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THE DUAL LATTICE OF AN EXTREME SIX-DIMENSIONAL LATTICE

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Abstract

The best lattice quantizers seem to be duals of extreme lattices. The quantizing constant associated with the dual lattice of Barnes's senary form ϕ_6 is found, together with a new type of quantizing technique. The quantizing constant is better than expected in the sense that it is better than D_6^* even though D_6 provides a denser packing. This is the smallest dimension for which this occurs.

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1. Introduction

In the past it has been found that the duals of lattices corresponding to extreme forms are good quantizing lattices.

In 1957 Barnes [2] enumerated all the extreme six-dimensional forms, six in all. The quantizing constants for four of the six dual lattices are known, namely those corresponding to E_6, E_6^*, A_6^* and D_6^*. In this paper the quantizing constant for one of the remaining lattices is calculated and is found to be better than its packing properties suggest. An innovative quantizing technique is also found for the lattice.

2. Preliminaries

A lattice Λ is usually considered to be the span over the integers of a set of linearly independent vectors (independent over the reals). An independent set of vectors which generate the lattice Λ is called a *basis*. There is a natural correspondence between lattices and positive definite quadratic forms.

From the above $\mathbf{y}$ is in Λ_n if and only if $\mathbf{y} = \mathbf{z}B$, where $\mathbf{z}$ is in $\mathbb{Z}^n$ and the rows of B form a basis for Λ_n. Hence for

$$
\begin{aligned}
&\mathbf{x}, \mathbf{y} \quad \text{in } \Lambda_n, \\
(\mathbf{x}, \mathbf{y}) &= \mathbf{x}\mathbf{y}^T \\
&= \mathbf{z}_\mathbf{x} B B^T \mathbf{z}_\mathbf{y} \quad \text{where } \mathbf{z}_\mathbf{x}, \mathbf{z}_\mathbf{y} \in \mathbb{Z}^n \\
&= \mathbf{z}_\mathbf{x} H \mathbf{z}_\mathbf{y}^T .
\end{aligned}
$$

We call H the *Gram matrix* associated with Λ and H obviously corresponds to a positive definite form [8].

A positive definite form is called *extreme* if M^n/D is a local maximum under perturbations of the coefficients of the Gram matrix, where n is the dimension of the lattice, D is the determinant of the Gram matrix H and M is the minimal length of non-zero lattice vectors. In an intuitive sense, we have made the short vectors of the lattice as long as possible without increasing the volume of the spanning parallelepiped. If an n-dimensional form has M^n/D being an absolute maximum then it is said to be absolutely extreme.

A lattice Λ_n is called *integral* if the inner product $(\mathbf{x}, \mathbf{y})$ is in $\mathbb{Z}$ for all $\mathbf{x}, \mathbf{y}$ in Λ_n. We call Λ_n even if norms $(\mathbf{x}, \mathbf{x})$ are always even. The *Dual Lattice* of an integral lattice Λ, denoted by Λ^*, lies in the real span of Λ (W, say) and is the set

$$\{\mathbf{v} \text{ in } W : (\mathbf{v}, \mathbf{x}) \in \mathbb{Z} \text{ for all } \mathbf{x} \text{ in } \Lambda\}.$$

The *quantizing constant* G of a region V about the origin is

$$G(V) = \frac{1}{n} \frac{\int_V (\mathbf{x}, \mathbf{x})\, d\mathbf{x}}{(\int_V d\mathbf{x})^{1+2/n}}$$

in n-dimensional space.

Loosely speaking, this is the average squared error per bit of information that is induced by approximating $\mathbf{x}$ in V by the origin. The constant is invariant to scale changes. In a lattice the quantizing constant is the average error induced by approximating a point in W by the closest lattice point.

The *Voronoi region* of Λ_n about $\mathbf{1}$ in Λ_n is the convex polytope

$$V(\mathbf{1}) := \{\mathbf{x} \text{ in } W : (\mathbf{x} - \mathbf{1}, \mathbf{x} - \mathbf{1}) \le (\mathbf{x} - \mathbf{1}', \mathbf{x} - \mathbf{1}') \text{ for all } \mathbf{1}' \text{ in } \Lambda_n\}.$$

The best-known lattice quantizers in dimensions up to six are given in Table 1 [7].

