ON A RESULT OF JOHNSON ABOUT SCHUR MULTIPLIERS

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The purpose of this short note is to give a new and shorter proof of the following theorem of Johnson [**1**], and to extend it somewhat.

THEOREM 1. *Let G be a finite non-cyclic p-group possessing a non-empty subset X such that, for each x in X, $\langle X \backslash \{x\} \rangle G'$ is a complement for $\langle x \rangle$ in G. Then the Schur multiplier of G is non-trivial.*

In particular, the theorem applies to p-groups generated by elements of order p. Johnson's proof is homological in flavour, and I have always believed (see [**2**]) that one should be able to produce a quick, purely group-theoretical argument. In fact, we have:

THEOREM 2. *Let G be a finite group that can be expressed as a factor-group $G = F/R$, where F is residually nilpotent, $R \neq 1$ and $R \leqslant F'$. Then $M(G) \neq 1$. If in addition F has trivial centre, there is an infinite sequence $G = G_1, G_2, \ldots$ of groups such that G_{i+1} is a non-trivial stem extension of G_i for $i \geqslant 1$.*

Proof. This is fairly standard, as is all the notation and terminology used here. Since $G \cong F/[R,F]/R/[R,F]$, it follows that $F/[R,F]$ has finite central factor-group, so that $F'/[R,F]$ is finite and thus $F/[R,F]$ is finite since $R \leqslant F'$. But then $R/[R,F]$ is an image of $M(G)$, and it is non-trivial since F is residually nilpotent: if $R = [R,F]$, then $R \leqslant \gamma_n(F)$ for every n (here, as usual, $\gamma_n(F)$ is the n-th term of the lower central series of G).

Now suppose that F has trivial centre, and write R_i for $[R, F, \ldots, F]$, with $i-1$ occurrences of F, so that $R_1 = R$. Set $G_i = F/R_i$. Then $G_i \cong F/R_{i+1}/R_i/R_{i+1}$, and as before R_i/R_{i+1} is an image of $M(G_i)$. Since $F/R_{i+1} = G_{i+1}$ and $R_i/R_{i+1} \leqslant G'_{i+1}$, all we have to do is to show that $R_i \neq R_{i+1}$. If $R_i = R_{i+1}$, that is, $R_i = [R_i, F]$, we have $R_i = 1$ since F is residually nilpotent. Hence $i \geqslant 2$ since $R \neq 1$ by assumption, so that $[R_{i-1}, F] = 1$ and thus $R_{i-1} = 1$ since F has trivial centre. This argument continues until we reach a contradiction to the fact that R is non-trivial.

To recover Theorem 1 from Theorem 2, proceed like this. Suppose that the set X figuring in Theorem 1 consists of elements $x_1, \ldots, x_n$ of orders precisely $p^{k_1}, p^{k_2}, \ldots, p^{k_n}$ respectively, and take for F the free product of cyclic groups $\langle z_1 \rangle, \ldots, \langle z_n \rangle$ of these same orders. Then F is residually nilpotent, and F and the kernel R of the homomorphism extending the map $z_1 \mapsto x_1, \ldots, z_n \mapsto x_n$ are readily seen to satisfy all the requirements of Theorem 2.

REFERENCES

1. David L. Johnson, A property of finite p-groups with trivial multiplicator, *Amer. J. Math.* **98** (1976), 105–108.

2. James Wiegold, The Schur multiplier: an elementary approach, in *Groups—St. Andrews, 1981, London Mathematical Society Lecture Note Series* **71** pp. 137–154.

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