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THE OPERATOR THEORY OF GENERALIZED BOUNDARY VALUE PROBLEMS

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1. Introduction. In this paper we develop a theory of maximal and minimal operators and their duals associated with the system

$$(1.1)\qquad l(y) = \sum_{i=0}^{n} a_i y^{(n-i)} = f,$$

$$(1.2)\qquad U_j(y, \ldots, y^{(n-1)}) = \sum_{i=0}^{n-1} \int_I dv_{n-i}{}^j y^{(i)} = 0, \quad j = 1, \ldots$$

We assume that the system is defined on an arbitrary interval $I = [a, b]$ in the extended reals such that the coefficients a_i are complex valued functions in $\mathscr{C}^{(n-i)}(I)$ with $a_0 > 0$ on I, and the side conditions $\{U_j\}$ are possibly infinite in number and represented by complex measures v_i which are (at least locally) of bounded variation.

Under these restrictions (1.1), (1.2) is said to determine a Stieltjes boundary value (s.b.v.) problem. We call the problem *regular* if I is compact and the set $\{U_j\}$ is finite; otherwise we say that the problem is *singular*. Clearly this vocabulary generalizes the notions of regular or singular b.v.p. for ordinary differential systems.

It is worth noting here that the class of s.b.v. problems defined above is quite large: Let $\mathscr{C}^{(n-1)}(I)$ be the space of $n - 1$ fold continuously differentiable functions on a compact interval I under the norm

$$|f| = \sum_{i=0}^{n-1} |f^{(i)}|_\infty.$$

Then by a mild generalization of the Riesz Representation Theorem [**7**, p. 344], every continuous functional $U : \mathscr{C}^{(n)} \to \mathbf{C}$ can be represented in the form (1.2). In particular, by choosing linear combinations of point evaluation measures for the $v_{n-i}{}^j$, all ordinary multipoint problems may be constructed. On the other hand, we exclude "interface" side conditions where the side conditions U_j involve left or right limits of derivatives at interior points of I.

Elsewhere [**4**; **5**] we have considered the operators on $\mathscr{L}^p(I)$, $1 \leqq p < \infty$, arising from the system (1.1), (1.2) in the regular case. In [**5**] for example,

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the expression (1.1) was assumed to be vector valued with nonsmooth coefficients (satisfying, however, conditions for the existence of Carathéodory solutions). Then (1.1), (1.2) naturally determines an operator $\mathscr{L}$ with domain and range in a space of $\mathscr{L}^p$ integrable functions $1 \leqq p < \infty$. Two distinct characterizations of the adjoint relation $\mathscr{L}^*$ (in $\mathscr{L}^q(I) \times \mathscr{L}^q(I), 1/p + 1/q = 1$) were derived and shown equivalent.

One useful application of $\mathscr{L}^*$ is to determine the structure of various classes of generalized splines [**6**].

As mentioned above the previous work dealt only with regular s.b.v. problems. There is need, however, for an approach which would extend the theory of singular ordinary differential operators (as developed, for example, by Naimark [**17**] in $\mathscr{L}^2$, and Goldberg [**8**] in $\mathscr{L}^p$ to singular s.b.v. problems.

Such an extension has so far been only slightly developed. Krall [**13**; **14**], Naimark [**16**], and Kim [**12**], for example, have investigated the $\mathscr{L}^2$ theory of various second order differential expressions on $[0, \infty]$ subject to end point conditions [**16**] and Stieltjes conditions [**12**; **13**; **14**]. Recently Coddington [**3**] has considered the selfadjoint extensions of symmetric operators defined on nondense domains in $\mathscr{L}^2(I)$. The boundary conditions determining these operators are special cases of (1.2). In several of these papers Green's functions are derived, adjoint operators are defined, eigenvalues and eigenfunctions are found and their convergence properties are discussed. Additional discussion of previous work may be found in Krall [**15**].

We present here an outline of the paper. After introducing the notion of an infinite dimensional matrix measure and fixing notation (§ 2), we outline in § 3 Arens' theory of linear relations and adjoint relations with respect to a dual pairing between two linear spaces. This theory is compatible with the usual theory of closed operators and their adjoints on Banach spaces.

In § 4 we use these ideas to reconstruct the theory of maximal and minimal ordinary differential operators on an arbitrary interval. We prove that each is the "adjoint" of the other. This allows us (§ 5) to define maximal and minimal operators and to investigate adjoint relations and Fredholm type alternative theorems for s.b.v. problems. At this point, however, we are still restricted to the regular case.

Next (§ 6) the results of § 5 are extended to an important subcategory of singular operators.

Finally, in the last section (§ 7) we illustrate our theory with a few examples, point out some applications to spline and minimum norm problems, and suggest some directions for future research.

2. Preliminaries. Observe first of all that the set of side conditions $\{U_v\}$ can be written as one "vector valued" side condition

$$(2.1) \qquad \mathscr{U}(y, \ldots, y^{(n-1)}) = \sum_{i=0}^{n-1} \int_I dv_{n-i} y^{(i)} = r,$$

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where v_{n-i} is the "vector valued" (v.v.) measure defined by

$$v_{n-1} = (w_{n-i}{}^1, w_{n-1}{}^2, \ldots, w_{n-i}{}^j, \ldots)^t,$$

r is the vector

$$(r_1, \ldots, r_j, \ldots)^t,$$

and the integrals

$$\int_I dv_{n-i}y^{(i)}$$

are defined componentwise. Similarly we consider matrix valued (m.v.) measures with respect to matrix or vector valued functions. Integration is defined componentwise provided the measure and the matrix are compatible in the sense of matrix multiplication. Notice that it is a matter of indifference whether or not such measures are finite or infinite dimensional. The various theorems of integration theory—for example, integration by parts, Fubini's theorem etc. are valid provided they are valid componentwise. Whenever possible therefore we will not distinguish between infinite or finite dimensional m.v. or v.v. measures. Since we do not need infinite two dimensional measures, however, it will be assumed throughout that all measures have at worst either infinitely many rows—"infinite row dimension" or infinitely many columns—"infinite column dimension", yet not both at the same time.

Using the Lebesgue decomposition theorem we write

$$v_i = v_{ic} + v_{is},$$

where v_{ic}, v_{is} are respectively absolutely continuous and singular with respect to Lebesgue measure; by analogy with the scalar case we will refer to the vector valued function obtained by taking the Radon-Nikodym derivatives of the components of v_{ic} as the Radon-Nikodym derivative of v_{ic}, Dv_{ic}. Having done this, the functional $\mathscr{U}(y, \ldots, y^{(n-1)})$ can be written

$$\mathscr{U}(y, \ldots, y^{(n-1)}) = \mathscr{U}_s(y, \ldots, y^{(n-1)}) + \mathscr{U}_c(y, \ldots, y^{(n-1)}), \tag{2.2}$$

where

$$\mathscr{U}_s(y, \ldots, y^{(n-1)}) = \sum_{i=0}^{n-1} \int_a^b dv_{n-is}y^{(i)},$$

and

$$\mathscr{U}_c(y, \ldots, y^{(n-1)}) = \sum_{h=0}^{n-1} \int_a^b Dv_{n-ic}y^{(i)}dt.$$

We introduce the further assumption is that v_{ic} is actually in $\mathscr{C}^{n-i+1}[I]$ that is, $Dv_{ic} \in \mathscr{C}^{n-i}[a, b]$). Then by repeated integration by parts (we integrate

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y and differentiate Dv_c), $\mathscr{U}(y, \ldots, y^{(n-1)})$ can be written in the "almost singular" form

$$(2.3) \quad \mathscr{U}(y, \ldots, y^{(n-1)}) = \sum_{i=0}^{n-1} \int_a^b dw_{n-i} y^{(i)} + \int_a^b \frac{dw^c}{du} y \, du,$$

where the w_{n-i}, $i = 0, \ldots, n-1$, are all singular and

$$\frac{dw^c}{du} = \sum_{i=0}^{n-1} (-1)^i D^i(v_{n-ic}).$$

At this point a few notational remarks are appropriate: If T is a linear operator or relation, $\mathscr{D}(T)$, $\mathscr{R}(T)$, $\mathscr{N}(T)$ will stand for its domain, range, and null space, respectively. If T and M are two linear operators or relations, M is an extension of T, in symbols: $M \supset T$ if $\mathscr{D}(M) \supset \mathscr{D}(L)$ and $M = L$ on $\mathscr{D}(L)$. T^* will denote the conjugate transpose, dual, or adjoint of a matrix, space operator, or relation depending on the context. (The mere transpose will be denoted by "t"). We represent the identity operator on the space X by the symbol I_X and the $m \times m$ identity matrix by I_m. $\mathbf{C}^m$ denotes m-dimensional space over the complex field under the standard Euclidean norm. The notation $\lambda[E](t)$ means the characteristic function acting on the set E, and $u(t_i)$ means the point evaluation measure concentrated at the point t_i. We employ the symbols π_{ij}, π_i to stand for the projection mappings which select respectively the component $\alpha^{(ij)}$ or the ith row of a matrix M (thus the jth column of M can be represented by $\pi_j(M^*)$).

Suppose $v_1, \ldots, v_n$ are measures, matrices or functions, then the symbol $\bar{v}$ will denote the measure, matrix, or v.v. function $(v_n, \ldots, v_1)$.

Let y be an $n-1$ fold differentiable function. We define $\hat{y}$ to be the n dimensional vector valued function $(y, \ldots, y^{(n-1)})^t$.

Employing the notation introduced above, $\bar{w}$ is the m.v. measure

$$\bar{w} = \left(w_n + \frac{dw^c}{du}, w_{n-1}, \ldots, w_1\right);$$

and the side condition (2.3) (which is itself equivalent to (1.2), (2.1) and (2.2)) may be written

$$\mathscr{U}(y, \ldots, y^{(n-1)}) = \int_I d\bar{w}\, \hat{y} = 0.$$

Notice that

$$\bar{w}_s = (w_n, \ldots, w_1), \bar{w}_c = \left(\frac{dw^c}{du} du, 0, \ldots, 0\right).$$

The spaces considered in the paper are the following: $\mathscr{L}_m(I)$ is the space of locally integrable m dimensional vector valued complex functions on I.

$BV_m^n(I) \subset \mathscr{L}_m(I)$ is the space of functions such that $y, \ldots, y^{(n-1)}$ are locally of bounded variation and $y^{(n)}$ exists a.e. Similarly $AC_m^n(I)$ is the collection of functions y in $BV_m^n(I)$ such that $y, \ldots, y^{(n-1)}$ are absolutely continuous. When the indices n, m are unity they will be omitted. The subscript "0" means "functions of the appropriate class with local support". Thus,

$$AC_0^n(I) = \{y : y \in AC^n(I); \text{supp. } y \subset (a, b)\}.$$

3. Pairings and formal adjoints. This section reviews certain results in Arens [**2**] and Kelley and Namioka [**11**].

3.1. *Definition.* A pair of linear spaces X, X' is a *dual* pair if there exists a nontrivial functional $\langle \cdot,\cdot \rangle : X \times X' \to \mathbf{C}$ linear in the first argument and semi-linear in the second argument.

