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A NEW BANACH SPACE WITHOUT THE KADEC-KLEE PROPERTY

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An example of a Banach space is given which does not contain isomorphically l_1 and does not admit the Kadec-Klee property.

1. INTRODUCTION

We say that a Banach space X admits a Kadec-Klee property if it has an equivalent norm $\|.\|$ such that the weak and norm sequence convergences coincide on the unit sphere $\{x \in X\colon \|x\| = 1\}$.

It is known that the space l_∞ and all spaces containing it isomorphically do not admit the Kadec-Klee property [4]. Alexandrov [1] has recently constructed a class of Banach spaces engendered by Boolean rings with a so-called subsequential interpolation property that have no Kadec-Klee property. Some of them do not have a subspace isomorphic to l_∞ but they contain l_1 [2]. In this paper we give an example of a Banach space which does not contain l_1 and does not admit the Kadec-Klee property.

2. DEFINITIONS, NOTATIONS AND REMARKS

Let L be the following union

$$L = \bigcup [0,\, \omega_1)^{[0,\alpha)},$$

all functions t whose domain $\operatorname{dom}(t)$ is some interval $[0,\, \alpha)$ for countable ordinal α and whose codomain is $[0,\, \omega_1)$ (ω_1 the first uncountable ordinal). When $\alpha = 0$, $[0,\, \omega_1)^{[0,0)}$ has one trivial element; we denote this element by 0. We may regard L as an uncountable branching tree all of whose branches have length ω_1. We give an order on L in the following way: if $s, t \in L$ then we say that $s \leqslant t$ if and only if $\operatorname{dom}(s) \subseteq \operatorname{dom}(t)$ and the restriction $t\,|_{\operatorname{dom}(s)} = s$.

If $t \in L$, we denote by t^+ the set of all immediate successors of t, that is $u \in t^+$ if and only if, if $s < u$ then $s \leqslant t$.

We equip L with the order-topology in a standard way: we take the intervals $(s,\, t] = \{u \in L\colon s < u \leqslant t\}$, $(\forall s \leqslant t)$, to be the basic neighbourhoods of the point

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$t > 0$ and declare the element 0 to be an isolated point. The topological space L with the order-topology is locally compact and scattered [**3**].

Let $C_0(L)$ be the space of all continuous functions vanishing at infinity.

We let $[t, \infty) = \{u \in L : u \geqslant t\}$ and let $\chi_{(s,t]}$ be the characteristic function of $(s, t]$.

3. Main result

Theorem. *The space $C_0(L)$ does not admit the Kadec-Klee property.*

Lemma. ([**3**]) *Let $\|.\|$ be an equivalent norm on $C_0(L)$, $x \in C_0(L)$ and $s \in L$. Then there exists $u > s$ such that for every $v > u$ we have*

$$\left\|x + \chi_{(x,v]}\right\| = \left\|x + \chi_{(s,u]}\right\|.$$

Proof: Let $u_0 = s$. If u_{n-1} has been defined, we set

$$\nu_n = \sup\{\left\|x + \chi_{(x,v]}\right\| : v > u_{n-1}\}$$

and choose $v_n > u_{n-1}$ with

$$\left\|x + \chi_{(s,v_n]}\right\| \geqslant \nu_n - 2^{-n}.$$

Let $g \in C_0(L)^*$ such that $\|g\| = 1$ and $g\left(x + \chi_{(s,v_n]}\right) = \left\|x + \chi_{(s,v_n]}\right\|$. The space $C_0(L)^* = l_1(L)$ and we may regard g as a summable scalar family $g = (g_l)_{l \in L}$. Since the uncountable family $\{[v, \infty) : v \in v_n^+\}$ is disjoint, we can choose $v_0 \in v_n^+$ such that $[v_0, \infty) \cap \{s \in L : g_l \neq 0\} = \emptyset$.

We set $u_n = v_0$ and note that for every $v > u_n$ we have

$$\left\|x + \chi_{(s,v]}\right\| \geqslant g\left(x + \chi_{(s,v]}\right) = g\left(x + \chi_{(s,v_n]}\right) = \left\|x + \chi_{(s,v_n]}\right\| \geqslant \nu_n - 2^{-n}.$$

We continue this process inductively and we construct a sequence $u_0 < u_1 < \ldots < u_n < \ldots$ in L and a sequence of positive scalars $\nu_0 \geqslant \nu_1 \geqslant \ldots \geqslant \nu_n \geqslant \ldots$ such that for every $v > u_n$ we have

$$\nu_n - 2^{-n} \leqslant \left\|x + \chi_{(s,v]}\right\| \leqslant \nu_n. \tag{1}$$

Let $\nu = \lim \nu_n$ and let $u > u_n$ for all n. Then from (1), we get

$$\left\|x + \chi_{(s,v]}\right\| = \nu$$

for all $v \geqslant u$ and the lemma is proved. ∎

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PROOF OF THEOREM: Let $\|.\|$ be any norm equivalent to the sup-norm $\|.\|_\infty$ in the space $C_0(L)$. We apply the lemma with the norm $\|.\|$, $x = 0$ and $s = 0$. Then there exists $u \in L$ such that

$$\left\|\chi_{(0,v]}\right\| = \left\|\chi_{(0,u]}\right\| \tag{2}$$

for all $v > u$.

We take a sequence $\{v_n\}_{n=1}^\infty \subseteq u^+$, $v_n \neq v_m\ (n \neq m)$.

From (2) we have

$$\left\|\chi_{(0,v_n]}\right\| = \left\|\chi_{(0,u]}\right\|$$

and since

$$\lim_n \chi_{(0,v_n]}(t) = \chi_{(0,u]}(t)$$

for all $t \in L$, then we find that the sequence $\chi_{(0,v_n]}$ tends to $\chi_{(0,u]}$ weakly.

On the other hand, obviously $\chi_{(0,v_n]}$ does not tend to $\chi_{(0,u]}$ in norm topology because

$$\left\|\chi_{(0,v_n]} - \chi_{(0,u]}\right\| = 1$$

for all n.

Consequently, the space $C_0(L)$ does not admit the Kadec-Klee property and the theorem is proved. □

REMARK. The main result is true for all Banach spaces $C_0(K)$, when K is a τ-branching tree ($\tau \geqslant$ card ω_1) all of whose branches have length greater than or equal to ω_1 with the order-topology.

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