M.-C. Liu and K.-M. Tsang
Nagoya Math. J.
Vol. 85 (1982), 241-249

ON THE DISTRIBUTION (MOD 1) OF POLYNOMIALS OF A PRIME VARIABLE

MING-CHIT LIU AND KAI-MAN TSANG

§1. Introduction

Throughout, ε is any small positive number, θ any real number, n, n_j, k, N some positive integers and p, p_j any primes. By $\|\theta\|$ we mean the distance from θ to the nearest integer. Write $C(\varepsilon), C(\varepsilon, k)$ for positive constants which may depend on the quantities indicated inside the parentheses.

Dirichlet's theorem says that for any θ, N there exists n such that

$$n \leqslant N \quad \text{and} \quad \|\theta n\| < N^{-1} . \tag{1.1}$$

Furthermore, as a direct consequence of (1.1), there are infinitely many n such that

$$\|\theta n\| < n^{-1} . \tag{1.2}$$

Improving an estimate of Vinogradov [12], Heilbronn [6] extended (1.1) by showing that for any θ, ε, N there are n and $C(\varepsilon)$ such that

$$n \leqslant N \quad \text{and} \quad \|\theta n^2\| < C(\varepsilon)N^{-1/2+\varepsilon} . \tag{1.3}$$

Later, Davenport [3] extended (1.3) by proving that if g is a polynomial of degree $k \geqslant 2$ with real coefficients and without constant term then for any ε, N there are n and $C(\varepsilon, k)$ such that

$$n \leqslant N \quad \text{and} \quad \|g(n)\| < C(\varepsilon, k)N^{-1/(2^k-1)+\varepsilon} . \tag{1.4}$$

The results of Heilbronn [6] and Davenport [3] sparked off a series of investigations (see [9]). In particular, recently Schmidt has made remarkable progress in [9, 10]. However all these developments concerning (1.1) have no parallel results for prime. This can be seen from the following example. Let q be any positive integer having at least two

Received June 6, 1980.

https://doi.org/10.1017/S0027763000019747 Published online by Cambridge University Press

distinct prime factors and $\{\alpha_j\}_1^\infty$ a sequence of irrationals which converges to the rational a/q with $(a, q) = 1$. Obviously

$$\|p^k a/q\| \geqslant 1/q \tag{1.5}$$

for any prime p. Suppose that when θ is irrational, (1.4) has a parallel result for prime, i.e. for any α_j, ε, N there are p and $C(\varepsilon, k)$ such that

$$p \leqslant N \quad \text{and} \quad \|\alpha_j p^k\| < C(\varepsilon, k)N^{-\delta+\varepsilon}\,, \tag{1.6}$$

where $\varepsilon < \delta$ and δ depends on k only. Now if $N^{\delta-\varepsilon} > q(C(\varepsilon, k) + 1)$ and α_j satisfies $|\alpha_j - a/q| < N^{-k-\delta}$, then by (1.6),

$$\|p^k a/q\| \leqslant \|\alpha_j p^k\| + p^k\,\|\alpha_j - a/q\| < N^{-\delta+\varepsilon}(C(\varepsilon, k) + 1) < 1/q\,.$$

This contradicts (1.5).

On the contrary, concerning (1.2) there is indeed a parallel result for prime. It was mentioned in [5] that by a result of Vinogradov [14, Chapter 9] for any ε and irrational α, there are infinitely many primes p such that $\|\alpha p\| < p^{-1/5+\varepsilon}$. Recently this inequality was improved by Vaughan [11] to $\|\alpha p\| < p^{-1/4}(\log p)^8$. The object of our present paper is to extend Dirichlet's theorem (1.2) to polynomials of a prime variable as that (1.3) and (1.4) extend Dirichlet's theorem (1.1). We shall prove

THEOREM. *If f is any polynomial of degree $k \geqslant 2$ with real coefficients and irrational leading coefficient then for any $\varepsilon > 0$ there are infinitely many primes p such that*

$$\|f(p)\| < p^{-A(k)+\varepsilon}\,,$$

where $A(k) = (3(k + 1)4^{k+1})^{-1}$.

By (1.5) we see that the irrationality in our theorem is essential. In our proof, unlike most previous work in this field, we make no use of the Heilbronn argument [6] presented by Davenport in [3] but we modify an earlier method due to Davenport and Heilbronn [1]. Also in § 4 we are able to use the full-strength of a result of Vinogradov [13] which determines the exponent, $A(k)$ in our Theorem.