3.2. *Definition.* Sets $S \subset X$, $S' \subset X'$ are said to be *total* if

$$\begin{aligned} &\langle s, x' \rangle = 0 \text{ for all } s \in S \Rightarrow X' = 0 \\ &\langle x, s' \rangle = 0 \text{ for all } s' \in S' \Rightarrow x = 0. \end{aligned} \tag{3.1}$$

If X, X' are themselves total, we say that the pairing $\langle .,. \rangle$ *distinguishes points.*

3.3. LEMMA. *Let Y, Y' be another dual pair over* $\mathbf{C}$. *Then the form*

$$\langle (x, y), (x', y') \rangle = \langle x, x' \rangle + \langle y, y' \rangle \tag{3.2}$$

makes $X \times Y$ and $X' \times Y'$ into a dual pair over $\mathbf{C}$. *The pairing distinguishes points if and only if the pairings on $X \times X'$ and $Y \times Y'$ distinguish points.*

Proof. Trivial calculation.

3.4. *Definition.* Let $S \subset X$, $S' \subset X'$. We define the annihilators $S^\perp$ and $S'^\perp$ by

$$S^\perp = \{x' : \langle s, x' \rangle = 0 \text{ for all } s \in S\}$$
$$S'^\perp = \{x : \langle x, s' \rangle = 0 \text{ for all } s' \in S'\}.$$

3.5. *Definition.* The *closure* of S is $S^{\perp\perp}$. S is said to be *closed* if $S^{\perp\perp} = S$ and *dense* if $S^{\perp\perp} = X$ (similarly with S').

The following implications are easy to see: S is total implies S is dense; S is dense implies S is total if and only if X is total. Thus in X or X', dense is equivalent to total if and only if $\langle .,. \rangle$ distinguishes points.

3.6. LEMMA. *Assume there are pairings on $X \times X'$ and $Y \times Y'$ which distinguish points. Then a finite dimensional subspace S in X, X', Y or Y' is closed.*

Proof. Without loss of generality we take $S \subset X$. The proof depends on the linear dependence principle [**11**, p. 7]. Let $f_1, \ldots, f_n$ be a basis of S and $\phi \in S^{\perp\perp}$. Define the functionals $\hat{f}_i : X' \to \mathbf{C}$ by $\hat{f}_i(x') = \langle f_i, x' \rangle$ and $\hat{\phi} : X' \to \mathbf{C}$ by

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$\hat{\phi}(x') = \langle \phi, x' \rangle$. Then

$$\mathcal{N}(\hat{\phi}) \supset S^{\perp} \quad \text{and} \quad \bigcap_{i=1}^{n} \mathcal{N}(f_i) = S^{\perp}.$$

By the linear dependence principle $\hat{\phi} = \sum \lambda_i \hat{f}_i$. Consequently

$$\left\langle \phi - \sum_{i=1}^{n} \lambda_i f_i, x' \right\rangle = 0 \text{ for all } x' \in X'.$$

But then $\phi = \sum_{i=1}^{n} \lambda_i f_i$. Thus $S^{\perp\perp} \subset S$ and S is closed.

The closure operation induces a topology on X. Consider the weak topology on X with respect to X' induced by the pairing. This topology is locally convex; it is not difficult to show that it is the same as the "closure" topology if and only if X' is total; thus, in particular a set is "closed" in the sense of Definition 3.5 if and only if it is weakly closed. Since similar remarks are true in X' *the closure and weak topologies in X or X' are identical, locally convex and Hausdorff if and only if the pairing distinguishes points.*

What happens if X has a topological structure of its own? We state some well known facts.

3.7. THEOREM. *Let X be a locally convex linear topological space. Choose $X' = X^*$ and define $\langle x, f \rangle$ by $\bar{f}(x)$. Call a set $S \subset X$ strongly closed if it is closed in the X topology, and a set $S' \subset X^*$ weakly closed if it is closed in the weak topology on X' induced by X. Then*

(i) *X^* is a locally convex linear topological space;*

(ii) *$\langle .,. \rangle$ distinguishes points;*

(iii) *for convex sets in X, closure $\Leftrightarrow$ weak closure $\Leftrightarrow$ strong closure;*

(iv) *for convex sets in X', closure $\Leftrightarrow$ weak closure $\Rightarrow$ strong closure; but strong closure $\Rightarrow$ weak closure if and only if X is reflexive.*

Proof. See [**11**, Chapter 17].

On occasion we will call the closure introduced in Definition 3.5 the "algebraic" or "$\perp$-closure" to distinguish it from the other closures. Whenever possible, however, we freely employ the words "closed", "dense", etc. and depend on the context to indicate which topology is met.

3.8. *Definition.* A *linear* relation $T : X \rightarrow Y$ where X, Y are linear spaces is a set valued mapping whose graph $\mathcal{G}(T)$ is a subspace of $X \times Y$.

For $\alpha \subset \mathcal{D}(T)$ we denote the image of α in $\mathcal{R}(T)$ by $T(\alpha)$. One verifies that $T(0)$ is a subspace of $\mathcal{R}(T)$ and that elements $\beta, \alpha \in T(\alpha)$ if and only if $\beta \equiv \alpha \bmod T(0)$. Thus if an arbitrary element in $T(\alpha)$ is α_T, $T(\alpha) = \alpha_T + T(0)$ and

$$\mathcal{G}(T) = \{(\alpha, \alpha_T + T(0)) : \alpha \in \mathcal{D}(T)\}.$$

To put it another way, the induced mapping $T' : X \rightarrow X/T(0)$ is algebraically an operator.

Let $T : X \to Y, S : Y \to Z$ be a linear relations. We define

$$S \circ T = \{(x, z) : \text{there exists } y \text{ such that } ((x, y) \in T \text{ and } (y, z) \in S)\}$$

(we write this simply as ST). If $\lambda \in \mathbf{C}$ the relation

(3.3) $\lambda T = \{(x, \lambda y) : (x, y) \in T\}.$

Equivalently this is the composition $\lambda_T \circ T$, where

$$\lambda_T = \{(x, \lambda x) : x \in \mathscr{R}(T)\}.$$

The addition of two linear relations S, T may be defined by

$$S + T = \{(x, y) : \text{there exist } s, t \text{ such that } y = s + t,\ (x, s) \in S, \text{ and } (x, t) \in T)\}.$$

Every relation T has an inverse T^{-1} such that

$$T^{-1}(\alpha) = \{\beta : \beta \in \mathscr{D}(T); \alpha \in T(\beta)\}.$$

A relation is said to be an operator if and only if it is single valued; that is, if and only if $T(0) = 0$.

The *null space* $\mathscr{N}(T)$ of a relation T will be the set in $\mathscr{D}(T)$ given by $T^{-1}T(0)$. This definition is equivalent to the statement

$$\mathscr{N}(T) = \{\alpha : (\alpha, 0) \in \mathscr{G}(T)\}.$$

Obviously $\mathscr{N}(T)$ is a subspace of X.

We call T *closed* if $\mathscr{G}(T)$ is closed in the sense of Definition 3.4. If T is an operator, T is said to be *closable* in case the closure $\bar{T}$ of T (i.e., the relation determined by $\overline{\mathscr{G}(T)}$) is also an operator.

3.9. *Definition.* The *adjoint* T^* of a relation T is the relation determined by the graph

$$\mathscr{G}(T^*) = (\mathscr{G}(-T)^{-1})^{\perp},$$

where $-T = -1T$ is defined according to (3.3).

The next theorem lists some properties of the adjoint, all of which are easily derived from the definition (for details see [**2**]).

3.10. THEOREM.

(i) $\mathscr{G}(T^*) = \{(\beta, \beta') : \langle \alpha_T, \beta \rangle = \langle \alpha, \beta' \rangle; \alpha \in \mathscr{D}(T); \alpha_T \in T(\alpha)\}$;

(ii) $S \subset T$ *implies* $T^* \subset S^*$;

(iii) $(T^{-1})^* = (T^*)^{-1}$;

(iv) $S \subset T^*$ *implies* $\bar{S} \subset T^*$;

(v) T^* *is an operator if and only if* $\overline{\mathscr{D}(T)} = X$;

(vi) $T^*(0) = \mathscr{D}(T)^{\perp}$;

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(vii) T^* *is closed;*

(viii) T^{**} *is the minimal closed extension containing* T*;*

(ix) $\mathscr{N}(T^*) = \underline{\mathscr{R}(T)^\perp}$;

(x) $\mathscr{N}(T^*)^\perp = \overline{\mathscr{R}(T)}$.

Suppose T *is closed. Then we also have:*

(xi) $\mathscr{N}(T) = \underline{\mathscr{R}(T^*)^\perp}$;

(xii) $\mathscr{N}(T)^\perp = \overline{\mathscr{R}(T^*)}$.

Finally if T, T^* *are both normally solvable* (*that is, have closed graphs and closed ranges*):

(xiii) $\mathscr{N}(T^*)^\perp = \mathscr{R}(T)$;

(xiv) $\mathscr{N}(T)^\perp = \mathscr{R}(T^*)$;

(xv) $\mathscr{N}(T^*) = \mathscr{R}(T^\perp)$;

(xvi) $\mathscr{N}(T) = \mathscr{R}(T^*)^\perp$.

Of course when $X' = X^*$, $Y' = Y^*$ and X, Y are locally convex, $\overline{\mathscr{D}(T)} = X$ and $\overline{\mathscr{D}(T^*)} = Y'$ if and only if $\mathscr{D}(T)$, $\mathscr{D}(T^*)$ are total. Moreover by Theorem 3.6 $\mathscr{N}(T)$ and $\mathscr{R}(T)$ are closed in X and Y if and only if they are strongly closed. If in addition X is reflexive, the closures of $\mathscr{G}(T)$ and $\mathscr{G}(T^*)$ coincide with their strong closures in $X \times Y$ and $Y^* \times X^*$ respectively (endowed with the product topology), and $\mathscr{R}(T^*)$, $\mathscr{N}(T^*)$ are closed if and only if they are strongly closed. If X, Y are Banach spaces more can be said:

3.11. THEOREM. (Banach Closed Range Theorem). *Let* $T : X \to Y$ *be a densely defined closed operator. Then the following statements are all equivalent:*

(i) $\mathscr{R}(T)$ *is closed;*

(ii) $\mathscr{R}(T^*)$ *is closed;*

(iii) $\mathscr{R}(T) = \mathscr{N}(T^*)^\perp$;

(iv) $\mathscr{R}(T^*) = \mathscr{N}(T)^\perp$.