TABLE 1

dim	Λ	$G(\Lambda)$	Λ^*
1	A_1	.0833..	A_1
2	A_2	.0801..	A_2
3	A_3^*	.0785..	A_3
4	D_4	.0766..	D_4
5	D_5^*	.0756..	D_5
6	E_6^*	.0742..	E_6

In the first three dimensions the lattice quantizers are known to be the best possible for that dimension [3]. The dual lattices are all extreme; indeed, they are absolutely extreme [10]. This, and the best-known results in higher dimensions (Table 1), give rise to the following conjecture.

CONJECTURE. The best quantizing lattice in a given dimension is the dual of the best packing lattice. More weakly, the best quantizing lattice is the dual of an extreme lattice.

All the six-dimensional extreme forms are known [2]. The main purpose of this account is to find the quantizing constant of the dual of the lattice associated with the extreme form labelled by Barnes [2] as ϕ_6.

3. The lattice Λ_6

Barnes [1] characterized the lattice Λ_6 associated with ϕ_6 (note that this is not the laminated lattice which Conway and Sloane call Λ_6 [6], [8]) as follows:

$$\Lambda_6 := \left\{\mathbf{y} \in \mathbb{Z}^7 : \sum y_i = 0,\ \sum i y_i \equiv 0 \pmod 7\right\}.$$

Conway and Sloane [8] characterized this lattice as a repeated difference. Now the vectors

$$\begin{aligned} \mathbf{m}_0 &:= (-1\ 1\ 0\ 0\ 0\ 0\ 0) \\ \mathbf{m}_1 &:= (0\ -1\ 1\ 0\ 0\ 0\ 0) \\ &\ \ \vdots \qquad\qquad \vdots \\ \mathbf{m}_6 &:= (1\ 0\ 0\ 0\ 0\ 0\ -1) \end{aligned}$$

generate A_6 and satisfy $\sum \mathbf{m}_i = \mathbf{0}$. Also Λ_6 is generated by $\mathbf{m}_i - \mathbf{m}_{i+1}$ for $i = 0, \dots, 6$, where $\mathbf{m}_7$ is understood to be $\mathbf{m}_0$.

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Craig [11] found an expression of Λ_6 in terms of algebraic integers and ideals. Let θ be a primitive root of unity of order seven; then the ideal $(1-\theta)$ corresponds to A_6 and the ideal $(1-\theta)^2$ corresponds to Λ_6.

The lattice Λ_6 corresponds to the three-dimensional complex laminated lattice Λ_3^{λ} of [6] and $Q_6(1)$ of [9].

Another characterization is the basis-free one which follows. Let Λ be any integral lattice embedded in $\mathbb{R}^{n+1}$ such that $\sum x_i = 0$. Define $c(\mathbf{x})$ to be $\mathbf{x}-\mathbf{x}'$ where $\mathbf{x}'$ is obtained from $\mathbf{x}$ by a cyclic shift one place to the right.

We define the lattice function c so that $c(\Lambda) := \{c(\mathbf{x}) : \mathbf{x} \in \Lambda\}$. The dual lattice to Λ_6, Λ^*, is generated by the vectors

$$\begin{array}{l} \tfrac{1}{7}(-3\ \ -2\ \ -1\ \ 0\ \ 1\ \ 2\ \ 3) \\ \tfrac{1}{7}(3\ \ -3\ \ -2\ \ -1\ \ 0\ \ 1\ \ 2) \\ \vdots \\ \tfrac{1}{7}(-2\ \ -1\ \ 0\ \ 1\ \ 2\ \ 3\ \ -3). \end{array}$$

