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EXTENSION THEOREMS FOR SMOOTH FUNCTIONS ON REAL ANALYTIC SPACES AND QUOTIENTS BY LIE GROUPS AND SMOOTH STABILITY

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Abstract

Extension theorems are proved for smooth functions on a coherent real analytic space for which local defining functions exist which are finitely determined in the sense of J. Mather, (1968), and for smooth functions invariant under the action of a compact lie group G, thus providing the main step in the proof that smooth infinitesimal stability implies smooth stability in the appropriate categories. In addition the remaining step for the category of C^∞ G-manifolds of finite orbit type is filled in.

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1. Introduction

Propositions 1 and 2 of Mather (1969), the essential parts of J. Mather's proof that C^∞ infinitesimal stability implies C^∞ stability for a C^∞ proper map $f: N \to P$ where N and P are manifolds, will go through for a C^∞ proper map between other "C^∞-objects" N and P which are "embedded" in R^n and R^p if there are continuous mappings (extensions) from the spaces $C^\infty(N)$ and $C^\infty(P)$ of C^∞ functions on N and P to $C^\infty(R^n)$ and $C^\infty(R^p)$ which are right inverses for the restriction maps $C^\infty(R^n) \to C^\infty(N)$ and $C^\infty(R^p) \to C^\infty(P)$ (see Mather (1969) page 283). For coherent real analytic spaces, for which everywhere locally a suitable set of defining functions can be found which have a contact finitely determined germ in the sense of Mather (see condition F below), such extensions are found (Section 5); similarly for C^∞ G manifolds of finite orbit type and their rings of G invariant C^∞ functions where G is a compact lie group (Section 6) — the "embedding" here is a G invariant map into a Euclidean space with trivial G action. The remaining condition needed to complete the proof that C^∞ infinitesimal stability implies C^∞ stability for a proper G invariant C^∞ map between C^∞ G manifolds of finite orbit type, namely that the spaces of C^∞ G invariant vector fields on the manifolds and "along f" be finitely generated over the appropriate rings of G invariant smooth functions, follows easily from Hilbert's invariant theory and Schwarz (1975) (Section 7).

Proofs in Sections 4 and 5 depend heavily on results from Malgrange's book, *Ideals of Differentiable Functions*, and the approach in Section 6 follows Schwarz. In Section 2 results needed on the local structure of real analytic sets are quoted, in Section 3 the "finitely determined" condition F for a real analytic set X is introduced and a proof given that a certain semi-analytic local stratification of X is "homogeneous" and in Section 4 an extension theorem is proved for the C^∞ Taylor fields on a real analytic set to be used in Section 5.

NOTATION. $E(A)$ = space of Taylor fields on $A \subset R^n$. $I(B, A)$ = subspace of $E(A)$ consisting of those fields which vanish to infinite order on $B \subset A$. $C(A)$ = space of smooth ($\equiv C^\infty$) functions on A. All are given the Whitney C^∞ topology based on compact supports.

If G is a group acting on (A, B) then a G-suffix will denote the subspace consisiting of G invariant elements, for example $E_G(A)$ = space of G invariant Taylor fields on $A \subset R^n$ when the group G acts on R^n and $G(A) = A$. $I(\phi, A) = E(A)$; $C(R^n) = E(R^n) \to E(A)$, the restriction is onto; $C(A) = C(R^n)/(S(A))$, where $S(A)$ is the ideal in $C(R^n)$ of functions vanishing on A.

2. Local analytic sets

Let X be an analytic subset of R^n, that is, X is closed and at each point x of X there is a neighbourhood U and an analytic function $f: U \to R^p$ such that $X \cap U = f^{-1}(0)$. Then by the local parametrization theorem (see Malgrange (1966) in particular pages 51–53 and 57–59) for analytic sets for a suitable neighbourhood V of $x = 0$ in X, $X \cap V = \bigcup_{i=1}^{s} X_i$ where the X_i are closed subsets and $X_i' = X_i - D_i$, where $D_i = \bigcup_{j<i} (X_i \cap X_j)$, is open in X_i and a non-singular locally closed analytic submanifold of V such that

(1) $X_i' = \{x = (x', \phi_i(x')); x' \in U_i\}$ for a suitable product neighbourhood of 0 of the form $V_i' \times V_i''$ where V_i' and V_i'' are neighbourhoods of 0 in $k(i)$ and $l(i)$ dimensional euclidean spaces respectively and a function $\phi_i: U_i \to V_i''$ where U_i is an open set in V_i' whose boundary is

contained in $\delta_i = \Delta_i^{-1}(0)$ for a suitable analytic $\Delta_i : V'_i \to R$. $0 \in \delta_i$ and $D_i \subseteq \delta_i \times V''_i$. The coordinate functions of ϕ_i are quasi-holderian and together with all their derivatives are multipliers for $I(V'_i - U_i, V'_i)$.