§ 2. Notation

Let δ be a small positive number (<1) and x a real variable. Write $e(x) = \exp(i2\pi x)$ and denote the integral part of x by $[x]$. Let α be the leading coefficient of the given polynomial f. Since α is irrational, by Theorem 183 [4] there are infinitely many convergents a/q such that

$$|\alpha - a/q| < (2q^2)^{-1} . \tag{2.1}$$

For sufficiently large q, put

$$\begin{aligned} X &= q^{1/(k-2/3)} , \qquad L = \log X , \\ I_j(x) &= \int_X^{2X} e((-1)^j xy^k)dy \qquad (j = 1, 2) , \end{aligned} \tag{2.2}$$

$$\begin{cases} S_j(x) = \sum_{X<n\leqslant 2X} e((-1)^j xn^k) \qquad (j = 1, 2) , \\ S_3(x) = \sum_{\delta X<p\leqslant 2\delta X} e(xf(p)) , \qquad S_4(x) = \sum_{\delta X<p\leqslant 2\delta X} e(xp^k) , \\ S_j(x) = \sum_{\delta X<n\leqslant 2\delta X} e(xn^k) \qquad (j = 5, 6, \cdots, s) , \end{cases} \tag{2.3}$$

where

$$s = 2^k + 2 . \tag{2.4}$$

Trivially,

$$|S_j(x)| \leqslant X \ (j = 1, \cdots, s) \quad \text{and} \quad |I_j(x)| \leqslant X \ (j = 1, 2) . \tag{2.5}$$

Furthermore we put

$$V(x) = \prod_{j=1}^{s} S_j(x) , \qquad W(x) = I_1(x)I_2(x) \prod_{j=3}^{s} S_j(x) , \tag{2.6}$$

$$A(k) = (3(k+1)4^{k+1})^{-1} , \tag{2.7}$$

$$\tau = X^{-A(k)+\varepsilon} , \tag{2.8}$$

$$K_\tau(x) = \begin{cases} \tau^2 & \text{if } x = 0 , \\ \left(\dfrac{\sin \pi\tau x}{\pi x}\right)^2 & \text{otherwise.} \end{cases}$$

Obviously,

$$K_\tau(x) \leqslant \tau^2 . \tag{2.9}$$

We partition the real line into

$$\begin{cases} E_1 = \{x: |x| \leqslant X^{-k+1/3}\} , \\ E_2 = \{x: X^{-k+1/3} < |x| \leqslant X^{2A(k)}\} , \\ E_3 = \{x: X^{2A(k)} < |x|\} . \end{cases} \tag{2.10}$$

If $Y>0$ we use $Z \ll Y$ (or $Y \gg Z$) to denote $|Z|<CY$ where C is some positive constant. The constants implied by $O, \ll, \gg$ may depend on the given constants, k, ε, δ and the coefficients of f only.

§ 3. Integration over E_1

LEMMA 1. *For any real y we have*

$$\int_{-\infty}^{\infty} e(xy)K_\tau(x)dx = \max(0, \tau-|y|) .$$

Proof. See Lemma 2 in [8].

LEMMA 2. *We have*

$$\int_{-\infty}^{\infty} W(x)K_\tau(x)dx \gg \tau^2 X^{s-k}L^{-2} .$$

Proof. Let B denote the cartesian product of the intervals, $X^k \ll y_j \ll (2X)^k$ $(j=1,2)$ and let the set B^* of (y_1, y_2) be defined by the following (3.1), (3.2), (3.3) and (3.4).

$$2X^k \leqslant y_1 \leqslant 3X^k , \tag{3.1}$$

$$y_2 = y_1 + \phi - f(p_3) - p_4^k - \sum_{5\leqslant j\leqslant s} n_j^k , \tag{3.2}$$

where

$$\delta X \leqslant p_3, p_4, n_5, \cdots, n_s \leqslant 2\delta X \tag{3.3}$$

and ϕ is a real variable satisfying

$$|\phi| < \tau/2 . \tag{3.4}$$

By (3.1), (3.2), (3.3) and (3.4) we see that

$$y_2 \leqslant 3X^k + \tau/2 + 2|\alpha|(2\delta X)^k + (2\delta X)^k + (s-4)(2\delta X)^k < 4X^k .$$