Taking the first six of these vectors (which form a basis) we get the Gram matrix

$$\frac{1}{7}\begin{vmatrix} 4 & 1 & -1 & -2 & -2 & -1 \\ 1 & 4 & 1 & -1 & -2 & -2 \\ -1 & 1 & 4 & 1 & -1 & -2 \\ -2 & -1 & 1 & 4 & 1 & -1 \\ -2 & -2 & -1 & 1 & 4 & 1 \\ -1 & -2 & -2 & -1 & 1 & 4 \end{vmatrix}.$$

The basis of Λ_6,

$$\begin{aligned} \mathbf{b}_1 &= (1\ \ -1\ \ -1\ \ 1\ \ 0\ \ 0\ \ 0), \\ \mathbf{b}_2 &= (0\ \ 0\ \ 0\ \ 1\ \ -1\ \ -1\ \ 1), \\ \mathbf{b}_3 &= (-1\ \ -1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 1), \\ \mathbf{b}_4 &= (0\ \ 0\ \ 1\ \ -1\ \ -1\ \ 1\ \ 0), \\ \mathbf{b}_5 &= (-1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 1\ \ -1), \\ \mathbf{b}_6 &= (0\ \ 1\ \ -1\ \ -1\ \ 1\ \ 0\ \ 0), \end{aligned}$$

has Gram matrix

$$\begin{vmatrix} 4 & 1 & -1 & -2 & -2 & -1 \\ 1 & 4 & 1 & -1 & -2 & -2 \\ -1 & 1 & 4 & 1 & -1 & -2 \\ -2 & -1 & 1 & 4 & 1 & -1 \\ -2 & -2 & -1 & 1 & 4 & 1 \\ -1 & -2 & -2 & -1 & 1 & 4 \end{vmatrix}.$$

Thus Λ_6 is isomorphic to its dual Λ_6^*. An integral lattice Λ is called unimodular if it is its own dual. Thus it is sufficient to investigate Λ_6 to study Λ_6^*.

It can quickly be checked, using spanning vectors $\mathbf{v}_i$ such that $\sum \mathbf{v}_i = 0$, that $c(\Lambda_6) = A_6^*$, $c(A_6^*) = A_6$ and $c(A_6) = \Lambda_6$. It is known that there are 2.6.7.8 automorphisms of Λ_6 [1].

Two automorphisms immediately present themselves:

$$m_1 : \mathbf{x} \to -\mathbf{x} \quad \text{order } 2,$$
$$m_2 : \mathbf{x} \to \mathbf{x}' \quad \text{order } 7.$$

The basis $\mathscr{B}$,

$$\begin{aligned}
\mathbf{b}_1 &= (\ 1\ -1\ -1\ \ 1\ 0\ \ 0\ \ 0),\\
\mathbf{b}_2 &= (\ 1\ -1\ \ 0\ \ 0\ 0\ -1\ \ 1),\\
\mathbf{b}_3 &= (\ 1\ \ 1\ -1\ \ 0\ 0\ \ 0\ -1),\\
\mathbf{b}_4 &= (-1\ \ 0\ \ 1\ \ 0\ 1\ \ 0\ -1),\\
\mathbf{b}_5 &= (\ 0\ \ 0\ \ 1\ -1\ 0\ -1\ \ 1),\\
\mathbf{b}_6 &= (-1\ \ 0\ \ 0\ -1\ 1\ \ 0\ \ 1),
\end{aligned}$$

has the Gram matrix

$$\begin{vmatrix}
4 & 2 & 1 & -2 & -2 & -2\\
2 & 4 & 0 & -2 & -1 & -2\\
1 & 0 & 4 & -1 & -2 & -2\\
-2 & -2 & -1 & 4 & 0 & 1\\
-2 & -1 & -2 & 0 & 4 & 2\\
-2 & -2 & -2 & 1 & 2 & 4
\end{vmatrix}$$

which gives rise to the automorphism

$$m_3 : \sum c_i \mathbf{b}_i \to \sum c_i \mathbf{c}_{7-i}$$

of order 2.