(2) For a positive integer m if

$$Q_i = \{(x', x''); x' \in U_i, \|x'' - \phi_i(x')\| < d(x', \delta_i)^m\}$$

then Q_i and Q_j intersect for $i > j$ only if $D_i \cap X'_j \neq \phi$. X'_i is contained in an analytic subset of X of dimension $k(i)$ which intersects Q_i in X'_i and in which it is a "sheet".

(3) There are $C > 0$ and $\alpha > 0$ such that $|\Delta_i(x')| \geqq Cd(x', \delta_i)^\alpha$ for $x' \in V'_i$. Multiplication by Δ_i gives a homeomorphism of $I(V'_i - U_i, V'_i)$.

(4) There are $B > 0$, $\beta > 0$ such that $d(x', \delta_i) \geqq Bd(x, D_i)^\beta$ for $x = (x', x'') \in X \cap V$. It can also be assumed that each U_i is connected.

Let X be now a (closed) semianalytic subset of R^n, that is, X is closed and at each point of R^n there is a neighbourhood U such that $X \cap U$ is the union of finitely many sets each of which is given by finitely many analytic equations and inequalities. From Mather (1973), for example, (where a finer Whitney stratification is obtained) the above parametrisation result also holds in the semianalytic case. In what follows we are concerned only with the germ of X at x so that V will be variable.

3. Condition F

Following Mather, $L^N(n)$ denotes the group under composition of N-jets at 0 of diffeomorphisms of $(R^n, 0)$ and the contact group $K^n(n, p)$ is the group under composition of N-jets at 0 of diffeomorphisms H of $(R^n \times R^p, 0)$ of the form $H(x, y) = (h(x), h'_x(y))$ with $h'_x(0) = 0$ and $h(0) = 0$. Suppose that at the point $x \in X$, where X is an analytic subset of R^n, the function f can be chosen such that

(a) f is analytically contact finitely determined (see Mather (1968)), that is, there is $N_1 > 0$ such that each local analytic vector field along f at x vanishing to order N_1 at x is of the form $tf(\alpha) + f^*(m)\theta$ where α is a local analytic vector field vanishing at x in R^n, tf is given by right composition with the derivative of f, m is a local analytic function in R^p vanishing at the origin and θ is a local analytic vector field along f,

(b) if $f^{(N)}: V \to J^N(n, p)$ is the N-jet of f then for V sufficiently small

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$f^{(N)}(V)$ is contained in a neighbourhood of $f^{(N)}(x)$ which meets only finitely many orbits of the contact group $K^N(n,p)$, for an $N \geqq N_1$.

(a) and (b) together make up the condition F on X at x. We can also assume that at points of V the germs of f are ($\leqq N_1$) contact finitely determined as this condition is an open one. In the "nice" range of dimensions (Mather (1970)), which includes $p < 7$, $n \geqq p + 3$, almost all maps f satisfy condition F but in general the orbit structure of $J^N(n,p)$ is infinite over a subset of codimension $< n$ (Mather (1973)). The orbits are non-closed semi-algebraic and their closures semi-algebraic so that the inverse images of the orbits under $f^{(N)}$ give a semi-analytic stratification of V, that is, $V = \bigcup_{i=1}^{t} V_i$ with the V_i having properties like the X_i in Section 2 and each V_i' mapping to a single orbit in $J^N(n,p)$. By intersecting this stratification with that for X we may suppose that each X_i' (in the notation of 2) maps to a single orbit (and f is ($\leqq N_1$) determined at all points of $X \cap V$).

Let $W = X_i'$, $\phi = \phi_i$, $t = x'$, $k = k(i)$ and take $z \in W$ with coordinates translated to make $z = 0$. Let B^k be a disk in $U = U_i$ and let $i: B^k \to W$, where $i(t) = (x', \phi(x'))$ in previous notation, $i(0) = 0$. As $f^{(N)}$ maps W to a single $K^N(n,p)$ orbit, by choosing an analytic section of the map from $K^N(n,p)$ to the orbit (and taking B^k sufficiently small), there is an analytic $B^k \to K^N(n,p)$, $t \to k_t = (h_t, h'_{t,x})$, where $h_t \in L^N(n)$, $h'_{t,x} \in L^N(p)$, $t \in B^k$ and x is near 0 in R^n, such that

(A) $\quad f^{(N)}(i(t)) = k_t(f^{(N)}(0))[= h'_{t,z}(f^{(N)}(0)(z))$ at x, where $z = h_t^{-1}(x)]$.

