13

$\overline{MS}$ scheme for QED

Related discussions can e.g. be found in [147,110,111]. QED which is an Abelian theory is much simpler than the non-Abelian QCD. As we have mentioned in the introduction, QED works well with the on-shell renormalization scheme and has been experimentally tested with a high degree of accuracy ($g-2, \ldots$), such that it is a priori useless to introduce a new scheme for studying it. However, it is interesting to know the relations of different observables in the on-shell and $\overline{MS}$ schemes.

13.1 The QED Lagrangian

As we have already mentioned previously, the QED Lagrangian is quite simple, as we do not need ghost fields for its quantization due to its Abelian character. Its expression is given in Eq. (5.10), which we repeat below:

$$\mathcal{L}_{\text{QED}} \equiv \mathcal{L}_\gamma + \mathcal{L}_l + \mathcal{L}_\text{I} \;, \tag{13.1}$$

with:

$$\begin{aligned}
\mathcal{L}_\gamma &= -\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x) - \frac{1}{2\alpha_G} \partial_\mu A^\mu(x) \partial_\nu A^\nu(x) \;, \\
\mathcal{L}_l &= \bar\psi(x)(i\partial_\mu \gamma^\mu - m)\psi(x) \;, \\
\mathcal{L}_\text{I} &= -e A_\mu(x) \bar\psi(x) \gamma^\mu \psi(x)
\end{aligned} \tag{13.2}$$

where $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ is the photon field strengths of the photon field $A_\mu(x)$; α_G is the gauge parameter with $\alpha_G = 0$ (Landau gauge) and $\alpha_G = 1$ (Feynman gauge); $\psi(x)$ and m is the lepton field and mass; e is the electric charge.

13.2 Renormalization constants and RGE

The renormalization constants of the fields are defined analogously to the case of QCD in Eqs. (9.20, 9.21, 9.22), and will not be repeated here. The QED Ward identity, analogous to the one in Eq. (9.24), gives:

$$Z_{1F} = Z_{2F} \;, \tag{13.3}$$

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which implies:

$$Z_\alpha = Z_{3YM}^{-1} \equiv Z_3^{-1} , \tag{13.4}$$

while the gauge invariance of the QED Lagrangian implies that there is no renormalization counterterm for gauge term of the Lagrangian. Then:

$$Z_G = Z_3 \tag{13.5}$$

The RGE of QED has been originally introduced by [75,76] for improving the QED perturbation series. It has the form in Eq. (11.32) with the solution in Eq. (11.33).

13.3 β function, running coupling and anomalous dimensions

The β function is known to order α^3. It reads [179]:

$$\beta(\alpha) \equiv \frac{\nu}{Z_3}\frac{dZ_3}{d\nu}$$
$$= \left(\beta_1 = \frac{2}{3}\right)\left(\frac{\alpha}{\pi}\right) + \left(\beta_2 = \frac{1}{2}\right)\left(\frac{\alpha}{\pi}\right)^2 + \left(\beta_3 = -\frac{121}{144}\right)\left(\frac{\alpha}{\pi}\right)^3 , \tag{13.6}$$

where, in the case of massless fermion, β_1 and β_2 are invariant under the renormalization schemes. It is important to notice the crucial difference between QCD and QED, as here the β function is positive, which means that the origin of the coupling is no longer an UV fixed point and the QED coupling increases with q^2. The running QED coupling is solution of the differential equation analogue to the one in Eq. (11.34). Its solution to one loop is:

$$\bar{\alpha}(q^2) = \frac{\alpha(\nu)}{1 - (\alpha(\nu)/\pi)\,\beta_1 \frac{1}{2}\ln(-q^2/\nu^2)} \equiv \frac{\pi}{\beta_1 \frac{1}{2}\ln\left(-q^2/\Lambda_{em}^2\right)} , \tag{13.7}$$

where in the last identity, we have introduced the RGI parameter Λ_{em} in analogy of QCD.

The anomalous dimensions are:

$$\gamma_F \equiv \frac{\nu}{Z_{2F}}\frac{dZ_{2F}}{d\nu} = \frac{\alpha_G}{2}\left(\frac{\alpha}{\pi}\right) + \mathcal{O}(\alpha^2) ,$$
$$\gamma_m \equiv \frac{\nu}{Z_m}\frac{dZ_m}{d\nu} = \left(\gamma_{\rm em} \equiv \frac{3}{2}\right)\left(\frac{\alpha}{\pi}\right) + \mathcal{O}(\alpha^2) ,$$
$$\gamma_\Gamma \equiv \frac{\nu}{Z_\Gamma}\frac{dZ_\Gamma}{d\nu} = -\frac{1}{2}[n_\gamma \beta(\alpha) + n_F \gamma_F] , \tag{13.8}$$

where n_γ and n_F are respectively the number of external photon and fermion fields.