Other generating automorphisms m_4, m_5, m_6 and m_7 (of orders 3, 2, 2 and 2 respectively) of the automorphism group were found by sending the basis $\mathscr{B}$ to a basis with the same Gram matrix. These automorphisms have no clear interpretation in the Λ_6 depiction of the lattice. However, in the corresponding complex integral laminated lattice Λ_3^λ [6] they are clearly visible as permutations and sign changes in the three complex coordinates. Conversely m_2 is not visible in Λ_3^λ and m_3 conjugates Λ_3^λ (which means it is not an automorphism of Λ_3^λ).

The Voronoi region about the origin was found using the following algorithm.

1. Find points $\mathbf{v}_j$ such that
   (a) $2(\mathbf{f}_i, \mathbf{v}_j) = (\mathbf{f}_i, \mathbf{f}_i)$ for six independent $\mathbf{f}_i \in \Lambda_6$,
   (b) $2(\mathbf{f}, \mathbf{v}_j) \le (\mathbf{f}, \mathbf{f})$ for all $f \in \Lambda_6$.
2. Generate the set of orbits of these points v_j under the automorphism group.

3. If this collection of points forms the vertices of a polytope with volume $\sqrt{|H|}$ then they are the complete set of vertices of the Voronoi region.

Using a program written by Worley which was used to investigate the Voronoi regions of E_7^* and E_6^* [13], [12], the following inequivalent (with respect to automorphisms keeping the origin fixed) Voronoi vertices were found:

$$v_1 = \tfrac{1}{7}(-6\ -4\ -2\ 0\ 2\ 4\ 6),$$

which is equidistant from
8 norm squared 4 points and
2 norm squared 8 points;

$$v_2 = \tfrac{1}{7}(4\ -4\ -5\ 1\ 7\ -1\ -2),$$

which is equidistant from
6 norm squared 4 points,
3 norm squared 6 points and
1 norm squared 8 point;

$$v_3 = \tfrac{1}{7}(-5\ -2\ 8\ -3\ 0\ 3\ -1),$$

which is equidistant from
5 norm squared 4 points,
4 norm squared 6 points and
1 norm squared 8 point.

The vectors v_3, v_2 and v_3 are equivalent with respect to translation by lattice vectors. For example

$$\begin{aligned} &\tfrac{1}{7}(-6\ -4\ -2\ 0\ 2\ 4\ 6) \\ =\ &(-1\ 0\ 0\ -1\ +1\ 0\ +1) \\ +\ &\tfrac{1}{7}(\ 1\ -4\ -2\ 7\ -5\ 4\ -1). \end{aligned}$$

There is also the obvious vertex

$$v_4 = (-1\ -1\ 0\ 0\ 0\ 0\ 0),$$

which is equidistant from
4 norm squared 4 points and
2 norm squared 6 points,

which the program failed to find because of the much higher probability of finding vertices with ten bounding planes than those with six bounding planes.

The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_4$ were then mapped by the automorphism group. The orbits were found to be of length 42, 336, 84 and 336 respectively.

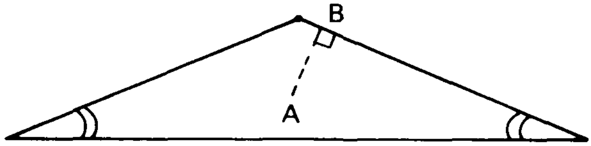

FIGURE 1

It is worth noting that these vectors (the conjectured set of Voronoi vertices about $\mathbf{0}$) are exactly the set S of points satisfying

$$\mathbf{x} \in \Lambda_6^* \text{ such that } (\mathbf{x}, \mathbf{x}) = 2 \text{ or } (\mathbf{x}, \mathbf{x}) = \tfrac{16}{7}.$$

A program was written which calculated the subpolytopes of the convex polytope formed by the above-mentioned set of points. Basically the algorithm was used

five-d subpolytopes are the intersection between S and one plane $2(\mathbf{x}, \mathbf{f}_i) = (\mathbf{f}_i, \mathbf{f}_i)$, four-$d$ subpolytopes are the intersection between S and two planes $2(\mathbf{x}, \mathbf{f}_i) = (\mathbf{f}_i, \mathbf{f}_i)$,

and so on.