Proof. See [**18**, p. 205; **8** p. 95].

We now discuss the specific pairing used in the remainder of the paper. X, X', Y, Y' are measurable locally integrable (that is, integrable on compact subsets of $[a, b]$) function spaces such that

$$(3.4) \qquad BV_0(I) \subset X \cap X' \cap Y \cap Y' \subset \mathscr{L}(I);$$

and

$$(3.5) \qquad \langle x, x' \rangle = \int_I x'^* x dt < \infty, \quad \langle y, y' \rangle = \int_I y'^* y dt < \infty,$$

for all $x \in X$, $x' \in X'$, $y \in Y$, $y' \in Y'$.

Obvious choices for X, X', Y and Y' given the pairings (3.5) are the various $\mathscr{L}^p$ spaces on I—for example, $X = \mathscr{L}^p(I)$, $X' = \mathscr{L}^q(I)$, $1 \leqq p \leqq \infty$, $1/p + 1/q = 1$. Other examples will be discussed in § 7.

If Δ is a compact subinterval of I and the notation X_Δ etc., f_Δ denotes the restriction of the functions in X or a particular function f to Δ.

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3.12. LEMMA.

$$(3.6)\qquad \begin{aligned} BV^n(\Delta) &\subset (X \cap X' \cap Y \cap Y' \cap BV(I))_\Delta \\ AC^n(\Delta) &\subset (X \cap X' \cap Y \cap Y' \cap AC(I))_\Delta. \end{aligned}$$

This is an easy exercise using (3.4); we omit the proof.

3.13. LEMMA. *For any integer n the space $AC_0^n(I)$ is a total subspace of X, X', Y, Y'.*

Proof. Since $AC_0^n(I) \subset BV_0(I)$,

$$AC_0^n(I) \subset X \cap X' \cap Y \cap Y'$$

by (3.4). Let Δ be a compact subinterval of I. Without loss of generality suppose

$$\langle s, x\rangle = 0 \text{ for all } s \in AC_0^n(I).$$

This implies

$$\langle s, x\rangle_\Delta = 0 \text{ for all } s \in AC_0^n(\Delta).$$

Now x, being integrable on Δ, is in $\mathscr{L}^1(\Delta)$. Since

$$\overline{AC_0^n(\Delta)} = \mathscr{L}^1(\Delta),$$

(3.6) implies $x_\Delta = 0$. Because Δ is arbitrary, x vanishes everywhere, proving the lemma.

Notice that the previous lemma is a modification of the so-called "Fundamental Lemma of the Calculus of Variations" [**1**, p. 20].

3.14. THEOREM. *The pairings (3.5) distinguish points. Moreover, for any integer n the spaces $BV^n(I) \cap X$, $BV^n(I) \cap X'$, $BV^n(I) \cap Y$, and $BV^n(I) \cap Y'$ are dense respectively in X, X', Y, and Y'.*

Proof. By Lemma 3.13, $AC_0^n(I)$ is total, that is, $AC_0^n(I)^\perp = \{0\}$. Since $X \cap X' \cap Y \cap Y' \supset AC_0^n(I)$, $X^\perp$ etc. $\subset AC_0^n(I)^\perp$. This shows that the spaces X, X', Y, Y' are total. Therefore the pairings separate points. Similarly since $BV^n(I) \cap X'$ etc. $\supset AC_0^n(I)$, these spaces are all total, hence dense.

3.15. COROLLARY. *$AC^n(I) \cap X$, $AC^n(I) \cap X'$, $AC^n(I) \cap Y$, and $AC^n(I) \cap Y'$ are dense in X, X', Y and Y'.*

Proof. Trivial.

4. Minimal and maximal ordinary differential operators and their adjoints. In this section we develop the adjoint theory of minimal and maximal ordinary differential operators on an arbitrary interval $I \subset \mathbf{R}$ with respect to a pairing. Our procedures and results resemble closely the Hilbert or $\mathscr{L}^p$

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space case considered by other writers, for example Naimark [**17**] or Goldberg [**8**].

In what follows I, X, X', Y, Y' are as defined at the end of the previous section and we employ the pairing (3.5).

4.1. *Definition.* Let L denote the operator obtained by restricting the expression $l(y)$ to the domain

$$\mathscr{D}(L) = \{y : y \in AC^n(I) \cap X; l(y) \in Y\},$$

and L_0' the operator obtained by restricting $l(y)$ to the domain

$$\mathscr{D}(L_0') = \{y : y \in AC_0^n(I); l(y) \in Y\}.$$

In some sense L and L_0' are "maximal" and "minimal" operators generated by the differential expression $l(y)$. The goal of this section will be to determine the adjoints in $Y' \times X'$.

4.2. *Definition.* In $BV^n(I)$, let $l_j^+(z)$, $j = 0, 1, \ldots, n$ be the $n + 1$ "partial adjoint" expressions:

$$l_0^+(z) = a_0^*z,$$

$$\vdots$$

$$l_j^+(z) = \sum_{i=0}^{j} (-1)^{j-1}(a_i^*z)^{j-1},$$

$$\vdots$$

$$l_n^+(z) = \sum_{i=0}^{n} (-1)^{n-i}(a_i^*z)^{n-i}.$$

Observe that $l_j^+(z)$ is simply the formal adjoint of the expression

$$l_j(z) = \sum_{i=0}^{j} a_i z^{j-i}$$

(thus in particular $l_n^+(z) = l^*(z)$), and that the recursion relation

$$l_{j+1}^+(z) = -\, l_j^+(z) + a_{j+1}^*z$$

holds. By Leibniz's rule and a reordering of the sums we can show that

$$l_j^+(z) = \sum_{r=0}^{j} \alpha_{jr}z^{(r)},$$

where

$$\alpha_{jr} = \sum_{i=0}^{j-r} (-1)^{j-i}\binom{j-i}{r} D^{j-i-r}(a_i^*).$$

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The coefficients α_{jr} of the $l_j^+(z)$ may then be exhibited by the lower triangular matrix

$$(4.1)\qquad \mathscr{B} = \begin{bmatrix} \alpha_{00} & 0 & \cdots & 0 \\ \alpha_{10} & \alpha_{11} & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \alpha_{n-1,0} & \alpha_{n-1,1} & & \alpha_{n-1,n-1} \end{bmatrix}.$$

Notice that $\mathscr{B}$ is invertible (since $\alpha_{ij} = (-1)^j a_{00}$ and $a_0 > 0$ in $\mathbf{R}$) and that

$$\pi_j(\mathscr{B}\hat{z}) = l_{j-1}^+(z), j = 1, \ldots, n.$$

The following result is well known:

4.3. Lemma (Green's Relation).

$$\int_\Delta y^* l^+(z)dt = [y, z](\beta) - [y, z](\alpha) + \int_\Delta l(y)^* z dt,$$

where $[y, \hat{z}](t) = (\mathscr{B}z)^*\hat{y}(t)$.

4.4. *Definition.* Let L^+, L_0^+, $L_0^{+\prime}$ respectively be the operators obtained by restricting $l^+(z)$ to

$$\mathscr{D}^+ = \{z : z \in AC^n(I) \cap Y'; l^+(z) \in X'\};$$

$$\mathscr{D}_0^+ = \{z : z \in \mathscr{D}^+; \lim_{s\to b} [y, z](s) - \lim_{s\to a} [y, z](t) = 0 \textit{ for all } y \in \mathscr{D}\};$$

$$\mathscr{D}_0^{+\prime} = \{z : z \in AC_0^n(I)\}.$$

4.5. *Definition.* Let L_0 be the restriction of $l(y)$ to

$$\mathscr{D}_0 = \{y : y \in \mathscr{D}; \lim_{s\to b} [y, z](s) - \lim [y, z](t) = 0 \textit{ for all } z \in \mathscr{D}^+\}.$$

We note that if I is compact it follows from the definition of the bilinear form $[y, z]$ that equivalent definitions of $\mathscr{D}_0$ and $\mathscr{D}_0^+$ are

$$\mathscr{D}_0 = \{y : y \in \mathscr{D}; \hat{y}(a) = \hat{y}(b) = 0\},$$

$$\mathscr{D}_0^+ = \{z : z \in \mathscr{D}^+; \hat{z}(a) = \hat{z}(b) = 0\}.$$

4.6. Theorem. *L, L_0, L_0', L^+, L_0^+, $L_0^{+\prime}$ are densely defined operators.*

Proof. Since $L \supset L_0 \supset L_0'$, it is necessary only to show that L_0' is densely defined. By assumption $Y \supset AC_0(I)$ and $X \supset AC_0^{n+1}(I)$, where n is the order of $l(y)$. Therefore, $\mathscr{D}(L_0') \supset AC_0^{n+1}(I)$. We conclude from Lemma 3.13 that $\mathscr{D}(L_0')$ is total and dense. It is clear that the same argument works for L^+, L_0^+, $L_0^{+\prime}$.

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4.7. THEOREM.
(i) $\langle L_0'(y), z\rangle = \langle y, L^+(z)\rangle$;
(ii) $\langle L(y), z\rangle = \langle y, L_0^{+\prime}(z)\rangle$;
(iii) $\langle L_0(y), z\rangle = \langle y, L^+(z)\rangle$;
(iv) $\langle L(y), z\rangle = \langle y, L_0^+(z)\rangle$.

Proof. Immediate from Lemma 4.3 and Definitions 4.4 and 4.5.

We briefly interrupt our development with a general result on closable operators.

4.8. LEMMA. *Let $T : X \to Y$ be an operator. Suppose $T^+ : Y' \to X'$ is a relation such that $\mathscr{D}(T^+)$ is total and $T^+ \subset T^*$. Then T is closable.*

Proof. By definition T is closable if and only if $\overline{\mathscr{G}(T)}$ is the graph of an operator. Assume therefore that $(0, \alpha) \in \overline{\mathscr{G}(T)}$ for a nontrivial $\alpha \in Y$. Since $T^+ \subset T^*$, $\mathscr{G}(T^+) \subset \mathscr{G}(-T^{-1})^\perp$. This implies $\mathscr{G}(-T^{+-1}) \subset \mathscr{G}(T)^\perp$. But then $\mathscr{G}(-T^{+-1})^\perp \supset \overline{\mathscr{G}(T)}$. Therefore $(0, \alpha) \in \mathscr{G}(-T^{+-1})^\perp$. Writing this out:

$$\langle (0, \alpha), (-T^+(z), z)\rangle = 0 = \langle \alpha, z\rangle$$

for all z in $\mathscr{D}(T^+)$. Since $\mathscr{D}(T^+)$ is total $\alpha = 0$, contradicting our assumption. Thus $\bar{T}$ is an operator.