Let C denote the set of analytic local maps from R^n to R^p sending 0 to 0 and let $g_t \in C$ be given by $g_t = k_t(f)$, where we identify elements of L^N in terms of the chosen coordinates with polynomial mappings of degree N. Let $B^k \times I \to C$, $(t,u) \to f_{t,u}$ be given by

$$f_{t,u}(x) = (1-u)f(x + i(t)) + ug_t(x).$$

$g_t(x)$ and $f(x + i(t))$ have the same N-jets (and $N \geqq N_1$) so that the vector field $g_t(x) - f(x + i(t))$ along $f_{t,u}$ for varying $(t,u) \in B^k \times I$ (obtained by differentiating $f_{t,u}$ with respect to u) is in the image of $(tf_{t,u}, \eta f_{t,u})$ where ηf denotes the map $M \times \theta \to \theta$, where M is the ideal of local analytic R^p functions vanishing at 0 and θ denotes the set of local analytic vector fields along f, given by $(m, \phi) \to f^*(m)\phi$ for $m \in M$, $\phi \in \theta$. Integrating the chosen vector fields, which can be supposed to depend analytically on (t,u) (see Mather (1968)) gives, for B^k sufficiently small, analytic local diffeomorphisms $H_{t,x}$ of R^p at 0 and G_t of R^n at 0 depending analytically on $t \in B^k$, $x \in R^n$ near 0 such that

$$H_{t,w}(f_{t,0}(w)) = f_{t,1}(x), \quad \text{where} \quad w = G_t^{-1}(x).$$

From (A) it follows that if $f^{(t)}$ denotes the germ of f near $i(t)$ then

$$f^{(t)}(x) = J_{t,w}(f^{(0)}(w)), \quad \text{where} \quad w = K_t^{-1}(x),$$

for K_t and $J_{t,x}$ locally defined analytic diffeomorphisms of R^n and R^p respectively, depending analytically on t and x, where K_t maps 0 to $i(t)$ and $J_{t,x}$ fixes 0. In particular K_t maps $f^{-1}(0)$ into itself.

Let Q' be a tubular neighbourhood of $W \mid i(B_1^k)$, where $B_1^k \subset B^k$ is a smaller ball, with transverse cells in the V'' direction. By choosing the transverse cells Q'_t suitably and sufficiently small it can be supposed that the map $L_t : Q'_0 \to Q'_t$ sending x'' to the point where $\bigcup_t K_t(x'')$ intersects Q'_t is an analytic diffeomorphism and so

PROPOSITION 1. *$L : B_1^k \times Q'_0 \to Q'$, where $L(t, x'') = L_t(x'')$, is an analytic diffeomorphism with $L(B_1^k \times (Q'_0 \cap X)) = Q' \cap X$.*
(That is the transverse germ of X along $W = X'_i$ is locally trivial or, in other words, the stratification is homogeneous.) Using the product structure functions in Q'_0 vanishing in $Q'_0 \cap X$ may be extended to functions vanishing on X in some neighbourhood (this is used in Section 5).

4. Extension for $E(X)$

In this section X denotes a closed semi-analytic set; the notation of Section 2 is used except that suffixes are dropped so that $V' = V'_i$, etcetera. For a sufficiently large even integer N and small enough $V' \subseteq \{x'; d(x', \delta) < 1\}$ we have $0 \leqq \Delta(x')^N \leqq d(x', \delta)^m$, for $x' \in V'$ where m is as in Section 2(2), and by (3) of Section 2 there is $p > 0$ such that $d(x', \delta)^p \leqq \Delta(x')^N$. If $g(x') = \Delta(x')^N$ then

$$d(x', \delta)^p \leqq g(x') \leqq d(x', \delta)^m \quad \text{for} \quad x' \in V'.$$

Multiplication by any power of g gives homeomorphisms of $I(V' - U, V')$, $I(\delta, Z)$, where $Z = U \cup \delta$, and of $I(D, Y)$, where $Y = W \cup D$ (see Malgrange (1966), chapter 4).