The second moments U and volumes V of the Voronoi region were calculated using the recursive formulae

$$V(P) = \sum_i \frac{N_i h_i}{n} V_{n-1}(i) \quad \text{and} \quad U(P) = \sum_i \frac{N_i h_i}{n+2} \{h_i^2 V_{n-1}(i) + U_{n-1}(i)\}.$$

Refer to [4].

The volume of the convex polytope S is the same as the volume of a spanning parallelepiped. Hence the convex polytope S is the Voronoi region of the origin.

Thus the quantizing constant for Λ_6 is

$$G(\Lambda_6) = \frac{3503}{360.49\sqrt{7}} = .07505723\ldots.$$

It is worth noting that the convex polytope S has some unusual asymmetrical features, namely that for some of the bounding polytopes the centroid of a subpolytope $((\frac{1}{m})\sum_1^m \mathbf{v}_i)$ is not the closest point to the centroid of the polytope it bounds.

For example, a non-equilateral, isosceles triangle is asymmetrical, as A is the centroid of the triangle but B is into the centroid of the bounding edge (see Figure 1).

It should be noted that the ranking of lattices according to packing density is not the same as the ranking according to the quantization constant of the dual lattices, as $(2/M)^6 D(D_6) < (2/M)^6 D(\Lambda_6)$ but $G(D_6^*) > G(\Lambda_6)$. This is the lowest dimension in which this happens.

TABLE 2. Known quantizing constants of extreme six-dimensional lattices.

lattice	$(2/M)^6 D$	lattice*	$G(\Lambda^*)$
A_6	7	A^*	.0765
Λ_6	$\frac{7}{2^6} = 5.35$	Λ_6	.07506
D_6	4	D^*	.07512
E^*	$\frac{3^5}{2^6} = 3.80$	E_6	.0743
E_6	3	E^*	.0742

In the verification of this, the quantizing constant of A_5^2, the dual of the extreme lattice A_5^3 [7], was calculated as

$$G(A_5^2) = 2^{-39/5}.3^{-11/5}.5^{-1}.947 = 0.07580\ldots.$$

4. A quantizing algorithm for Λ_6

The quantizing algorithm for Λ_6, which is outlined below, is a new type of algorithm.

The algorithm uses $A_6 \supseteq \Lambda_6$ rather than the sublattice quantizing algorithm which uses $\Lambda_6 \supseteq \sqrt{7}A_6$. This 'new' algorithm is twice as fast as the sublattice algorithm.

As $c(\Lambda_6^*) = A_6^*$, $c(A_6^*) = A_6$ and $c(A_6) = \Lambda_6 \cong \Lambda_6^*$ there is a sublattice of Λ_6 isomorphic to A_6. Furthermore, as the lattice function c gives a sublattice of index seven, the sublattice isomorphic to A_6 has index forty-nine. This can be utilized to find a quantizing algorithm for Λ_6 [5]. There is, however, an alternate quantizing algorithm which, rather than using the fact that A_6 is a sublattice of Λ_6, uses the fact that Λ_6 is itself a sublattice of A_6.

First we need some notation.

By $Q_\Lambda(\mathbf{x}) = \boldsymbol{\lambda}$ we mean that the closest lattice point of Λ to $\mathbf{x}$ is $\boldsymbol{\lambda}$; A is the set of elements of A_6^* of squared length less than or equal to $\frac{12}{7}$. An element of A is either the origin or a Voronoi vertex of the origin with respect to the lattice A_6.

It can be shown by exhaustion that for all Voronoi vertices $\boldsymbol{\lambda}$ of Λ_6 about the origin, there exists an $\mathbf{a}$ in A such that $Q(\boldsymbol{\lambda} + \mathbf{a} : A_6) = 0$.