Similarly $y_2 > X^k$. So

$$B^* \subset B . \tag{3.5}$$

By (2.6), (2.2) and (2.3) we have

$$\int_{-\infty}^{\infty} W(x)K_\tau(x)dx = \sum\nolimits_1 \int_{-\infty}^{\infty} \Big(\prod_{j=1}^{2} \int_{X^k}^{(2X)^k} (ky_j^{1-1/k})^{-1} e((-1)^j xy_j)dy_j\Big) \times e\Big(x\Big\{f(p_3) + p_4^k + \sum_{5\leqslant j\leqslant s} n_j^k\Big\}\Big)K_\tau(x)dx ,$$

where $\sum_1$ is a summation taken over all p_j, n_j satisfying (3.3). Then by Lemma 1, (3.5), (3.2), (3.4) and (3.1) we have

$$\begin{aligned}\int_{-\infty}^{\infty} W(x)K_\tau(x)\,dx &\gg X^{2(1-k)} \sum\nolimits_1 \int_B \max\Big(0,\, \tau - \Big|-y_1 + y_2 + f(p_3) + p_4^k \\ &\qquad\qquad + \sum_{5\leqslant j\leqslant s} n_j^k\Big|\Big)dy_1dy_2 \\ &\gg X^{2(1-k)} \sum\nolimits_1 (\tau - (\tau/2)) \int_{B^*} dy_1dy_2(\phi) \\ &\gg X^{2(1-k)} \sum\nolimits_1 \tau(\tau X^k) \\ &\gg \tau^2 X^{2-k} X^{s-4}(X/L)^2 .\end{aligned}$$

The last inequality follows from (3.3) and the prime number theorem. This proves Lemma 2.

LEMMA 3. *If* $|x| \ll X^{-k+1/3}$ *then for* $j = 1, 2$

$$S_j(x) = I_j(x) + O(1) .$$

Proof. This is essentially the Corollary in [2, p. 85].

LEMMA 4. *We have*

$$\int_{E_1} V(x)K_\tau(x)dx \gg \tau^2 X^{s-k} L^{-2} .$$

Proof. By (2.6), Lemma 3 and (2.5) we have, when $x \in E_1$

$$\begin{aligned}|V(x) - W(x)| &= |S_1(S_2 - I_2) + I_2(S_1 - I_1)| \prod_{j=3}^{s} |S_j(x)| \\ &= O(X) \prod_{j=3}^{s} |S_j(x)| = O(X^{s-1}) .\end{aligned}$$

So in view of (2.9) and (2.10)

$$\left|\int_{E_1} V(x)K_\tau(x)dx - \int_{E_1} W(x)K_\tau(x)dx\right| \ll \tau^2 X^{s-1} \int_{E_1} dx \ll \tau^2 X^{s-k-2/3} . \tag{3.6}$$

On the other hand, by integration by parts and (2.2) if $x \neq 0$ we have

$$I_j(x) = O(|x|^{-1} X^{-k+1}) . \tag{3.7}$$

It follows from (2.9), (3.7), (2.5) and (2.10) that

$$\int_{x\notin E_1} W(x)K_\tau(x)dx \ll \tau^2 X^{s-2} X^{2(1-k)} \int_{x\notin E_1} |x|^{-2}\,dx \ll \tau^2 X^{s-k-1/3} . \tag{3.8}$$

Lemma 4 follows from Lemma 2, (3.6) and (3.8).

https://doi.org/10.1017/S0027763000019747 Published online by Cambridge University Press

§ 4. Integration over E_2

LEMMA 5. *Let $\lambda_3 = \alpha$ (the leading coefficient of f) and $\lambda_4 = 1$. Suppose that for $j = 3$ or 4 there are integers a_j, q_j with $(a_j, q_j) = 1$, $1 \leqslant q_j$ and*

$$|\lambda_j x - a_j/q_j| \leqslant q_j^{-2} .$$

If

$$Q = \min(q_j, [2\delta X]^k/q_j) , \qquad U = \min(Q, [2\delta X]^{1/3}) \tag{4.1}$$

and

$$Q \geqslant (k \log [2\delta X])^{(2k+1)4^{3k-1}} \tag{4.2}$$

then

$$S_j(x) \ll XU^{-3A(k)} ,$$

where $A(k)$ is defined in (2.7).

Proof. This is the Theorem in [13].