4.9. THEOREM. *L, L_0, L_0', L^+, L_0^+, L_0' are all closable.*

Proof. By Theorem 4.7 $L^+ \subset L_0^* \subset L_0'^*$; $L_0^+ \subset L^*$. By Lemma 3.13 $\mathscr{D}_0^{+\prime} = AC_0^n(I)$ is total and thus so is $\mathscr{D}_0^+ \supset \mathscr{D}_0^{+\prime}$. From Lemma 4.8 therefore we conclude that L_0, L_0', and L are closable. Similar reasoning shows that L_0^+, $L_0^{+\prime}$, and L^+ are closable.

At this point we introduce some new notation. Let $\Delta = [\alpha, \beta]$ be a fixed compact subinterval of I. We have already let f_Δ, X_Δ, etc. signify the restriction of a function or space of functions defined on I to Δ. Now we let $(L_0')_\Delta$, $(L_0^{+\prime})_\Delta$, $(L)_\Delta$, etc. denote the restrictions of L_0', $L_0^{+\prime}$, L, etc. to Δ. On the other hand $L_{0,\Delta}'$, $L_{0,\Delta}^+$, L_Δ, etc. will represent the operators L_0', L_0^+, L, etc. defined on $I = \Delta$. Also we signify the pairing functional $\langle .,. \rangle$ on Δ by $\langle .,. \rangle_\Delta$.

4.10. THEOREM.
(i) $\mathscr{R}(L_0')^\perp = \mathscr{N}(L^+)$;
(ii) $\mathscr{R}(L_0)^\perp = \mathscr{N}(L^+)$;
(iii) $\mathscr{R}(L_0^{+\prime})^\perp = \mathscr{N}(L)$;
(iv) $\mathscr{R}(L_0^+)^\perp = \mathscr{N}(L)$.

Proof. Since there is complete symmetry between (i) and (iii) and (ii) and (iv), we need only prove (i) and (ii). Let $f \in \mathscr{N}(L_\Delta^+)^\perp$. Let $\{z_v\}$ be a basis for $\mathscr{N}(L_\Delta^+)$ such that

$$(4.2) \qquad z_v^{(k-1)}(\beta) = \begin{cases} 0 & v \neq k \\ & \text{for} \\ 1 & v = k\,. \end{cases}$$

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Now there exists a unique $y \in AC^n(\Delta)$ such that $l(y) = f$ and $\hat{y}(\alpha) = 0$. Since $l^+(z_v) = 0$, we have by Lemma 4.3 that $[y, z_v](\beta) = 0$. By (4.2) and the definition of $[y, z_\iota]$ this can happen if and only if $\hat{y}(\beta) = 0$. In other words, $f \in N(L_\Delta{}^+)^\perp$ if and only if $f \in \mathscr{R}(L_{0,\Delta})$. Now $\mathscr{N}(L_\Delta{}^+)$, being finite dimensional, is closed. Hence $\mathscr{R}(L_{0,\Delta})^\perp = \mathscr{N}(L_\Delta{}^+)$. Next let $\Delta' = [\alpha', \beta']$ be a compact subinterval of Δ. Define

$$\mathscr{N}(L^+)_{(0,\Delta')}{}^\perp = \{f : f \in \mathscr{N}(L^+)^\perp; \text{supp.}\, f \subset \Delta'\}.$$

Repeating the previous reasoning we see that

$$\mathscr{N}(L^+)_{(0,\Delta')}{}^\perp = \mathscr{R}(L_{0,\Delta'}) \subset \mathscr{R}(L_0').$$

It follows that $\mathscr{N}(L^+)_{(0,\Delta')}{}^\perp \subset \mathscr{R}(L_{0,\Delta}')$. Because Δ' is arbitrary in Δ, $\mathscr{N}(L_\Delta{}^+)^\perp \subset \mathscr{R}(L_{0,\Delta}')$. Hence $\mathscr{N}(L_\Delta{}^+) \supset \mathscr{R}(L_0')^\perp$. But $\mathscr{N}(L_\Delta{}^+) \subset \mathscr{R}(L_{0,\Delta}')^\perp$ by Theorem 4.7 (i). We conclude that $\mathscr{N}(L_\Delta{}^+) = \mathscr{R}(L_{0,\Delta}')^\perp$. We have now completed the proof of both (i) and (ii) for a compact subinterval Δ of I. It is easy to see that $\mathscr{N}(L_\Delta{}^+) = \mathscr{N}(L^+)_\Delta$.

4.11. COROLLARY.

$$\overline{\mathscr{R}(L_0')} = \overline{\mathscr{R}(L_0)}; \quad \overline{\mathscr{R}(L_0'^+)} = \overline{\mathscr{R}(L_0{}^+)}.$$

Moreover if the operators L_0, $L_0{}^+$ are regular, their ranges are closed.

4.12. THEOREM. $L_0'^* = L^+$; $L_0{}^* = L^+$.

Proof. By Theorem 4.7, $L^+ \subset L_0'^*$ and $L^+ \subset L_0{}^*$.

To prove the reverse inclusions, we begin with a compact subinterval Δ of I. Let $\xi \in \mathscr{D}(L_{0,\Delta}'^*)$. Put $g = L_{0,\Delta}'^*(\xi)$. (It is clear that $L_{0,\Delta}'^*$ is an operator because $\mathscr{D}(L_0')$ is dense). We can choose $z \in \mathscr{D}(L_0{}^+)$ such that $l^+(z) = g$. Since $(z, l^+(z)) \in \mathscr{G}(L_{0,\Delta}'^*)$ and $g \in \mathscr{R}(L_{0,\Delta}'^*)$, $(z - \xi, 0) \in \mathscr{G}(L_{0,\Delta}'^*)$, i.e., $z - \xi \in \mathscr{N}(L_{0,\Delta}'^*)$. Thus (Theorem 3.10 (xi)) $z - \xi \in \mathscr{R}(L_{0,\Delta}')^\perp$. But then from the previous theorem $z - \xi \in \mathscr{N}(L_\Delta{}^+)$. Since $\mathscr{N}(L_\Delta{}^+) \subset AC^n(\Delta)$ and $z \in AC^n(\Delta)$, so does ξ. Thus $g = L_{0,\Delta}{}^*(\xi)$ is just $l^+(\xi)$. Since ξ was arbitrary, we conclude that $L_{0,\Delta}{}^* \subset L_\Delta{}^+$. Hence $L_{0,\Delta}'^* = L_\Delta{}^+$. To show that $L_{0,\Delta}{}^* = L_\Delta{}^+$, choose $\xi \in \mathscr{D}(L_{0,\Delta}{}^*)$. Repeating the reasoning above, we find that $z - \xi \in \mathscr{R}(L_{0,\Delta})^\perp$. Again from the previous theorem, $z - \xi \in \mathscr{N}(L_\Delta{}^+)$. We leave the remaining steps to the reader.

We now proceed to extend (i) and (ii) to the noncompact case. Let $z \in \mathscr{D}(L_0'^*)$ and $y \in \mathscr{D}(L_{0,\Delta}')$. Since $\mathscr{D}(L_{0,\Delta}') \subset \mathscr{D}(L_0')$, clearly

$$\langle L_0'(y), z\rangle = \langle y, L_0'^*(z)\rangle = \langle L_{0,\Delta}'(y), z_\Delta\rangle_\Delta.$$

From what we have just shown $z_\Delta \in \mathscr{D}(L_\Delta{}^+)$; i.e., $z \in AC^n(\Delta)$ and $(L_0'^*(z))_\Delta = l^+(z)$. Since Δ is arbitrary, $z \in AC^n(I)$ and $L_0'^*(z) = l^+(z)$. Therefore $L_0'^* \subset L^+$ and the two operators are equal. Repeating the argument with $z \in \mathscr{D}(L_0{}^*)$ shows that $L_0{}^* \subset L^+$. We omit the details. This completes the proof of the theorem.

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4.13. COROLLARY.
(i) $L_0^{+\prime *} = L$;
(ii) $L_0^{+*} = L$;
(iii) $L^{+*} = L_0$;
(iv) $L^* = L_0^+$;
(v) $\bar{L}_{0'} = L_0$;
(vi) $\bar{L}_0^{+\prime} = L_0^+$.

Proof. By interchanging L_0' and L_0 with $L_0^{+\prime}$ and L_0^+ and reproving Theorem 4.12 it is possible to show (i) and (ii). In a similar way (iii), (iv) and (v), (vi) are each dual pairs of statements. Since $L^+ \supset L_0^+$, $L^{+*} \subset L$; in particular $\mathscr{D}(L^{+*}) \subset \mathscr{D}(L)$. From Lemma 4.3 and the fact that Green's relation must be satisfied $\mathscr{D}(L^{+*}) \subset \mathscr{D}(L_0)$. Obviously, however, $L_0 \subset L^{+*}$. Therefore the two operators are equal. This proves (iii) and thus, by "duality", (iv). We have shown that $L_0'^* = L^+$, $L^{+*} = \overline{L_0'}$. Hence $\overline{L_0'} = L_0$ proving (v). The "dual" argument gives (vi).

4.14. COROLLARY. $\overline{\mathscr{R}(L)} = Y$; $\overline{\mathscr{R}(L^+)} = X'$.

Proof. Again by duality we need only deal with the first statement. From Corollary 4.13 (ii) and Theorem 3.10 (xi) it follows that $\mathscr{R}(L)^\perp = \mathscr{N}(L_0^+)$. Since $L_0^* = \overline{L_0^{+\prime}}$, $\mathscr{N}(L_0^+) = \overline{\mathscr{N}(L_0^{+\prime})}$. But $\mathscr{N}(L_0^{+\prime}) = \{0\}$. Therefore $\mathscr{N}(L_0^{+\prime}) = \{0\}^{\perp\perp} = \{0\}$. Hence

$$\begin{aligned}\overline{\mathscr{R}(L)} &= \mathscr{R}(L)^{\perp\perp}\\ &= \{0\}^\perp\\ &= Y,\end{aligned}$$

as was to be shown.

Much simpler to prove is

4.15. THEOREM. *If I is compact, $\mathscr{R}(L) = Y$ and $\mathscr{R}(L^+) = X'$.*

Proof. If $g(t, s)$ is the Green's function for $l(y)$ and $f(s) \in Y$,

$$\int_I g(t, s) f(s) ds$$

exists and is in $\mathscr{D}(L)$. The same argument (using $g\overline{(s, t)}$) works for the second assertion.

