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ALMOST ABELIAN NUMBERS

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Abstract

We introduce the concept of almost $\mathcal{P}$-numbers where $\mathcal{P}$ is a class of groups. We survey the existing results in the literature for almost cyclic numbers, and give characterisations for almost abelian and almost nilpotent numbers proving these two concepts are equivalent.

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1. Introduction

Throughout this article, let $\mathcal{P}$ be a class of groups. We will denote the dihedral group of order n by D_n and the cyclic group of order n by C_n. For standard notation and definitions, see [3].

DEFINITION 1.1. A positive integer n is called a *$\mathcal{P}$-number* if all groups of order n are in the class $\mathcal{P}$.

Some of the most obvious particular cases for this definition (cyclic, abelian and nilpotent) use the concept of nilpotent factorisation.

DEFINITION 1.2 [4]. A positive integer $n = p_1^{n_1} \cdots p_k^{n_k}$, where the p_i are distinct primes, is said to have *nilpotent factorisation* if $p_i^l \not\equiv 1 \bmod p_j$ for all positive integers i, j and l with $1 \leq i, j \leq k$ and $1 \leq l \leq n_i$.

Pakianathan and Shankar established the following characterisations.

PROPOSITION 1.3 [4]. *A positive integer n is:*

- *a nilpotent number if and only if it has nilpotent factorisation;*
- *an abelian number if and only if it is a cube-free number with nilpotent factorisation;*
- *a cyclic number if and only if it is a square-free number with nilpotent factorisation.*

The characterisation of cyclic numbers can also be given as follows.

PROPOSITION 1.4 [4]. *A positive integer n is cyclic if and only if the number and its Euler totient $\varphi(n)$ are coprime, that is,* $\gcd(n, \varphi(n)) = 1$.

EXAMPLE 1.5. All prime numbers are cyclic numbers.

If we loosen the hypothesis in Definition 1.1, we reach the following definition.

DEFINITION 1.6. A positive integer n is called an *almost $\mathcal{P}$-number* if all but one group of order n (up to isomorphism) are in the class $\mathcal{P}$.

This definition is suggested by the following result given in [1].

THEOREM 1.7. *Let G be a group of order $n = p_1^{n_1} \cdots p_k^{n_k}$, where $p_1 < p_2 < \cdots < p_k$ are distinct primes. Then, there are exactly two groups (up to isomorphism) of order n if and only if $k \geq 2$ and one of the following scenarios occurs: either*

$$\begin{aligned} &n_l = 1 \quad \textit{for all } l \in \{1, \ldots, k\} \textit{ and} \\ &\textit{there exists a unique pair } (i, j) \in \{1, \ldots, k\}^2 \textit{ such that } p_i \mid p_j - 1, \end{aligned} \tag{1.1}$$

or

$$\begin{aligned} &\textit{there exists a unique } j \in \{1, \ldots, k\} \textit{ such that } n_j = 2 \\ &\textit{and } n_l = 1 \quad \textit{for all } l \in \{1, \ldots, k\} \setminus \{j\}, \textit{ and} \\ &\textit{there exists a unique } i \in \{1, \ldots, k\} \setminus \{j\} \textit{ such that } p_i \mid p_j - 1,\ p_i \nmid p_j + 1 \\ &\textit{and } p_\alpha \nmid p_\beta - 1 \quad \textit{for all } (\alpha, \beta) \in \{1, \ldots, k\}^2 \setminus \{(i, j)\}. \end{aligned} \tag{1.2}$$

Theorem 1.7 is equivalent to the following statement.

COROLLARY 1.8. *A positive integer $n = p_1^{n_1} \cdots p_n^{n_k}$ is almost cyclic if $k \geq 2$ and either (1.1) or (1.2) hold.*

In what follows, we establish results similar to Corollary 1.8 for abelian groups and nilpotent groups.