LEMMA 6. *We have*

$$\sup_{x \in E_2} \min(|S_3(x)|, |S_4(x)|) \ll X^{1-A(k)} .$$

Proof. Let $\lambda_3 = \alpha$, which is the leading coefficient of the polynomial f, and $\lambda_4 = 1$. By Dirichlet's theorem [4, p. 30] for each $x \in E_2$ there are integers a_j, q_j with $(a_j, q_j) = 1$ and

$$1 \leqslant q_j \leqslant \delta^{-1} X^{k-1/3} \tag{4.3}$$

such that

$$|\lambda_j x - a_j/q_j| \leqslant \delta X^{-k+1/3} q_j^{-1} \qquad (j = 3, 4) .$$

By the same argument as that in Lemma 13 of [8] we can prove that $\max(q_3, q_4) \geqslant X^{1/3}$. In the proof we need (2.1), that is the irrationality of α. Then Lemma 6 follows from Lemma 5.

LEMMA 7. *For $j \neq 3, 4$ we have*

$$\int_{-\infty}^{\infty} |S_j(x)|^{2^k} K_\tau(x) dx \ll \tau X^{2^k - k} L^c ,$$

where c is some positive constant depending on k only.

Proof. This is a consequence of Hua's Lemma [Theorem 4, 7]. See Lemma 21 in [8].

https://doi.org/10.1017/S0027763000019747 Published online by Cambridge University Press

LEMMA 8. *We have*

$$\int_{E_2} |V(x)|\, K_\tau(x)dx \ll \tau^2 X^{s-k} L^{-3}\,.$$

Proof. By Lemma 6 we have

$$\begin{aligned}\int_{E_2} |V(x)|\, K_\tau(x)dx &\leqslant \sup_{x\in E_2} \min\,(|S_3(x)|, |S_4(x)|)\\ &\qquad \times \left\{\int_{E_2} (|S_3(x)| + |S_4(x)|) \left|\prod_{j\neq 3,4} S_j(x)\right| K_\tau(x)dx\right\} \qquad (4.4)\\ &\ll X^{1-A(k)}\{J_1 + J_2\}\,, \qquad \text{say.}\end{aligned}$$

Note that by (2.4) there are 2^k factors in the above product $\prod_{j\neq 3,4} S_j(x)$. We denote the products taken over the first 2^{k-1} and last 2^{k-1} factors by $\prod_1$ and $\prod_2$ respectively. By (2.5) and Hölder's inequality we have

$$\begin{aligned}J_1 &\ll X\int_{E_2} \left|\prod_{j\neq 3,4} S_j(x)\right| K_\tau(x)dx\\ &\ll X\left\{\prod\nolimits_1 \left(\int_{-\infty}^{\infty} |S_j(x)|^{2^k} K_\tau(x)dx\right)^{2^{1-k}}\right\}^{1/2} \left\{\prod\nolimits_2 \left(\int_{-\infty}^{\infty} |S_j(x)|^{2^k} K_\tau(x)dx\right)^{2^{1-k}}\right\}^{1/2}.\end{aligned}$$

The same argument holds for J_2, then by Lemma 7 we have

$$J_1 \quad \text{and} \quad J_2 \ll \tau X^{2^k-k+1} L^c\,.$$

This, together with (4.4), (2.4) and (2.8), proves Lemma 8.

§ 5. Completion of the proof

LEMMA 9. *Let $\Omega(x) = \sum e(x\omega(y_1, \cdots, y_n))$, where ω is any real-valued function and the summation is over any finite set of values $y_1, \cdots, y_n$. Then for any $R > 4/\tau$ we have*

$$\int_{|x|>R} |\Omega(x)|^2\, K_\tau(x)dx \ll (R\tau)^{-1} \int_{-\infty}^{\infty} |\Omega(x)|^2\, K_\tau(x)dx\,.$$

Proof. This follows from Lemma 2 in [2]. See Lemma 16 in [8].

LEMMA 10. *We have*

$$\int_{E_3} |V(x)|\, K_\tau(x)dx \ll \tau^2 X^{s-k} L^{-3}\,.$$

Proof. By (2.5), Lemma 9 with $R = X^{2A(k)}$ and a similar argument as in the proof of Lemma 8, we have

$$\int_{E_3} |V(x)|\, K_\tau(x)dx \ll X^2(X^{2A(k)}\tau)^{-1}\left\{\prod\nolimits_1 \left(\int_{-\infty}^{\infty} |S_j(x)|^{2^k} K_\tau(x)dx\right)^{2^{1-k}}\right\}^{1/2}$$
$$\times \left\{\prod\nolimits_2 \left(\int_{-\infty}^{\infty} |S_j(x)|^{2^k} K_\tau(x)dx\right)^{2^{1-k}}\right\}^{1/2}$$
$$\ll \tau^2 X^{s-k} L^{-3} .$$

This proves Lemma 10.