By convexity, for any point $\mathbf{x}$ such that $Q(\mathbf{x} : \Lambda_6) = 0$ there exists a vector $\mathbf{a}$ in A such that $Q(\mathbf{x} + \mathbf{a} : A_6) = 0$.

Hence the only candidates for $Q(\mathbf{x} : \Lambda_6)$ are $\{(Q(\mathbf{x} + \mathbf{a} : A_6)$; where $a \in A\}$ and the closest of these to $\mathbf{x}$ is $Q(\mathbf{x} : \Lambda_6)$.

This search (amongst the $Q(\mathbf{x}+\mathbf{v})A_6)$'s can be made systematic by subdividing these $\mathbf{v}$'s (in A) into cosets $(\operatorname{mod} A_6)$ and then further subdividing these cosets into subcosets $(\operatorname{mod} \Lambda_6)$ and arranging these subcosets so that $\sum ix_i$ increases by 1 with each new coset.

The following algorithm makes use of the fact that for $\mathbf{v}, \mathbf{w}$ in A and $\mathbf{v} \cong \mathbf{w} \pmod{A_6}$

> if $Q(\mathbf{x}+\mathbf{v}, A_6) \in \Lambda_6$, then $Q(\mathbf{x}+\mathbf{w}, A_6) \in \Lambda_6$ if and only if $\mathbf{v} \cong \mathbf{w} \pmod{\Lambda_6}$ in which case $Q(\mathbf{x}+\mathbf{x}, A_6) = Q(\mathbf{x}+\mathbf{v}, A_6) + (\mathbf{w}-\mathbf{v})$.

Thus the algorithm is as follows:

1. Form $Q(\mathbf{x}, A_6) = \mathbf{a}$. If $\mathbf{a} \in \Lambda_6$, then $Q(\mathbf{x}, \Lambda_6) = \mathbf{a}$, otherwise perform steps 2 to 7.
2. Look at the first coset $(\operatorname{mod} A_6)$ of the $\mathbf{v}$'s in A.
3. Look at the first subcoset $(\operatorname{mod} \Lambda_6)$.
4. Quantize one $\mathbf{x}+\mathbf{v}_i$ to form $\mathbf{a} = Q(\mathbf{x}+\mathbf{v}_i, A_6)$.
5. Let $\sum ia_i \equiv m \pmod 7$, where $\mathbf{a} = (a_1, a_2, \dots, a_7)$. If $m \equiv 0 \pmod 7$, then $\mathbf{a} \in \Lambda_6$ and $\mathbf{a}$ is a candidate for $Q(\mathbf{x}, \Lambda_6)$. Determine other candidates by examining the rest of the subcoset (and performing 4).

   If $m \not\equiv 0 \pmod 7$, then $\mathbf{a} \notin \Lambda_6$ and we are in the wrong subcoset $(\operatorname{mod} \Lambda_6)$. The subcosets are arranged so that $\sum ia_i$ increases by 1 with each successive subcoset. Thus candidates $\mathbf{a}$ are found by examining the $(8-m)$th subcoset.
6. Repeat steps 3, 4, 5 for all cosets $(\operatorname{mod} \Lambda_6)$ to determine the set of all possible candidates for $Q(\mathbf{x}, \Lambda_6)$.
7. The closest of these candidates to $\mathbf{x}$ is $Q(\mathbf{x}, \Lambda_6)$.

In the following example, the first element of a subcoset is expressed in the usual way, for example, $\frac{1}{7}(-3\ -3\ -3\ -3\ 4\ 4\ 4)$ with the remaining elements being written as elements of Λ_6, for example $(1\ 0\ 1\ 0\ -1\ -1\ 0)$. This vector represents the sum of itself with the subcoset leader. For example, $(1\ 0\ 1\ 0\ -1\ -1\ 0)$ represents

$$\tfrac{1}{7}(4\ -3\ 4\ -3\ -3\ -3\ 4) = \tfrac{1}{7}(-3\ -3\ -3\ -3\ 4\ 4\ 4) + (1\ 0\ 1\ 0\ -1\ -1\ 0).$$

For brevity's sake the algorithm is illustrated on the fourth coset $(\operatorname{mod} A_n)$ only, and the first and fourth subcosets only.

