

CORRECTIONS TO MY PAPER "ON THE STRUCTURE OF COMPLETE LOCAL RINGS"¹⁾

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The proof of Proposition 2 and that of Corollary to Proposition 3 in my previous paper "On the structure of complete local rings"¹⁾ are not correct.²⁾ Here we want to correct them.

Proof of Proposition 2.

Since the previous proof of Proposition 2 is valid when R/\mathfrak{m} is perfect, we treat only the case when R/\mathfrak{m} is not perfect.

Starting from $K_0 = R/\mathfrak{m}$, we obtain K_n ($n = 1, 2, \dots$) from K_{n-1} by adjoining all p -th roots of elements of K_{n-1} .

Definition. Let a local ring R_1 with maximal ideal \mathfrak{m}_1 be a subring of another local ring R_2 with maximal ideal \mathfrak{m}_2 . We say that R_2 is unramified with respect to R_1 if $\mathfrak{m}_2 = \mathfrak{m}_1 R_2$ and $\mathfrak{m}_2^k \cap R_1 = \mathfrak{m}_1^k$ for every positive integer k .

(1) Equal characteristic case.

We construct a sequence of local rings $R = R^{(0)} \subset R^{(1)} \subset \dots$ such that (1) $R^{(n)}$ is unramified with respect to R , (2) $R^{(n)}/\mathfrak{m}R^{(n)} = K_n$ and (3) $(R^{(n)})^p \subseteq R^{(n-1)}$.

The existence of such a sequence obviously follows from Zorn's Lemma if we observe that a monic polynomial $f(x)$ over a local ring, say R^* , is irreducible modulo its maximal ideal, then $R^*[x]/(f(x))$ is unramified with respect to R^* . (We may use the p -basis).

Let S be the union of all $R^{(n)}$. Then S is a local ring unramified with respect to R . For every element a^* of R/\mathfrak{m} , we construct a sequence (a_n) as follows: Let b_n be a representative of $a^{*p^{-n}}$ in R_n and set $a_n = b_n^{p^n}$. Then $a_n \in R$ and the limit a , which is the multiplicative representative of a^* , is in R . Thus we have Proposition 2 in this case.

(2) Unequal characteristic case.

As in above, we construct a sequence of local rings $R = R^{(0)} \subset R^{(1)} \subset \dots$ satisfying the above conditions (1) and (2) as follows: Let $\mathfrak{M} = \mathfrak{M}^{(0)}$ be a sys-

Received 26 July, 1952.

¹⁾ Nagoya Math. Journ. 1 (1950), pp. 63-70.

²⁾ Prof. I. S. Cohen (Massachusetts Institute of Technology, U.S.A.) pointed out the error of the proof of Proposition 2. I am grateful to him for his kind communication.

tem of representatives of a p -basis of $M/R/m$. Let $\mathfrak{M}^{(n)}$ be, when $\mathfrak{M}^{(n-1)}$ is already given, a set such that (1) for every element of $\mathfrak{M}^{(n-1)}$, $\mathfrak{M}^{(n)}$ contains one and only one p -th root of it and (2) $\mathfrak{M}^{(n)}$ consists merely of p -th roots of elements of $\mathfrak{M}^{(n-1)}$. Set $R^{(n)} = R[\mathfrak{M}^{(n)}]$.

Let S be the union of all $R^{(n)}$ and let \bar{S} be its completion. Then we see easily that the multiplicative representative of an arbitrary element of \mathfrak{M} is itself. Let R_0 be the absolutely unramified local ring which is generated by multiplicative representatives for R/m . Now, for our purpose, it is sufficient to prove the following.

Lemma. For every element a of R_0 , there exists an element a_n of R such that $a \equiv a_n \pmod{m^n \bar{S}}$.

Proof. For $n = 1$, our assertion is evident. We assume that this is true for $n = r$ and we prove the case $n = r + 1$. Since $R_0/(\mathfrak{p}) = R/m = (R_0/(\mathfrak{p}))^{p^r}(M)$, we can find an element $c_1 = \sum_i b_i^{p^r} m_i$ (where $b_i \in R_0$ and m_i is a monomial on elements of \mathfrak{M}) such that $a = c_1 + \mathfrak{p}c_2$ ($c_2 \in R_0$). Let b'_i be an element of R such that $b_i \equiv b'_i \pmod{m \bar{S}}$ and let c' be an element of R such that $c \equiv c' \pmod{m^r \bar{S}}$. Then $a_n = \sum_i b_i'^{p^r} m_i + \mathfrak{p}c'$ is a required element.

Proof of the Corollary to Proposition 3.

As is obvious, we have only to treat the case when \bar{R}_0 is of characteristic 0 and $p \neq 0$. Let B be a complete valuation ring (of characteristic 0) such that $B/(\mathfrak{p}) = \bar{R}_0/(\mathfrak{p})$.

(1) When $\bar{R}_0/(\mathfrak{p})$ is perfect:

Let $\{y_\lambda\}$ be a transcendental basis for $\bar{R}_0/(\mathfrak{p})$ over the prime field. Then we can find its multiplicative representative systems $\{y_\nu\}$, $\{z_\lambda\}$ in \bar{R}_0 and B . Then we can identify z_λ with y_λ . The same holds for $\{y_\lambda^{p^{-n}}\}$ and the similar identification allows the above identification of y_λ and z_λ . Therefore we may consider that \bar{R}_0 and B contains the same complete valuation ring B_1 such that its residue field is the least perfect field containing $\{y_\lambda\}$. Since $\bar{R}_0/(\mathfrak{p})$ is separably algebraic over $B_1/(\mathfrak{p})$ and since B is complete, we see that B and \bar{R}_0 are isomorphic over B_1 .

(2) General case:

Considering \bar{R}_0 as R in the above proof of Proposition 2, we construct the valuation ring \bar{S} . Let K be the largest perfect subfield of $\bar{R}_0/(\mathfrak{p})$. Then using multiplicative representatives for K in \bar{R}_0 and B , we see that \bar{R}_0 and B contain, respectively, complete valuation rings B_1 and B'_1 with the same residue field K . Then by (1), we may identify B'_1 with B_1 . Further, we may assume without loss of generality that \mathfrak{M} (= a system of representatives of p -basis in \bar{R}_0) is also contained in B . Then our assertion follows immediately by our above

proof of Proposition 2.

Errata:

p. 63, l. 21 and p. 64, l. 27; For “form” read “forms”, p. 66, Proposition 2;
For “with maximal ideal” read “with maximal ideal m ”, p. 69, Proposition 7;
For “With these conditions” read “If these conditions”.

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