2. Main results

THEOREM 2.1. *A positive integer $n = p_1^{n_1} \cdots p_n^{n_k}$ is almost abelian if and only if either (1.1) holds or*

$$\begin{aligned} &\textit{there exists a unique } j \in \{1, \ldots, k\} \textit{ such that } n_j = 2 \\ &\textit{and } n_l = 1 \quad \textit{for all } l \in \{1, \ldots, k\} \setminus \{j\}, \textit{ and} \\ &\textit{there exists a unique } i \in \{1, \ldots, k\} \setminus \{j\} \textit{ such that } p_i \mid p_j + 1 \\ &\textit{and } p_\alpha \nmid p_\beta - 1 \quad \textit{for all } (\alpha, \beta) \in \{1, \ldots, k\}^2 \setminus \{(i, j)\}. \end{aligned} \tag{2.1}$$

PROOF. $\Rightarrow$ Let us suppose that n is almost abelian. It follows that

$$\text{for any divisor } d \text{ of } n, \text{ there is at most one nonabelian group of order } d. \tag{2.2}$$

First, suppose that there exists $r \in \{1, \ldots, k\}$ such that $n_r \geq 3$. This contradicts (2.2), since there are at least two nonabelian groups of order $p_r^{n_r}$ when $n_r \geq 3$.

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Since n is not abelian, the only remaining possibility is that

$$\begin{aligned}&\text{there exist } i,j \in \{1,\dots,k\} \text{ such that } p_i \mid p_j^{n_j}-1\\ &\text{and } n_r \leq 2 \quad \text{for all } r \in \{1,\dots,k\}.\end{aligned} \tag{2.3}$$

Let us analyse which of the exponents can be 2. From (2.3), it follows that there exists a nontrivial semi-direct product $(C_{p_j})^{n_j} \rtimes C_{p_i}$. If $n_r = 2$, for $r \in \{1,2,\dots,k\} \setminus \{i,j\}$, it follows that there are two nonabelian groups of order $p_i p_j^{n_j} p_r^2$, namely,

$$((C_{p_j})^{n_j} \rtimes C_{p_i}) \times (C_{p_r})^2 \quad \text{and} \quad ((C_{p_j})^{n_j} \rtimes C_{p_i}) \times C_{p_r^2},$$

which gives a contradiction. If $n_i = 2$, then we will have two nonabelian groups of order $p_i^2 p_j^{n_j}$,

$$(C_{p_j})^{n_j} \rtimes C_{p_i^2} \quad \text{and} \quad ((C_{p_j})^{n_j} \rtimes C_{p_i}) \times C_{p_i},$$

which again gives a contradiction.

It follows that $n_r = 1$ for all $r \neq j$. If in (2.3) there are two distinct pairs (i,j) and (α,β) such that $p_i \mid p_j^{n_j} - 1$ and $p_\alpha \mid p_\beta^{n_\beta} - 1$, then there are two distinct nonabelian groups of order $p_i p_j^{n_j} p_\alpha p_\beta^{n_\beta}$,

$$((C_{p_\beta})^{n_\beta} \rtimes C_{p_\alpha}) \times C_{p_i p_j^{n_j}} \quad \text{and} \quad C_{p_\alpha p_\beta^{n_\beta}} \times ((C_{p_j})^{n_j} \rtimes C_{p_i}),$$

which is a contradiction. Consequently, there is at most one pair.

If $n_j = 1$, then (1.1) is satisfied. If $n_j = 2$, let us assume that $p_i \mid p_j - 1$. It follows that there are two distinct nonabelian groups of order $p_i p_j^2$,

$$(C_{p_j})^2 \rtimes C_{p_i} \quad \text{and} \quad C_{p_j{}^2} \rtimes C_{p_i},$$

which contradicts our hypothesis. Consequently, $p_i \nmid p_j - 1$ and since $p_i \mid p_j^2 - 1$, we have $p_i \mid p_j + 1$, which gives (2.1).

$\Leftarrow$ We will prove by induction over n that

$$\begin{aligned}&\text{if } n \text{ has at least two nonprime factors and satisfies either (1.1) or (2.1),}\\ &\text{then there is a unique nonabelian group of order } n.\end{aligned} \tag{2.4}$$

The base case is $n = 6$. There is just one nonabelian group of order 6, the symmetric group S_3.

Let us proceed to the inductive step. Assume that (2.4) holds for any positive integer with at least two factors $n' < n$. Let G be a nonabelian group of order n. We can assume that $p_1 < p_2 < \cdots < p_k$. It follows that $j \geq 2$. Indeed, if $j = 1$, there are two possibilities. If $p_i \mid p_1 - 1$, then $p_i < p_1$, which is a contradiction. If $p_i \mid p_1 + 1$, then since $p_i > p_1$, we have $p_i = p_1 + 1$ which implies $p_1 = 2$ and $p_i = p_2 = 3$ so that $p_j \mid p_i - 1$, which is a contradiction.