5. Regular generalized differential operators and their adjoints. We are finally in a position to study the adjoint theory of operators determined by the system (1.1) and (1.2).

Again throughout this section I, unless otherwise stated, is an arbitrary interval $[a, b]$ in the extended reals.

5.1. *Definition.* Let $\mathscr{L}$ denote the operator obtained by restricting $l(y)$ to the domain

$$\mathscr{D}(\mathscr{L}) = \{y : y \in AC^n(I) \cap X \cap \mathscr{N}[\mathscr{U}(y, \ldots, y^{n-1})];\ l(y) \in Y\}.$$

$\mathscr{L}$ (or any restriction thereof) is called a *generalized differential operator*; it is said to be *regular* or *singular* according to whether or not the system (1.1), (1.2) is a regular or singular s.b.v. problem.

5.2. *Definition.* If ω is the row dimension of the measure $\bar{w}$, define

$$\text{(5.1)}\qquad \mathscr{D}^* = \bigcup_{\phi \in \mathbf{C}^\omega} \Big\{ z : z \in BV^n(I) \cap Y';\ \text{for all } \tau, t (\tau \in (a, b) \text{ and } t > \tau \Rightarrow z(t) + \mathscr{B}^{-1}\bar{w}_s^*[\tau, t]\phi \in AC_n[\tau, b));\ l^+(z) + \frac{dw^{c*}}{du}\phi \in X' \Big\},$$

and the relation $\mathscr{L}^+ \subset Y' \times X'$ such that

$$\mathscr{G}(\mathscr{L}^+) = \left\{ \left(z, l^+(z) + \frac{dw^{c*}}{du}\phi \right) : z \in \mathscr{D}^* \right\}.$$

From the definition of $\mathscr{B}$ (see (4.1)) it is not difficult to show that an alternative characterization of $\mathscr{D}^*$ is

$$\text{(5.2)}\qquad \mathscr{D}^* = \bigcup_{\phi \in \mathbf{C}^\omega} \Big\{ z : z \in BV^n(I) \cap Y';\ \text{for all } \tau, t (\tau \in (a, b) \text{ and } t > \tau \Rightarrow l_j^+(z)[t] + w_{js}^*[\tau, t]\phi \in AC[\tau, b));\ l^+(z) + \frac{dw^{c*}}{du}\phi \subset X' \Big\}.$$

In the context of the s.b.v. problem (1.1), (1.2), $\mathscr{L}$ and $\mathscr{L}^+$ are the maximal operators (or relations) corresponding to L and L^+.

5.3. *Definition.* Let $\mathscr{H}_j, \mathscr{H} : AC^n(I) \times \mathscr{D}^* \to \mathbf{C}$ be the bilinear forms

$$\text{(5.3)}\qquad \mathscr{H}_j[y, z](s, t) = [l_{j-1}^{+*}(z) - \phi^* w_{js}[s]] y^{(n-j)}(s) - [l_{j-1}^{+*}(z) + \phi^* w_{js}[t]] y^{(n-j)}(t), \quad j = 1, \ldots, n,$$

where ϕ is the (not necessarily unique) element in $\mathbf{C}^\omega$ corresponding to z in (5.1).

From Lemma 4.3 we see that $\mathscr{H}[y, z](s, t)$ can be written:

$$([y, z](s) - \phi^*\bar{w}[s]\hat{y}[s]) - ([y, z](t) + \phi^*\bar{w}[t]\hat{y}(t))$$

Thus $\mathscr{H}$ can be thought of as a "generalized bilinear concomitant." With it we construct the Stieltjes analogues of L_0 and L_0^+.

5.4. *Definition.* Let $\mathscr{L}_0 \subset \mathscr{L}$ be the operator defined by $l(y)$ on

$$\text{(5.4)}\qquad \mathscr{D}(\mathscr{L}_0) = \{y : y \in \mathscr{D}(\mathscr{L});\ \mathscr{H}[y, z](b^-, a^+) = 0, \text{ for all } z \in \mathscr{D}^*\}.$$

5.5. *Definition.* Let

$$\text{(5.5)}\qquad \mathscr{D}_0^* = \bigcup_{\phi \in \mathbf{C}^\omega} \{z : z \in \mathscr{D}^*;\ \mathscr{H}[y, z](b^-, a^+) = 0, \text{ for all } y \in \mathscr{D}(\mathscr{L})\};$$

and $\mathscr{L}_0^+ : Y' \to X'$ be the relation such that

$$\mathscr{G}(\mathscr{L}_0^+) = \left\{\left(z, l^+(z) + \frac{dw^{c*}}{du}\phi\right) ; z \in \mathscr{D}_0^*\right\}.$$

Since $L_0^+ \subset \mathscr{L}_0^+$ and $L^+ \subset \mathscr{L}^+$ it is obvious that $\mathscr{L}_0^+$ and $\mathscr{L}^+$ are defined on total sets and thus dense. Under what conditions are $\mathscr{L}^+$ and $\mathscr{L}_0^+$ operators?

5.6. THEOREM. *$\mathscr{L}^+$ and $\mathscr{L}_0^+$ are operators if and only if*

$$\bigcap_{a<\tau<t<b} \mathscr{N}(\bar{w}_s^*[\tau, t]) \subset \mathscr{N}\left(\frac{dw^{c*}}{du}(t)\right) \text{ a.e.} \tag{5.6}$$

Proof. Clearly $\mathscr{L}^+$ and $\mathscr{L}_0^+$ can fail to be operators if and only if there exists distinct elements ϕ and ϕ' in $\mathbf{C}^\omega$ corresponding to a given z in $\mathscr{D}^*$ or $\mathscr{D}_0^*$. The definition of these domains and the singularity of the measure $\bar{w}_s$ forces

$$\phi - \phi' \in \bigcap_{a<\tau<t<b} \mathscr{N}(\bar{w}_s^*(\tau, t]).$$

If $\phi - \phi' \in \mathscr{N}(dw^{c*}/du\ (t))$ a.e., then $\mathscr{L}_0^+(z)$ and $\mathscr{L}^+(z)$ are single valued. On the other hand if the operators are single valued for z then

$$\frac{dw^{c*}}{du}(t)\phi = \frac{dw^{c*}}{du}(t)\phi' \text{ a.e.,}$$

which implies

$$\phi - \phi' \in \frac{dw^{c*}}{du}(t) \text{ a.e.}$$

5.7. COROLLARY.

$$\mathscr{L}^+(0) = \mathscr{L}_0^+(0) = \left\{\frac{dw^{c*}}{du}\phi ; \phi \in \bigcap_{a<\tau<t<b} \mathscr{N}(\bar{w}_s^*[\tau, t)\right\}.$$

Note that (5.6) is not unduly restrictive; for example it is automatically satisfied if the measures v_j are singular. On the other hand if the v_j have no singular parts, it is clear that $\mathscr{L}^+, \mathscr{L}_0^+$ are never operators.

5.8. LEMMA.

$$\langle l(y), z\rangle = \langle l^+(z), y\rangle + \mathscr{H}[y, x](b^-, a^+) - (\phi^*\mathscr{U}(y, \ldots, y^{(n-1)}))$$

for all $y \in \mathscr{D}(L)$, $\phi \in \mathbf{C}^\omega$ *such that* $\phi^*\mathscr{U}(y, \ldots, y^{(n-1)}) < \infty$.

Proof. Let $\Delta = [\alpha, \beta] \subset I$ be compact. Then

$$\int_\alpha^\beta y^{(n-j)*} l_j^+(z) dt = -\int_\alpha^\beta y^{(n-j)*}[l_{j-1}^+(z) + w_{js}^*[\alpha, t]\phi]' + \int_\alpha^\beta (a_j y^{(n-j)})^* z\, dt \tag{5.7}$$

(since w_{js}^* is singular, $w_{js}^{*\prime}[\alpha, t] = 0$ a.e.). We integrate the first term on the right by parts. Then the right side of (5.7) becomes

$$-y^{(n-j)*}(t)(l_{j-1}^{+}(z)+w_{js}^{*}[\alpha,t]\phi)|_{\alpha}^{\beta-}+\int_{\alpha}^{\beta}y^{(n-j+1)*}l_{j-1}^{+}(z)dt$$
$$+\int_{\alpha}^{\beta}y^{(n-j+1)*}w_{js}^{*}[\alpha,t]\phi dt+\int_{\alpha}^{\beta}(a_{j}y^{(n-j)})^{*}z\,dt.$$

Integrating the third term of the above expression by parts (and simplifying a bit) we get

$$(5.8)\qquad -\mathscr{H}_{j}^{*}[y,z](\beta^{-},\alpha^{+})-\int_{\alpha}^{\beta}(dw_{js}y^{n-j})^{*}\phi+\int_{\alpha}^{\beta}y^{(n-j+1)*}l_{j-1}^{+}(z)dt+\int_{\alpha}^{\beta}(a_{j}y^{(n-j)})^{*}z\,dt.$$

Combining (5.7) and (5.8) we derive

$$(5.9)\qquad \int_{\alpha}^{\beta}y^{(n-j)*}l_{j}^{+}(z)dt=-(\mathscr{H}_{j}^{*}[y,z](\beta^{-},\alpha^{+})-\int_{\alpha}^{\beta}(dw_{js}y^{(n-j)})^{*}\phi+\int_{\alpha}^{\beta}y^{(n-j+1)*}l_{j-1}^{+}(z)dt+\int_{\alpha}^{\beta}(a_{j}y^{(n-j)})^{*}z\,dt.$$

(5.9) is a recursion relation. Applying it successively for $j=n,\ldots,1$ the identity

$$(5.10)\qquad \int_{\alpha}^{\beta}y^{*}l^{+}(z)dt=-\sum_{j=1}^{n}\mathscr{H}_{j}[y,z](\beta^{-},\alpha^{+})+-\sum_{j=1}^{n}\int_{\alpha}^{\beta}(dw_{js}y^{(n-j)})^{*}\phi+\int_{\alpha}^{\beta}l(y)^{*}z\,dt$$

immediately follows. By (5.3)

$$\sum_{j=1}^{n}\mathscr{H}_{j}[y,z](t)\equiv\mathscr{H}[y,z](t).$$

As $\beta\to b$ and $\alpha\to a$ the integrals

$$\int_{a}^{b}y^{*}l^{+}(z)=\langle l^{+}(z),y\rangle\quad\text{and}$$

$$\int_{a}^{b}l(y)^{*}z\,dt=\langle z,l(y)\rangle$$

exist because $y\in\mathscr{D}(L)\subset X$, $l^{+}(z)\in X'$, $l(y)\in Y$, and $z\in\mathscr{D}^{*}\subset Y'$. The last term of (5.10) tends to $\mathscr{U}(y,\ldots,y^{n-1})^{*}\phi$. This forces

$$\mathscr{H}[y,z](b^{-},a^{+})$$

to exist. Taking transposes of both sides of (5.10) we arrive at the identity stated in the lemma.