In terms of the coordinates (x', x'') elements of $I(D, Y)$ are the form $F = \{f^\lambda; \lambda \in N^n = N^k \times N^l\}$ (where N is the natural numbers) and each $f^\lambda(x)$ tends to 0 as x in Y tends to D faster than any positive power of $d(x, D)$. Malgrange (1966, page 64) defines a continuous (and obviously injective) map $\pi : I(D, Y) \to I_l(\delta, Z)$, where $I_l(\delta, Z)$ denotes the N^l-fold cartesian power of $I(\delta, Z)$, by $\{f^\lambda; \lambda \in N^n\} \to \{h^\mu(x') = f^{0 \times \mu}(x', \phi(x')); \mu \in N^l\}$. [Notice that

$0 \times \mu \in 0 \times N^l \subset N^k \times N^l = N^n$; (1) and (4) of Section 2 are used in showing that π is well defined and continuous].

Let

$$M = \{(x', x''); \|x''\| < g(x'), x' \in U\} \subset V' \times R^l.$$

$$\pi' : I(\delta \times 0;\ V' \times 0) \to I_l(\delta, Z),$$

where $V' \times 0 \subset V' \times R^l$, is defined similarly to π and is obviously continuous. Since the components of ϕ and their derivatives are multipliers for $I(\delta, Z)$ the map $I(V' \times R^l - M, V' \times R^l) \to I(V - Q, V)$ induced by $(x', x'') \to (x', x'' - \phi(x'))$ is continuous and so in order to construct a continuous "extension" $E: I(D, Y) \to I(V - Q, V)$ such that E composed with the restriction $I(V - Q, V) \to I(D, Y)$ is the identity it is sufficient to construct a continuous $E: I_l(\delta, Z) \to I(V' \times R^l - M, V' \times R^l)$ which when composed with the restriction to $I(\delta \times 0, Z \times 0)$ followed by π' gives the identity.

PROPOSITION 2. *$\pi: I(D, Y) \to I_l(\delta, Z)$ is a homeomorphism and there is a continuous $E: I(D, Y) \to I(V - Q, V)$ such that the restriction map $E(V) \to I(D, Y)$ is a left inverse for jE, where j is the inclusion $I(V - Q, V) \subseteq E(V)$.*

PROOF. Malgrange (1966, page 65) shows that π is onto: the method is easily refined to give a continuous inverse. Put $x = x'$, $y = x''$ for convenience. Let $B(r)$ be a smooth non-negative function $R \to R$ identically one in a neighbourhood of 0 and identically zero for $r \geqq 1$ and let $S(r)$ be a smooth monotone function defined on the nonnegative real line into R such that $S(r) \leqq \min(1/r, 1)$ for all r, $= 1$ for $r \leqq 1/2$ and $= 1/r$ for $r \geqq 2$. If $A = \{a_\mu(x); \mu \in N^l\} \in I_l(\delta, Z)$ the function $E(A)$ will be constructed of the form

$$\sum_{m=0}^{\infty} B\left(\frac{\|y\|}{g(x)\alpha_m(x)}\right)\left(\sum_{|\mu|=m} a_\mu(x) y^\mu\right) = \sum_{m=0}^{\infty} G_m,$$

say, for suitable smooth $\alpha_m(x)$, $0 < \alpha_m(x) \leqq 1$, chosen to depend on $\{a_\mu(x) : |\mu| = m\}$.

We can suppose that V' is compact in order to avoid taking k-norms with respect to varying compact sets. Evaluating a derivative of G_m of order n gives an expression which is a linear combination with integer coefficients of products of derivatives of orders $\leqq n$ of the a_μ (with $|\mu| = m$), $g\alpha_m$ and B with y^λ and negative powers of $g\alpha_m$ (the highest negative power being $-2n$), and if $n \leqq m - 1$ then

$$\|G_m\|_{(x,y),n} \leqq CL(\alpha_m) H(a_\mu) \|y\|^{m-n}, \quad \text{for} \quad \|y\| < g(x)\alpha_m(x)$$

$$\text{and} \quad = 0 \qquad\qquad \text{for} \quad \|y\| \geqq g(x)\alpha_m(x)$$

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where C is a constant independent of n and A, $H(a_\mu) = \Sigma_{|\mu|=m} \| a_\mu \|_{x,m}$, and $L(\alpha_m) = \Sigma_{i=0}^{m-1} \| \alpha_m \|_m^i$. If $\alpha'_m(x) = S(m 2^m CH(a_\mu))$ then

$$CH(a_\mu) \| y \|^{m-n} < \frac{2^{-m}}{m} \quad \text{for} \quad \| y \| < g(x) \alpha'_m(x), \text{ and 0 otherwise.}$$