13.4 Effective charge and link between the $\overline{MS}$ and on-shell scheme

One can relate the electric charge in the QED on-shell scheme to the one in the $\overline{MS}$ scheme, by using the fact that the QED effective charge is invariant under the choice

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of renormalizations:[1]

$$\alpha_{\text{eff}}(q^2) = \frac{\alpha_B}{1 + e_B^2 \Pi_{\text{em}}^B} = \frac{\alpha_R}{1 + e_R^2 \Pi_{\text{em}}^R} , \tag{13.9}$$

where the indices R and B denote renormalized and bare quantities; $\alpha = e^2/4\pi$ is the fine structure constant; Π_{em} is the electromagnetic vacuum polarization defined as:

$$\Pi_{\text{em}}^{\mu\nu}(q) = i \int d^4x \, e^{iqx} \langle 0|\mathcal{T} J^\mu(x) J^\nu(0)^\dagger |0\rangle = -(g^{\mu\nu} q^2 - q^\mu q^\nu)\Pi_{\text{em}}(q^2) , \tag{13.10}$$

where $J^\mu = \bar{l}\gamma^\mu l$ is the local current built from the lepton field. Using the Feynman rule given in Appendix E and the Dirac algebra in n dimensions in Appendix D, its expression can be easily obtained, if one follows closely the derivation of the pseudoscalar two-point function discussed previously. The *bare* two-point correlator reads to lowest order:

$$\Pi_{\text{em}}^B(q^2) = \frac{1}{12\pi^2} \left\{ \frac{2}{\epsilon} + (\ln 4\pi) - \gamma - 3 \int_0^1 dx \; 2x(1-x) \ln \frac{-q^2 x(1-x) + m^2 - i\epsilon'}{\nu^2} \right\} . \tag{13.11}$$

From this expression, one can deduce to leading order in m^2/q^2:

$$\Pi_{\text{em}}^B(-q^2 \gg m^2) = \frac{1}{12\pi^2} \left\{ \frac{2}{\epsilon} + (\ln 4\pi) - \gamma - \ln \frac{-q^2}{\nu^2} + \frac{5}{3} \right. \\ \left. - 6 \frac{m^2}{-q^2} + 6 \left(\frac{m^2}{-q^2} \right)^2 \ln \frac{-q^2}{m^2} \right\} , \tag{13.12}$$

and:

$$\Pi_{\text{em}}^B(q^2 = 0) = \frac{1}{12\pi^2} \left\{ \frac{2}{\epsilon} + (\ln 4\pi) - \gamma - \ln \frac{m^2}{\nu^2} \right\} . \tag{13.13}$$

The renormalized *vacuum polarization* can be easily obtained:

$$\begin{aligned} \Pi_{\text{em}}^{\text{o.s.}} &\simeq \frac{1}{12\pi^2} \left\{ -\ln \frac{-q^2}{m^2} + \frac{5}{3} + \cdots \right\} \\ \Pi_{\text{em}}^{\overline{MS}} &= \frac{1}{12\pi^2} \left\{ -\ln \frac{-q^2}{\nu^2} + \frac{5}{3} + \cdots \right\} \end{aligned} \tag{13.14}$$

Using the fact that $\Pi_{\text{em}}^{\text{on-shell}}(q^2 = 0) = 0$ in Eq. (13.9), one can relate the charge in the on-shell and $\overline{MS}$ scheme as.[2]

$$\alpha(\nu) = \alpha_{\text{o.s.}} \left\{ 1 + \left(\frac{\alpha_{\text{o.s.}}}{\pi} \right) \frac{1}{3} \ln \frac{\nu^2}{m^2} \right\} \tag{13.15}$$

[1] The last equality comes from the fact that the charge renormalization constant Z_α and the photon field renormalization constant Z_3 are related to each others from Eq. (13.4).

[2] For a more elegant notation, we shall not put index for the $\overline{MS}$ coupling.

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Using this relation into Eq. (13.7), one can deduce the running coupling in the $\overline{MS}$ scheme in terms of the one in the on-shell scheme evaluated at $\nu = m$:

$$\bar{\alpha}(q^2) = \frac{\alpha_{\rm o.s.}}{1 - (\alpha_{\rm o.s.}/\pi)\,\beta_1 \frac{1}{2} \ln(-q^2/m^2)} \,. \tag{13.16}$$

Identifying this result with the one in Eq. (13.7), one can deduce for one fermion:

$$\Lambda_{\rm em} = m \;\exp\left(\frac{\pi}{\beta_1 \alpha_{\rm o.s.}}\right) . \tag{13.17}$$

This result can be easily generalized to n fermions of mass m_i:

$$\Lambda_{\rm em} = \left(\prod_{i=1}^{n} m_i\right) \exp\left(\frac{\pi}{n\beta_1 \alpha_{\rm o.s.}}\right) . \tag{13.18}$$

For the three observed charged leptons $e,\ \mu,\ \tau$, this leads to:

$$\Lambda_{\rm em} = 5.2 \times 10^{93}\ {\rm GeV}\,, \tag{13.19}$$

which is an astronomical number. This is the scale at which one expects that the QED series expansion breaks down, and is commonly called as the *Landau pole*. Using its definition in Eq. (13.9), the effective QED charge can be expressed in terms of the running charge. It reads, in the $\overline{MS}$ scheme:

$$\alpha_{\rm eff}(q^2) \simeq \bar{\alpha}(q^2)\left\{1 - \left(\frac{\bar{\alpha}}{\pi}\right)\frac{1}{3}\left(\frac{5}{3}\right) + \mathcal{O}\left(\frac{m^2}{q^2}\right)\right\} . \tag{13.20}$$

Analogous relation from Eq. (13.9) can also be obtained in the on-shell scheme. The identification of the two relations for $\alpha_{\rm eff}$ leads to the relation between the running coupling in the $\overline{MS}$ and on-shell schemes:

$$\bar{\alpha}(q^2) = \bar{\alpha}_{\rm o.s.}(q^2)\left\{1 + \left(\frac{\bar{\alpha}_{\rm o.s.}}{\pi}\right)\frac{1}{3}\ln\frac{-q^2}{m^2} + \mathcal{O}\left(\frac{\bar{\alpha}_{\rm o.s.}}{\pi}\right)^2\right\} . \tag{13.21}$$

This relation is useful in the analysis of electroweak processes where the Green's functions are often evaluated using the $\overline{MS}$ scheme.