From Lemma 5.8 combined with (5.4) and (5.5) we have immediately

$$(5.11)\quad \begin{aligned}\langle \mathscr{L}(y), z\rangle &= \langle y, \mathscr{L}_0^+(z)\rangle\\ \langle \mathscr{L}_0(y), z\rangle &= \langle y, \mathscr{L}^+(z)\rangle.\end{aligned}$$

5.9. THEOREM. *If $\mathscr{L}_0^+$, $\mathscr{L}^+$ are operators, they are closable.*

Proof. Since $\mathscr{D}^*$, $\mathscr{D}_0^*$ are total and $\mathscr{L}_0^+ \subset \mathscr{L}^*$, $\mathscr{L}^+ \subset \mathscr{L}_0^*$ by (5.11), the hypotheses of Lemma 4.8 are satisfied.

In the remainder of this section we assume that $\mathscr{L}$ is regular.

5.10. LEMMA. $\mathscr{L}^{+*} = \mathscr{L}_0$.

Proof. Since $\mathscr{L}^+ \supset L^+$, $\mathscr{L}^{+*} \subset L^{+*} \subset L$. Let $y \in \mathscr{D}(\mathscr{L}^{+*})$, and $z \in \mathscr{D}_0^*$. By the above inclusion, the definition of $\mathscr{D}_0^*$ (5.4), and Lemma 5.8, it follows that

$$\phi^*\mathscr{U}(y, \ldots, y^{(n-1)}) = 0.$$

Now ϕ is arbitrary over $\mathbf{C}^\omega$ since $\mathscr{N}(\mathscr{L}^+) \subset \mathscr{D}^*$ contains all functions of the form

$$\pi_n \left[\int_a^b (d\bar{w}(s)g(s, t))^*\phi\right]$$

(verification of this fact is a tedious computation similar to [**5**, Theorem 5.1]. Hence $\mathscr{U}(y, \ldots, y^{(n-1)})$ vanishes and $y \in (\mathscr{L}_0)$. The theorem is proved.

5.11 COROLLARY. $\mathscr{R}(\mathscr{L}^+)^\perp = \mathscr{N}(\mathscr{L}_0)$.

The interesting question, however, is to determine circumstances under which $\mathscr{L}^* = \mathscr{L}_0^+$ and $\mathscr{L}_0^* = \mathscr{L}^+$. In this section we show that this is always the case for *regular* operators; and in the next section we shall extend our results to a large subclass of singular operators.

5.12 LEMMA. *Let $T^+ : Y' \to X'$, $T : X \to Y$ be relations such that $(T^+)^* = T$ and $\mathscr{R}(T^+) = X'$. Then the following statements are equivalent:*
(i) *$\mathscr{N}(T^+)$ is closed;*
(ii) *$T^* = T^+$;*
(iii) *T^+ is closed;*
(iv) *$\mathscr{R}(T)^\perp = \mathscr{N}(T^+)$.*

Proof. The implications (ii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i) are trivial. (ii) implies (iii) by (vii) of Theorem 3.10. One easily checks that the closure of a relation implies the closure of its null space so (iii) implies (i). Finally, (i) implies (iv)

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because

$$(T^+)^* = T \text{ implies } \overline{T^+} = T^* = \mathscr{R}(T)^\perp = \mathscr{N}(\bar{T}^+) = \overline{\mathscr{N}(T^+)}.$$

Therefore to complete the argument it is required only to show (iv) implies (ii).

Let ξ be an arbitrary member of $\mathscr{D}(T^*)$ and put $g \in T^*(\xi)$. Since $\mathscr{R}(T^+) = X'$, there exists a function z in $\mathscr{D}(T^+)$ such that $g \in T^+(z)$. Because $T^+ \subset T^*$, it follows that $(z - \xi, 0) \in \mathscr{G}(T^*)$, that is, that $z - \xi \in \mathscr{N}(T^*)$. By Theorem 3.10 (x), $z - \xi \in \mathscr{R}(T)^\perp$. But then (iv) implies $z - \xi \in \mathscr{N}(T^+) \subset \mathscr{D}(T^*)$. Since $z \in \mathscr{D}(T^+)$, so does ξ. Therefore $(\xi, T(\xi)) \in T^+$. Since ξ was arbitrary, this shows that $T^* \subset T^+$. However, $(T^+)^* = T$ by assumption. Hence $T^* = \overline{T^+}$ (Theorem 3.10 (viii)), that is, $T^+ \subset T^*$. We conclude that the two operators are equal.

5.13. LEMMA. *$\operatorname{Dim} \mathscr{N}(\mathscr{L}^+) \leqq m(n + 1)$, where m (assumed $\geqq 1$) is the row dimension of the measure $\bar{w}$.*

Proof. Under the hypothesis that $\mathscr{L}$ is regular and m is the row dimension of $\bar{w}$,

$$\mathscr{N}(\mathscr{L}^+) = \left\{z : \hat{z} + \mathscr{B}^{-1} w_s^*[a, t]\phi \in AC(I); l^+(z) + \frac{dw^{c*}}{du}\phi = 0\right\},$$

where $\phi \in \mathbf{C}^m$.

Let

$$\mathscr{N}(\mathscr{L}_i^+) = \left\{z : \hat{z} + \mathscr{B}^{-1} w_s^*[a, t]\epsilon_i \in AC(I); l^+(z) + \frac{dw^{c*}}{du}\epsilon_i = 0\right\}, \quad i = 1, \ldots, m,$$

where ϵ_i is the ith unit basis vector in $\mathbf{C}^m$. We show first that $\dim \mathscr{N}(\mathscr{L}_i^+) \leqq n + 1$. To see this, note that if $z_1, z_2 \in \mathscr{N}(\mathscr{L}_i^+)$ with $z_1 \neq z_2$, then $\hat{z}_1 - \hat{z}_2 \in AC(I) \cap Y'$ (equivalently $z_1 - z_2 \in AC^n(I) \cap Y'$), and

(5.12) $\quad z_1 \equiv z_2 \bmod \mathscr{N}(\mathscr{L}_{i,c}^+)$,

where $\mathscr{L}_{i,c}^+$ is the operator determined by the restriction of $\mathscr{L}^+$ to $AC^n(I) \cap \mathscr{D}^*$. If $\dim \mathscr{N}(\mathscr{L}_i^+) > n + 1$, there exists $\{z_i\}_{i=1}^{n+2}$ linearly independent elements in $\mathscr{N}(\mathscr{L}_i^+)$. By (5.12) $z_1 - z_{n+2}, z_{n+1} - z_{n+2}, \ldots$ are all elements of $\mathscr{N}(\mathscr{L}_{i,c}^+)$. Since $\mathscr{L}_{i,c}^+$ is an "ordinary" differential operator, it follows from standard theory that $\dim \mathscr{N}(\mathscr{L}_{i,c}^+) \leqq n$ (the order of l^+). Therefore there exist constants $c_1, \ldots, c_{n+1}$ not all zero such that

$$\sum_{i=1}^{n+1} c_i(z_i - z_{n+2}) = \sum_{i=1}^{n} c_i z_i + \left(-\sum_{i=1}^{n+1} c_i\right) z_{n+2} = 0$$

contradicting the assumption that the $\{z_i\}$ were linearly independent. This

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contradiction implies that $\dim \mathcal{N}(\mathcal{L}_i^+) \leqq n + 1$. Because

$$\mathcal{N}(\mathcal{L}^+) = \sum_{i=1}^{m} \mathcal{N}(\mathcal{L}^+)_{(i)},$$

$\dim \mathcal{N}(\mathcal{L}^+) \leqq m(n+1)$ and the lemma is proved.

Note that in case $m = 0$, i.e., if no side condition is present, $\mathcal{L}^+ = L^+$ so that $\dim \mathcal{N}(\mathcal{L}^+) = \dim \mathcal{N}(\mathcal{L}^+) = n$.

5.14. THEOREM. *$\mathcal{L}^* = \mathcal{L}_0^+$ and $\mathcal{L}_0^* = \mathcal{L}^+$.*

Proof. Since I is compact, L^+ is surjective (Corollary 4.16) and thus so is $\mathcal{L}^+$. Also $\mathcal{L}^{+*} = \mathcal{L}_0$ (Lemma 5.10 (i)). By Lemma 5.13 $\mathcal{N}(\mathcal{L}^+)$ is finite dimensional and hence closed (Lemma 3.6). The second part of the theorem now follows from Lemma 5.12. We only sketch a proof for the first statement: observe that $\mathcal{L} \supset \mathcal{L}_0$ implies $\mathcal{L}^* \subset \mathcal{L}_0^* = \mathcal{L}^+$. Then using Lemma 5.8 we argue that $\mathcal{L}_0^* = \mathcal{L}_0^+$.

5.15 COROLLARY. *$\mathcal{L}^+$ and $\mathcal{L}_0^+$ are operators if and only if $\mathcal{L}$ and $\mathcal{L}_0$ are densely defined. Thus Theorem* 5.6 *yields a test for the density of $\mathcal{D}(\mathcal{L})$ and $\mathcal{D}(\mathcal{L}_0)$ in the $\perp$-topology.*

6. Locally regular operators. Here we extend Theorem 5.14 from the regular case to a certain class of singular operators.

6.1. *Definition.* A singular operator $\mathcal{L}$ is said to be *locally regular* on a set $S \subset I$ if on every compact subinterval $\Delta \subset S$ the set

$$\Gamma_\Delta = \{j : \text{supp. } \pi_j(\bar{w}) \cap \Delta \neq \{\phi\}\}$$

is finite.

The above definition is another way of saying that the side condition $\mathcal{U}(y, \ldots, y^{(n-1)})$ may be represented by measures which are locally finite dimensional.

If $\mathcal{L}$ is locally regular and $\Delta = [\alpha, \beta]$ is any compact subinterval of $I - S$, define $\mathcal{L}_{0,\Delta}$ as the restriction of $L_{0,\Delta}$ satisfying the side condition

$$\int_\Delta d\bar{w}_\Delta \hat{y} = 0, \tag{6.1}$$

where

$$\bar{w}_\Delta = [\pi_{j_1}(\bar{w}), \ldots, \pi_{j_k}(\bar{w})]^t$$

for $j_i \in \Gamma_\Delta$. Observe that $\mathcal{L}_{0,\Delta}$ is regular; in fact it is the minimal $\mathcal{L}$-type operator defined in Δ with Stieltjes side condition (6.1).

