is as follows:

THEOREM 9.4. *If k satisfies* $\hat{F}$ *over* $U^\mu = I^\mu(\mathbf{0}, \mathbf{i})$ *and has null left boundary limits, then* J_0, *as defined by* (9.15) *with k replacing g, satisfies the relation*

$$|J_0| \leqslant P^\mu[k, U^\mu]\, M_1 \ldots M_\mu. \tag{9.22}$$

https://doi.org/10.4153/CJM-1950-033-1 Published online by Cambridge University Press

If g is derived from k as in the preceeding lemma, Theorem 9.2 holds while J_0 is unchanged in value if g is replaced by k in (9.15). Moreover,

$$P^{\mu}[k, U^{\mu}] = P^{\mu}[g, \bar{U}^{\mu}],$$

according to (9.19), so that (9.22) follows from (9.15)$'$.

Theorem 9.1 can be similarly generalized with k replacing g, U^{μ} replacing $\bar{U}^{\mu}$, and null boundary limits replacing null boundary values, with the conclusion

$$P^{r}[J_r, U^{r}] \leqslant P^{\mu}[k, U^{\mu}]\, M_{r+1} \ldots M_{\mu}.$$

References

[1] S. Banach, *Théorie des opérations linéaires* (Warsaw, 1932).

[2] S. Bochner, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc., vol. 40 (1936), 175-207.

[3] S. Bochner and K. Chandrasekharan, *On the localization property for multiple Fourier series*, Ann. of Math., vol. 49 (1948), 966-978.

[4] L. Cesari, *Sulle funzioni di due variabili a variazione limitata secondo Tonelli e sulla convergenza delle relative serie doppie di Fourier*, Rendiconti del Seminario della R. Università di Roma, 1937.

[5] M. Fréchet, *Sur les fonctionnelles bilinéaires*, Trans. Amer. Math. Soc., vol. 16 (1915), 215-234.

[6] J. J. Gergen, *Convergence criteria for double Fourier series*, Trans. Amer. Math. Soc., vol. 35 (1933), 29-63.

[7] M. Morse and W. Transue, *Functionals of bounded Fréchet variation*, Can. J. Math., vol. 1 (1949), 153-165.

[8] ——— *Functionals F bilinear over the product A × B of two pseudo-normed vector spaces, I. The representation of F*, Ann. of Math., vol. 50 (1949).

[9] ——— *Integral representations of bilinear functionals*, Proc. Nat. Acad. Sci., vol. 35 (1949), 136-143.

[10] ——— *Functionals F bilinear over the product A × B of two pseudo-normed vector spaces,* II. *Admissible spaces A*, Ann. of Math., vol. 51 (1950).

[11] ———*A characterization of the bilinear sums associated with the classical second variation*, Annali di Matematica Pura ed Applicata Ser. 4, vol. 28 (1949).

[12] ———*Norms of distribution functions associated with bilinear functionals* (to be published).

[13] ———*The Fréchet variation and a generalization of the Jordan test for multiple Fourier series*, Rivista Matematica, vol. 1 (1950).

[14] ——— *Calculus for Fréchet variations* (to be published).

[15] ——— *The Fréchet variation and the convergence of multiple Fourier series*, Proc. Nat. Acad. Sci., vol. 35 (1949).