Thus, $j \geq 2$ and therefore $n_1 = 1$. Hence, the p_1-Sylow subgroups of G are cyclic of order p_1. By the Burnside normal p-complement theorem, G has a p_1-normal complement, that is, there exists $H \triangleleft G$ with $|H| = n/p_1$ such that $G = H \rtimes C_{p_1}$. We identify two cases.

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Case 1: n satisfies (1.1). If $i = 1$, it follows that H is cyclic so that

$$G = C_{n/p_1} \rtimes C_{p_1} \cong (C_{p_j} \rtimes C_{p_1}) \times C_{n/p_1p_j}.$$

If $i \geq 2$, there are again two possibilities.

- If H is abelian, then H is cyclic which is analogous to the case $i = 1$.
- If H is nonabelian, then $H \cong (C_{p_j} \rtimes C_{p_i}) \times C_{n/p_1p_ip_j}$ which implies

$$G \cong ((C_{p_j} \rtimes C_{p_i}) \times C_{n/p_1p_ip_j}) \rtimes C_{p_1} \cong ((C_{p_j} \rtimes C_{p_i}) \rtimes C_{p_1}) \times C_{n/p_1p_ip_j}.$$

  Since $|\mathrm{Aut}(C_{p_j} \rtimes C_{p_i})| = p_j(p_j - 1)$ and $p_1 \nmid p_j(p_j - 1)$, the semidirect product $(C_{p_j} \rtimes C_{p_i}) \rtimes C_{p_1}$ is trivial; therefore,

$$G \cong ((C_{p_j} \rtimes C_{p_i}) \times C_{p_1}) \times C_{n/p_1p_ip_j} \cong (C_{p_j} \rtimes C_{p_i}) \times C_{n/p_ip_j}.$$

  It follows that in this case, there is a single nonabelian group of order n.

Case 2: n satisfies (2.1). If $i = 1$, then H is abelian and there are two possibilities.

- If $H \cong C_{p_j^2} \times C_{n/p_1p_j^2}$, then $G \cong (C_{p_j^2} \rtimes C_{p_1}) \times C_{n/p_1p_j^2}$. Since $p_1 \nmid |\mathrm{Aut}((C_{p_j^2})| = p_j(p_j - 1)$, it follows that $C_{p_j^2} \rtimes C_{p_1} = C_{p_j^2} \times C_{p_1}$. Therefore, $G \cong C_n$, which is a contradiction.
- If $H \cong (C_{p_j})^2 \times C_{n/p_1p_j^2}$, then $G \cong ((C_{p_j})^2 \rtimes C_{p_1}) \times C_{n/p_1p_j^2}$, which is nonabelian.

If $i \geq 2$, then we again have two cases.

- If $H \cong ((C_{p_j})^2 \rtimes C_{p_i}) \times C_{n/p_1p_ip_j^2}$, then $G \cong (((C_{p_j})^2 \rtimes C_{p_i}) \rtimes C_{p_1}) \times C_{n/p_1p_ip_j^2}$. From [2], it follows that $|\mathrm{Aut}((C_{p_j})^2 \rtimes C_{p_i}| = 2(p_j^2 - 1)p_j^2 \nmid p_1$. Therefore,

$$G \cong (((C_{p_j})^2 \rtimes C_{p_i}) \times C_{p_1}) \times C_{n/p_1p_ip_j^2} \cong ((C_{p_j})^2 \rtimes C_{p_i}) \times C_{n/p_ip_j^2}.$$

- Finally, if H is abelian, then we identify two possibilities. The first possibility is $H \cong C_{p_2} \times \cdots \times C_{p_j} \times \cdots \times C_{p_k} \cong C_{n/p_1}$, which implies that G cyclic, and this is a contradiction. The second possibility is $H \cong C_{p_2} \times \cdots C_{p_j}^2 \times \cdots \times C_{p_k} \cong (C_{p_j})^2 \times C_{n/p_1p_j^2}$. We observe that

$$p_1 \nmid |\mathrm{Aut}((C_{p_j})^2 \times C_{n/p_1p_j^2})| = (p_j^2 - 1)(p_j^2 - p_j) \cdot \varphi\Big(\frac{n}{p_1p_j^2}\Big). \tag{2.5}$$

  Thus, $G \cong ((C_{p_j})^2 \times C_{n/p_1p_j^2}) \rtimes C_{p_1} \cong ((C_{p_j})^2 \times C_{n/p_1p_j^2}) \times C_{p_1} \cong (C_{p_j})^2 \times C_{n/p_j^2}$, where the final congruence comes from (2.5). This means G is abelian, which is a contradiction.