By viewing functions in $\mathcal{D}(L_{0,\Delta})$ as defined on I but having support on Δ, $\mathcal{L}_{0,\Delta}$ is naturally associated with a new operator $\mathcal{L}_{[0,\Delta]}$ which is a restriction of

$\mathscr{L}$. More precisely, $\mathscr{L}_{[0,\Delta]} \subset \mathscr{L}$ and

$$\mathscr{D}(\mathscr{L}_{[0,\Delta]}) = \{y : y \in \mathscr{D}(\mathscr{L}); \text{supp. } y \subset \Delta\}.$$

$$\text{Let } \frac{dw_\Delta{}^{c*}}{du} = \left[\pi_{j_1}\left(\frac{dw^{c*}}{du}\right), \ldots, \pi_{j_k}\left(\frac{dw^{c*}}{du}\right)\right].$$

Since $\mathscr{L}_{0,\Delta}$ is regular, using Theorem 5.14 it is not hard to see that

$$\mathscr{G}(\mathscr{L}_{0,\Delta}{}^*) = \left(z, l^+(z) + \frac{dw_\Delta{}^{c*}}{du}\,\phi\right),$$

where $z \in BV^n(\Delta)$ and $\hat{z}(t) + \mathscr{B}^{-1}\bar{w}_{\Delta_s}{}^*[\alpha,\ t]\phi \in AC^n(\Delta)$. Since $BV^n(\Delta) = (BV^n(I) \cap Y')_\Delta$ and $AC^n(\Delta) = (AC^n(I) \cap Y')_\Delta$ (cf. 3.5), $\mathscr{G}(\mathscr{L}_{0,\Delta}{}^*)$ is in fact the restriction of $\mathscr{G}(\mathscr{L}^+)$ to Δ. We call the relation determined by this graph the *restriction of* $\mathscr{L}^+$ *to* Δ, $(\mathscr{L}^+)_\Delta$.

Having thus characterized $\mathscr{L}_{0,\Delta}{}^*$, what is $\mathscr{L}_{[0,\Delta]}$?

6.2. LEMMA $\mathscr{L}_{[0,\Delta]} = \mathscr{L}_{[\Delta]}{}^+$, *where*

$$\mathscr{G}(\mathscr{L}_{[\Delta]}{}^+) = \{\mathscr{S} : \mathscr{S} \subset Y' \times X'; \mathscr{S}_\Delta = \mathscr{G}(\mathscr{L}^+)_\Delta\}.$$

Proof. We outline the main steps of the proof but omit details. Observe first of all that $\mathscr{L}_{[\Delta]}{}^+ \supset L^+$. Hence $(\mathscr{L}_{[\Delta]}{}^+)^* \subset L_0$. Let $y \in \mathscr{D}(L_0)$, $u \in \mathscr{D}(\mathscr{L}_{[\Delta]}{}^+)$ and $v \in \mathscr{R}(\mathscr{L}_{[\Delta]}{}^+)$. Then $\langle l(y), u\rangle = \langle y, v\rangle$. Now by definition

$$\mathscr{G}(\mathscr{L}_{[\Delta]}{}^+)_{I-\Delta} = (Y' \times X')_{I-\Delta}.$$

The ajdoint of this "relation" with respect to the pairing $\langle .,. \rangle_{I-\Delta}$ is trivial. Hence

$$y_{I-\Delta} = l(y)_{I-\Delta} = 0.$$

Since $y \in \mathscr{D}(\mathscr{L}_0) \subset AC^n(I)$,

$$\mathscr{H}[y, u_\Delta](\beta^-, \alpha^+) = 0.$$

This implies that $\hat{y}(\beta) = \hat{y}(\alpha) = 0$, in other words that supp. $y \subset \Delta$. Because the pairing is bi-additive we can write

$$\langle l(y), u\rangle_{I-\Delta} + \langle l(y), u\rangle_\Delta = \langle y, v\rangle_{I-\Delta} + \langle y, v\rangle_\Delta$$

or

$$\langle l(y), u\rangle_{I-\Delta} - \langle y, v\rangle_{I-\Delta} = \langle l(y), u\rangle_\Delta + \langle y, v\rangle_\Delta. \tag{6.2}$$

Since, restricted to Δ, $(u, v) \in \mathscr{G}(\mathscr{L}_{0,\Delta}{}^*)$, by Lemma 5.8 the right side of (6.2) can be replaced by

$$\mathscr{H}[y, u_\Delta](\beta^-, \alpha^+) - \int_\Delta d\bar{w}_\Delta \hat{y}. \tag{6.3}$$

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Then, because (6.3) and the left side of (6.2) are defined on *disjoint* intervals,

$$(6.4) \qquad \langle l(y), u\rangle_{I-\Delta} - \langle y, v\rangle_{I-\Delta} = 0,$$

$$(6.5) \qquad \mathscr{H}[y, u_\Delta](\beta^-, \alpha^+) - \int_\Delta d\bar{w}_\Delta \mathfrak{y} = 0.$$

Furthermore, from our characterization of $\mathscr{L}_{[\Delta]}{}^+$,

$$\mathscr{N}(\mathscr{L}_{[\Delta]}{}^+) = \{v : u \in Y',\ u_\Delta \in \mathscr{N}[(\mathscr{L}^+)_\Delta]\}.$$

Now $\mathscr{N}[(\mathscr{L}^+)_\Delta]$ is closed (Lemmas 3.6, 5.13). It is left as an exercise to see that $\mathscr{N}(\mathscr{L}_{[\Delta]}{}^+)$ is too. From (6.4), (6.5) and (6.6) we conclude that

$$\int_\Delta d\bar{w}_\Delta y = 0.$$

These facts together mean that $y \in \mathscr{D}(\mathscr{L}_{[0,\Delta]})$, proving that

$$\mathscr{D}(\mathscr{L}_{[0,\Delta]}) \supset \mathscr{D}(\mathscr{L}_{[\Delta]}{}^+)^*.$$

Because the reverse inclusion is trivial, $(\mathscr{L}_{[\Delta]}{}^+)^* = \mathscr{L}_{[0,\Delta]}$. To finish the lemma we observe that $\mathscr{L}_{[\Delta]}{}^+$ is surjective (trivial from its definition). Applying Lemma 5.14 the desired conclusion $\mathscr{L}_{[0,\Delta]}{}^* = \mathscr{L}_{[\Delta]}{}^+$ follows.

6.3. LEMMA. *Let S be a closed set of measure zero. Then*

$$\mathscr{G}(\mathscr{L}^+) = \bigcap_{\Delta \subset I-S} \mathscr{G}(\mathscr{L}_{[\Delta]}{}^+) \text{ a.e.}$$

Proof. Since

$$\mathscr{G}(\mathscr{L}_{[\Delta]}{}^+) = \{\mathscr{S} : \mathscr{S} \subset Y' \times X';\ \mathscr{S}_\Delta = \mathscr{G}(\mathscr{L}^+)_\Delta\},$$

and S is closed (so that $I - S$ is the union of open intervals), we deduce that

$$\mathscr{G}(\mathscr{L}^+) = \bigcap_{\Delta \subset I} \mathscr{G}(\mathscr{L}_{[\Delta]}{}^+) \subset \bigcap_{\Delta \subset I-S} \mathscr{G}(\mathscr{L}_{[\Delta]}{}^+).$$

Also $\mathscr{G}(\mathscr{L}^+)$ can differ from

$$\bigcap_{\Delta \subset I-S} \mathscr{G}(\mathscr{L}_{[\Delta]}{}^+)$$

at most on the points of S. Since S has measure zero, the lemma follows.

6.4. THEOREM. *If $\mathscr{L}$ is locally regular on $I - S$ where S is a closed set of measure zero, $\mathscr{L}_0{}^* = \mathscr{L}^+$ and $\mathscr{L}^* = \mathscr{L}_0{}^+$*

Proof. Let $\Delta \subset I - S$. Then $\mathscr{L}_{0,\Delta}$ is regular and $\mathscr{L}_{[0,\Delta]} \subset \mathscr{L}_0$. Thus $\mathscr{L}_{[0,\Delta]}{}^* \supset \mathscr{L}_0{}^*$. Since this holds for *all* Δ in $I - S$,

$$\mathscr{G}(\mathscr{L}_0{}^*) \subset \bigcap_{\Delta \subset I-S} \mathscr{G}(\mathscr{L}_{[0,\Delta]}{}^*).$$

By Lemma 6.2 $\mathscr{L}_{[0,\Delta]}{}^* = \mathscr{L}_{[\Delta]}{}^+$, and by Lemma 6.3

$$\mathscr{G}(\mathscr{L}^+) = \bigcap_{\Delta \subset I-S} \mathscr{G}(\mathscr{L}_{[\Delta]}{}^+).$$

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Therefore, $\mathscr{G}(\mathscr{L}_0^*) \subset \mathscr{G}(\mathscr{L}^+)$. Since the reverse inclusion is trivial (cf. 5.11), the first statement of the theorem is proved. The proof of the second statement is the same as in Theorem 5.16.

7. Conclusion. This paper has been a preliminary effort to develop some apparatus for a theory of singular operators under Stieltjes side conditions which would be analogous to the theory of singular ordinary differential operators. We have purposely chosen to work in the framework of "dual pairings" rather than in particular spaces to make the results as widely applicable as possible. This approach also has the advantage of showing that the whole theory of differential operators (ordinary or generalized) in whatever space rests on an appropriate Green's Relation and a principle related to the "Fundamental Lemma of the Calculus of Variations" (cf. Lemmas 5.8, 3.13 and Theorem 3.14).

We now close the paper with some examples designed to make the theory developed in the previous sections more concrete. We also develop a few applications to the theory of splines and suggest a few areas that might be profitably developed.