We come now to the proof of our Theorem. By Lemma 1 we have

$$J = \int_{-\infty}^{\infty} V(x)K_\tau(x)dx = \sum\nolimits_2 \max\left(0, \tau - \left|n_1^k - n_2^k + f(p_3) + p_4^k + \sum_{5\leqslant j\leqslant s} n_j^k\right|\right) ,$$

where the summation $\sum_2$ is taken over all s-tuples $(n_1, n_2, p_3, p_4, n_5, \cdots, n_s)$ lying in

(5.1) $$X \leqslant n_1, n_2 \leqslant 2X ; \qquad \delta X \leqslant p_3, p_4, n_5, \cdots, n_s \leqslant 2\delta X .$$

Then

(5.2) $$J \leqslant \tau N ,$$

where N is the number of $(n_1, n_2, p_3, p_4, n_5, \cdots, n_s)$ satisfying (5.1) and

(5.3) $$\left|n_1^k - n_2^k + f(p_3) + p_4^k + \sum_{5\leqslant j\leqslant s} n_j^k\right| < \tau = X^{-A(k)+\varepsilon} .$$

Now, by Lemmas 4, 8 and 10 we have

$$J = \sum_{\nu=1}^{3} \int_{E_\nu} V(x)K_\tau(x)dx \gg \tau^2 X^{s-k} L^{-2} .$$

So by (5.2)

(5.4) $$N \gg \tau X^{s-k} L^{-2} \longrightarrow \infty \qquad \text{as } X \longrightarrow \infty .$$

Since $n_1^k - n_2^k + p_4^k + \sum_{5\leqslant j\leqslant s} n_j^k$ is an integer and $\delta X \leqslant p_3 \leqslant 2\delta X$, by (5.3), (5.4) we see that

$$\|f(p_3)\| < p_3^{-A(k)+\varepsilon}$$

has infinitely many solutions in primes p_3. This proves our Theorem.

References

[1] H. Davenport and H. Heilbronn, On indefinite quadratic forms in five variables, J. London Math. Soc., **21** (1946), 185–193.

[2] H. Davenport and K. F. Roth, The solubility of certain diophantine inequalities, Mathematika, **2** (1955), 81–96.

[3] H. Davenport, On a theorem of Heilbronn, Quart. J. Math. Oxford, (2), **18** (1967), 339–344.
[4] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 4th ed. Oxford, 1965.
[5] S. Hartman and S. Knapowski, Bemerkungen über die Bruchteile von $p\alpha$, Ann. Polon. Math., **3** (1957), 285–287.
[6] H. Heilbronn, On the distribution of the sequence $n^2\theta\,(\mathrm{mod}\,1)$, Quart. J. Math. Oxford, **19** (1948), 249–256.
[7] L. K. Hua, Additive Theory of Prime Numbers, Translations of Mathematical Monographs Vol. 13, Amer. Math. Soc., Providence, R.I. 1965.
[8] M. C. Liu, Approximation by a sum of polynomials involving primes, J. Math. Soc. Japan, **30** (1978), 395–412.
[9] W. M. Schmidt, Small Fractional Parts of Polynomials, Regional Conference Series in Mathematics No. 32, Amer. Math. Soc. Providence, R.I. 1977.
[10] ——, On the distribution modulo 1 of the sequence $\alpha n^2+\beta n$, Canad. J. Math., **29** (1977), 819–826.
[11] R. C. Vaughan, On the distribution of αp modulo 1, Mathematika, **24** (1977), 135–141.
[12] I. M. Vinogradov, Analytischer Beweis des Satzes über die Verteilung der Bruchteile eines ganzen Polynoms, (in Russian), Bull. Acad. Sci. USSR (6), **21** (1927), 567–578.
[13] ——, A new estimate of a trigonometric sum containing primes, (in Russian with English summary), Bull. Acad. Sci. USSR ser. Math., **2** (1938), 3–13.
[14] ——, The Method of Trigonometrical Sums in the Theory of Numbers, New York: Interscience 1954.

Department of Mathematics
University of Hong Kong
Hong Kong