Therefore, also in this case, there is only one nonabelian group of order n. □

COROLLARY 2.2. *Let $n = p_1^{n_1} \cdots p_k^{n_k}$, where $2 = p_1 < \cdots < p_k$ are primes. Then, n is almost abelian if and only if $k = 2$ and $n_1 = n_2 = 1$.*

COROLLARY 2.3. *Let $n = p_1 \cdots p_k$, where $p_1 < \cdots < p_k$ are primes. Then, n is almost abelian if and only if there is a unique pair $(i, j) \in \{1, \ldots, k\}^2$ such that $p_i \mid p_j - 1$.*

REMARK 2.4. If n is almost cyclic, n is either abelian or almost abelian. The converse is false. For example, 75 is almost abelian, but 75 is neither almost cyclic nor cyclic.

THEOREM 2.5. *A number $n = p_1^{n_1} \cdots p_k^{n_k}$ is almost nilpotent if and only if n is almost abelian, that is, $k \geq 2$ and n satisfies (1.1) or (2.1).*

PROOF. $\Leftarrow$ The conclusion follows from Theorem 2.1 since all the groups constructed in the proof are nonnilpotent.

$\Rightarrow$ Let us assume n is almost nilpotent. It follows that for all $d \mid n$, there exists at most one nonnilpotent group of order d. Since n is nonnilpotent, there are integers $(i, j) \in \{1, \ldots, k\}^2$ and d_j with $1 \leq d_j \leq n_j$ such that $p_i \mid p_j^{d_j} - 1$. It follows that $\alpha_j = n_j$. Otherwise, there would be two nonnilpotent nonisomorphic groups of order $p_j^{n_j} p_i$, namely

$$((C_{p_j})^{\alpha_j} \rtimes C_{p_i}) \times C_{p_j^{n_j - d_j}} \quad \text{and} \quad (C_{p_j})^{n_j} \rtimes C_{p_i}.$$

Furthermore, the pair (i, j) is unique. Otherwise, if there were two pairs $(i', j') \neq (i, j)$ such that $p_{i'} \mid p_{j'}^{n_j} - 1$, again there would be two nonnilpotent nonisomorphic groups of order n,

$$((C_{p_j})^{n_j} \rtimes C_{p_i}) \times C_{n/p_i p_j^{n_j}} \quad \text{and} \quad ((C_{p_{j'}})^{n_{j'}} \rtimes C_{p_{i'}}) \times C_{n/p_{i'} p_{j'}^{n_{j'}}}.$$

Let us observe that $n_r = 1$ for all $r \neq i, j$. Otherwise, there would exist at least two distinct groups P_r and Q_r of order $p_r^{n_r}$, which would give two nonnilpotent, nonisomorphic groups of order $p_j^{n_j} p_i p_r^{n_r}$,

$$((C_{p_j})^{n_j} \rtimes C_{p_i}) \times P_r \quad \text{and} \quad ((C_{p_j})^{n_j} \rtimes C_{p_i}) \times Q_r.$$

Analogously, we can show that $n_i = 1$.

If $n_j = 1$, then (1.1) holds. If $n_j \geq 2$, then $n_j = 2$, since otherwise, we would have two nonnilpotent, nonisomorphic groups of order $p_j^{n_j} p_i$. Thus,

$$p_i \mid p_j^2 - 1. \tag{2.6}$$

In addition, if $p_1 \mid p_j - 1$, there are two nonnilpotent, nonisomorphic groups of order $p_j^2 p_i$,

$$(C_{p_j})^2 \rtimes C_{p_i} \quad \text{and} \quad C_{p_j^2} \rtimes C_{p_i}.$$

It follows that $p_i \nmid p_j - 1$ and, by (2.6), $p_i \mid p_j + 1$. This yields (2.1) and concludes the proof. □

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References

[1] H. I. Binjedaen, 'When there is a unique group of a given order and related results', Graduate Thesis, Missouri State University, 2016. https://bearworks.missouristate.edu/theses/2952/.

[2] E. Campedel, A. Caranti and I. Del Corso, 'The automorphism groups of groups of order p^2q', *Int. J. Group Theory* **10**(3) (2021), 149–157.

[3] I. M. Isaacs, *Finite Group Theory* (American Mathematical Society, Providence, RI, 2008).

[4] J. Pakianathan and K. Shankar, 'Nilpotent numbers', *Amer. Math. Monthly* **107**(7) (2000), 631–634.

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