Fourth coset $(\operatorname{mod} A_n)$

First coset $(\operatorname{mod} \Lambda_6)$

$$\tfrac{1}{7}(-3, -3, -3, -3, 4, 4, 4); \quad (1, 0, 1, 0, -1, -1, 0);$$
$$(1, 0, 0, 1, -1, 0, -1);$$
$$(0, 1, 1, 0, -1, 0, -1); \quad (0, 1, 0, 1, 0, -1, -1).$$

EXAMPLE. Quantize $\mathbf{x} = \frac{1}{21}(-34, 13, 60, -29, -16, -41, 47)$. The first subcoset leader is

$$\mathbf{v} = \tfrac{1}{7}(-3, -3, -3, -3, 4, 4, 4).$$

Therefore $\mathbf{x}+\mathbf{v} = \frac{1}{21}(-43, 4, 51, -38, -4, -29, 59)$. Thus $Q(\mathbf{x}+\mathbf{v}, A_6) = (-2, 0, 2, -2, 0, -1, 3) = \mathbf{a}$ with $m = \sum ia_i \equiv 4 \pmod 7$. Hence we look at the $(8-4)$th subcoset mod (Λ_6).

The fourth subcoset leader is $\mathbf{v} = \frac{1}{7}(4, -3, -3, -3, -3, 4, 4)$, which gives $\mathbf{x}+\mathbf{v} = \frac{1}{21}(-22, 4, 51, -38, -25, -29, 59)$ with $Q(\mathbf{x}+\mathbf{v}, A_6) = (-1, 0, 2, -2, -1, -1, 3) = \boldsymbol{\lambda} \in \Lambda_6$ and the other candidates from this subcoset (formed by $\boldsymbol{\lambda} + \mathbf{w}$ where $\mathbf{w} \in \Lambda_6$ in fourth subcoset) are

$$(-2, 1, 2, -2, 0, -2, 3); \quad (-2, 0, 3, -1, -1, -2, 3);$$
$$(-2, 0, 3, -2, 0, -1, 2); \quad (-1, 1, 2, -1, -1, -2, -2).$$

These candidates are at squared distances of

$$\frac{919}{11^2}, \frac{1134}{11^2}, \frac{563}{11^2}, \frac{1092}{11^2} \quad \text{and} \quad \frac{672}{11^2},$$

from $\mathbf{x}$ respectively. It turns out that if we look at all the cosets $(\mathrm{mod}\, A_6)$, we find that $Q(\mathbf{x}, \Lambda_6) = (-2, 0, 3, -1, -1, -2, 3)$.

The quantizing algorithm given above requires approximately one half the number of calculations as the sublattice quantizing using A_6. It is conjectured that the quantizing algorithm would work in other dimensions.

CONJECTURE. If $Q(\mathbf{x}, c(A_n)) = \boldsymbol{\lambda}$, then $Q(\mathbf{x}+\mathbf{v}, A_n) = \boldsymbol{\lambda}$ where $\mathbf{v}$ is some Voronoi vertex (or $\mathbf{0}$) in A_n.

If the conjecture is true, then the savings of the Voronoi vertex quantizing method over the sublattice quantizing method would be even greater for larger n when $n+1$ is prime.

In general $c^n(A_n) = (n+1)(A_n) \cong A_n$, and hence we would naively have expected A_6 to be a sublattice of $c(A_6)$ of order 7^5, not 7^2 as was dictated by $c(A_6) \cong \{c(A_6)\}^*$. This circumstance does not repeat itself in higher dimensions for $n+1$ prime. This is because $c(A_n)$ is extreme for $n > 6$, $n+1$ prime [10], but $\{c(A_n)\}^*$ is not, that is, $\{c(A_n)\}^* \not\cong c(A_n)$.

So it would seem likely that the Voronoi vertex quantizing algorithm would be a lot better than a sublattice quantizing algorithm in these higher dimensions.

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