7.1. *Examples.*

1. Let

$$X = \mathscr{L}^p[0, 1],\ X' = \mathscr{L}^q[0, 1],$$
$$Y = \mathscr{L}^{p'}[0, 1],\ Y' = \mathscr{L}^{q'}[0, 1],$$

where $1 \leqq p,\ p' < \infty$, $1/p + 1/q = 1$, $1/p' + 1/q' = 1$. We choose $l(y) = y^{(n)}$ and

$$U_j(y, \ldots, y^{(n-1)}) = y(t_i),$$

where $T = \{t_i\}$ is an infinite set of points ($t_0 = 0$, $t_1 = 1$) with finitely many limit points $\{t_{i_j}\}$ in $[0, 1]$. Then in the sense of Definition 6.1 the operator $\mathscr{L}$ generated by this system is locally regular. Now

$$\bar{w} = \begin{bmatrix} u(t_0) & \cdots & u(t_i) & \cdots & u(t_1) \\ 0 & \cdots & 0 & \cdots & 0 \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}^t,$$

$$\frac{d\bar{w}_c}{du} = 0.$$

Also $\mathscr{B}$ is a diagonal matrix with entries $\alpha_{ij} = (-1)^j$. Hence from (5.2) it is

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readily seen that

$$l^+(z) = (-1)^n z^{(n)}$$

$$(7.1)\qquad \mathscr{D}^* = \{z : z \in \mathscr{C}^{(n-2)}[0, 1]; z^{(n-1)} \in BV[0, 1]; \\ z^{(n-1)}(t_i^+) - z^{(n-1)}(t_i^-) = \phi_i; z^{(n)} \in \mathscr{L}^q[0, 1]\},$$

for arbitrary scalar parameters ϕ_i. Since $\mathscr{L}$ is defined on a compact interval, by (5.5), by the definition of $\mathscr{H}[y, z]$ it is not difficult to see that

$$\mathscr{D}_0^* = \{z : z \in \mathscr{D}^*; z^{(n-1)}(0^+) = -\phi_0; z^{(n-1)}(1^-) = \phi_1\}.$$

By Theorem 6.4

$$\mathscr{L}^* = \mathscr{L}_0^+; \quad \mathscr{L}_0^* = \mathscr{L}^+.$$

Since $\mathscr{L}_0^+, \mathscr{L}^+$ are surjective (see Theorem 5.14), it follows from the Banach Closed Range Theorem (Theorem 3.11) that the ranges of $\mathscr{L}$ and $\mathscr{L}_0$ are closed.

What are $\mathscr{N}(\mathscr{L}^+)$ and $\mathscr{N}(\mathscr{L}_0^+)$? It is clear from (7.1) that $\mathscr{N}(\mathscr{L}^+)$ consists of functions in null space of $l^+(z)$ on $I - \{t_i\}$-in this case polynomials of degree $n - 1$-in $\mathscr{C}^{(n-2)}(I)$, and such that

$$z^{(n-1)}(t_i^+) - z^{(n-1)}(t_i^-) = \phi_i, t_i \in T - \{t_{ij}\}.$$

$\mathscr{N}(\mathscr{L}_0^+) \subset \mathscr{N}(\mathscr{L}^+)$ consists of functions satisfying the additional end point conditions

$$z^{(i)}(0^+) = z^{(i)}(1^-) = 0, i = 0, \ldots, n-2.$$

Functions of this type are also known as polynomial splines of order n.

2. Let $I = \mathbf{R}$, T be a countable point set, X, X', Y, and $l(y)$ as in the previous example. For each $t_i \in T$ assign an integer $1 \leqq j_i \leqq n - 1$. Put

$$U_j(y, \ldots, y^{(n-1)}) = (y(t_i), \ldots, y^{(j_i)}(t_i))^t = 0,$$

that is, the first j_i derivatives of y are set equal to zero. The measure $\bar{w}$ may be written as a matrix of infinite row dimension with the jth row having its first j_i entries the point evaluation measure $u(t_j)$ and the rest zero measures. In this case, $z \in \mathscr{C}^{(n-1)}$ on the intervals between successive t_j. At t_j however, z has $n - 1 - (j_i + 1)$ successive smooth derivatives. The null spaces $\mathscr{N}(\mathscr{L}^+), \mathscr{N}(\mathscr{L}_0^+)$ consist of polynomial splines with Hermite ties at the knot set $\{t_j\}$.

3. Let I be arbitrary. Take

$$X = AC^n(I),\ Y = \mathscr{L}(I),$$
$$X' = \mathscr{L}_0(I),\ Y' = AC_0^n(I),$$

and $l(y)$ a regular operator on I. Then pairings can be naturally defined on $X \times X'$ and $Y \times Y'$. If L, L^+, etc., are constructed according to Definitions

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4.1 and 4.4,

$$\mathscr{D}^+ = \{z : z \in AC_0^n(I); l(z) \in \mathscr{L}_0(I)\}.$$

Since every function in $\mathscr{D}^+$ has compact support

$$\lim_{s\to b} [y, z](s) - \lim_{t\to a} [y, z](t) = 0$$

for all y in $\mathscr{D}(L)$ and z in $\mathscr{D}^+$. This shows that $L = L_0$ and that $L^+ = L_0^+$. Moreover, $L = \bar{L}_0'$ and $L^+ = \bar{L}_0^{+\prime}$.

4. Let $I = [0, 2\pi]$, $X = X' = Y = Y' = \mathscr{L}^2(\mathbf{R})$, and $l(y) = y''$. Instead of imposing a singular side condition we let

$$(7.2) \qquad \mathscr{U}(y, \ldots, y^{(n-1)}) = \int_0^{2\pi} y \cos(t)dt = 0.$$

Here $\mathscr{D}^*$ possesses no interior point discontinuities. Indeed

$$\mathscr{D}^* = \mathscr{D}(L) = \{y : y \in AC^2(I); y'' \in \mathscr{L}^2(I)\}.$$

Since $\mathscr{D}(\mathscr{L})$ is not dense, $\mathscr{L}^+$ is a relation rather than an operator and

$$\mathscr{G}(\mathscr{L}^+) = \{(y, y'' - \sin t\phi) : y \in \mathscr{D}(L); \phi \in \mathbf{C}\}.$$

$\mathscr{N}(\mathscr{L}^+)$ is one dimensional and spanned by $\sin t$. The equation $y'' = f$ has a solution satisfying (7.2) if and only if

$$\int_0^{2\pi} f \sin t\, dt = 0.$$

In the first two examples above we have shown that $\mathscr{N}(\mathscr{L}^+)$, $\mathscr{N}(\mathscr{L}_0^+)$ are structurally "spline functions". There is a more striking connection, however, between the operators $\mathscr{L}$, $\mathscr{L}_0$ and their duals and the theory of splines. Splines are often (e.g. Jerome and Schumaker [**10**]) defined as solutions to the minimization of a differential operator $l(y)$ in a suitable Hilbert space norm subject to the constraint

$$(7.3) \qquad \mathscr{U}(y, \ldots, y^{(n-1)}) = r.$$

If I is compact, $X = Y = \mathscr{L}^2(I)$, and $\mathscr{L}_r$ denotes the nonlinear translate of $\mathscr{L}$ satisfying the nonhomogenous condition (7.3), then it can be shown that the equation

$$(7.4) \qquad \mathscr{L}^*\mathscr{L}_r[f] = 0$$

characterizes the solution (or "spline") f to the minimization problem. Using (7.4) it is then possible to analyze the local and global smoothness prospect in terms of the measure $\bar{w}$ and the coefficients of $l(y)$. This analysis gives many known results as special cases as well as new ones which would be hard to prove using existing methods (see [**6**] for the derivation of (7.4) and details).

To consider the noncompact case ($I = \mathbf{R}$, for example), we need to generalize slightly our notion of an adjoint operator. Let us put

$$\hat{\mathscr{D}}(L) = \{y : y \in AC^n(I); l(y) \in \mathscr{L}^2(\mathbf{R})\}.$$

There is no pairing now on $\mathscr{D}(L)$. However, an operator $\hat{\mathscr{L}}$ may still be defined on $\hat{\mathscr{D}}(L) \cap \mathscr{N}(\mathscr{U}(y, \ldots, y^{(n-1)}))$ with range in $\mathscr{L}^2(\mathbf{R})$. A "formal adjoint" $\hat{\mathscr{L}}_0^+ : BV^n(I) \to \mathscr{L}(I)$ can also be defined having the same structure as $\mathscr{L}_0^+$. Green's relation no longer holds. However, if we consider the $\mathscr{L}^2$ solutions of $\hat{\mathscr{L}}_0^+$, that is, the collection of functions z in $\mathscr{D}(\hat{\mathscr{L}}_0^+) \cap \mathscr{L}^2(\mathbf{R})$ satisfying $\mathscr{L}_0^+(z) = 0$, it is possible to repeat the argument of Lemma 5.8. We then obtain

$$(7.5) \qquad \langle Ly, z\rangle = -\phi^*\mathscr{U}(y, \ldots, y^{(n-1)}),$$

for all y in $\hat{\mathscr{D}}(L)$ such that $\phi^*\mathscr{U}(y, \ldots, y^{(n-1)})$ exists. (7.5) now implies by essentially the same argument used in Theorem 5.10 and elsewhere that

$$(7.6) \qquad \mathscr{R}(\hat{\mathscr{L}}) = (\mathscr{N}(\hat{\mathscr{L}}_0^+) \cap \mathscr{L}^2(\mathbf{R}))^{\perp}.$$

Thus $\hat{\mathscr{L}}_0^+$ may be considered the "adjoint" of $\mathscr{L}$ in the sense that the Fredholm Alternative (7.6) holds. Also $\mathscr{R}(\mathscr{L})$ is closed (in $\mathscr{L}^2(\mathbf{R})$ as well as with respect to the $\perp$ operation) and a version of (7.4) remains valid:

7.2. THEOREM. *A necessary and sufficient condition that f minimize $l(y)$ in $\mathscr{L}^2(\mathbf{R})$ for all $y \in \hat{\mathscr{D}}(L)$ satisfying condition* (7.3) *is that*

$$\hat{\mathscr{L}}^*\hat{\mathscr{L}}_r(f) = 0.$$

Saying this in a slightly different way,

$$\hat{\mathscr{L}}_r(f) = z,$$

where z is an $\mathscr{L}^2$ solution of $\hat{\mathscr{L}}_0^+$.

Since we have made no use of "local regularity," Theorem 7.2 is true for a general singular operator. As an illustration of the theorem it is clear that the solution of the minimization problem associated with Examples 1 and 2 (assuming $X = Y = \mathscr{L}^2(\mathbf{R})$), whether or not $\{t_i\}$ has limit points, is a polynomial spline of order $2n - 1$ with Hermite ties at the knots.

Similar minimization problems have also been treated by Golomb and Jerome [**9**].

7.3 *Remarks.* We have only scratched the surface in this paper. The nature of the spectrum, resolvents, eigenfunctions, and their dependence on the underlying spaces and measures etc. are all open problems. In the singular $\mathscr{L}^2$ case it would be particularly interesting to determine boundary values and to characterize the operators intermediate between $\mathscr{L}_0$ and $\mathscr{L}$ which are solvable. Also if $l(y) = l^+(y)$, $\mathscr{L}_0 \subset \mathscr{L}^+$ and $\mathscr{L}_0$ is formally symmetric in $\mathscr{L}^2$. By suitably restricting $\mathscr{L}^+$ one might obtain precise characterizations of the selfadjoint extensions of $\mathscr{L}_0$, generalizing the results of Coddington.

In another area, deeper applications to the theory of splines than those given here would be desirable.

In subsequent papers we hope to explore these directions in more detail.

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