If $\alpha_m(x) = \alpha'_m(x)/L(\alpha'_m)$ then $\alpha_m(x) < \alpha'_m(x)$ and $L(\alpha_m) < m$ (as $\| \alpha_m \|_m \leqq 1$) so that

$$\begin{aligned} \| G_m \|_{m-1} &< 2^{-m} \quad \text{for} \quad \| y \| < g(x) \alpha_m(x) \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Hence $E(A)$ converges to a smooth function on $V' \times R^l$. $E(A)$ depends continuously on $A \in E(Z)^{N^l}$. Since each $a_\mu(x) \in I_l(\delta, Z)$ and g^{-1} is a multiplier for $I_l(\delta, Z)$, any derivative of each partial sum $\Sigma_{m \leqq M} G_m$ tends to zero faster than any power of $d(x, \delta)$ as x approaches δ.Note that the method, when A is in $E(Z)^{N^l}$ and not necessarily in $I_l(\delta, Z)$ (for example A a constant function), will give $E(A)$ not necessarily smooth along δ but at least a multiplier for $I_l(\delta, Z)$.

An extension $E: E(V \cap X) \to E(V)$ which is a right inverse for the restriction can be constructed inductively over $Y_k = \bigcup_{i \leqq k} X_i$, $k = 1, \cdots, s$ (in the notation of §2) in the obvious way: if E_{k-1} is an extension for $E(Y_{k-1})$, E is the extension constructed above for $I(Y_{k-1}, Y_k)$ and $g \in E(Y_k)$ then put $E_k(g) = E(g - r'' E_{k-1} r'(g)) + E_{k-1}(r'(g))$ where r' is the restriction $E(Y_k) \to E(Y_{k-1})$ and r'' is the restriction $E(V) \to E(Y_k)$.

Proposition 3. *There is a continuous* $E: E(V \cap X) \to E(V)$ *such that* $r \circ E = 1$ *where* r *is the restriction* $E(V) \to E(V \cap X)$.

Corollary. *If* $\tilde{X}$ *is a semi-analytic subset of the set* X, $x \in X$ *and* V *is a suitably chosen neighbourhood of* x *then there is an extension* $E: I(V \cap \tilde{X}, V \cap X) \to I(V \cap \tilde{X}, V)$.

Proof. A local decomposition as in Section 2 can be made with $X_1 = \tilde{X} \cap V$ and then the above method gives the result.

Now by choosing a locally finite partition of unity on X with supports in sets of the form $V \cap X$ as above and "piecing together" in the usual way gives, since the topology is based on compact supports:

Theorem 1. *If* $X \supset \tilde{X}$ *are closed semi-analytic subsets of* R^n *then there is a continuous* $E: I(\tilde{X}, X) \to I(\tilde{X}, R^n)$ *which is a right inverse for the restriction* $I(\tilde{X}, R^n) \to I(\tilde{X}, X)$.

5. Extension for $C(X)$; X coherent of type F

Suppose now that X satisfies condition F on V and is also coherent on V. (Coherence means that there are analytic functions on V vanishing on X whose germs at each point of $X \cap V$ generate the ideal of local analytic functions vanishing on X.) Let W be as in Section 3 and $\pi: E(W) \to E(U)^{N'}$ as in Section 4. If G in $E(W)$ has x''-component G'' then $\pi(G) = T^*(G'')$ where $T(x', x'') = (x', x'' + \phi(x'))$ for $(x', x'') \in U \times V''$. G'' can be regarded as a collection of elements of $R_{x'}$, the power series ring in $x'' - \phi(x')$, for $x' \in U$: under T^* it is possible to identify R_x, with the power series ring in x''. $R_{x'} = \Pi_{m=0}^{\infty} R_{m,x'}$ where $R_{m,x'}$ is the homogeneous part of degree m of $R_{x'}$.

We let $S_{x'} \subset R_{x'}$ denote the formal local ideal of power series vanishing on $X \cap (x' \times V'')$ at $(x', \phi(x'))$: S_x, is the completion of the local ideal of smooth functions vanishing on $X \cap (x' \times V'')$ at $(x', \phi(x'))$ with respect to the local ring at $(x', \phi(x'))$ of smooth functions on $x' \times V''$. We let $S \subset I = I_l(\delta, Z)$ be the subspace of smooth functions vanishing on X and $I_m = \{\{g^\mu(x')\} \in I;\ g^\mu = 0 \text{ unless } |\mu| = m\}$, so that $I = \Pi_{m=0}^{\infty} I_m$. By Malgrange (1966, Chapter 6) and Proposition 1,

$$S = \{\{g^\mu(x')\} \in I;\ g^\mu(x') \in S_{x'} \text{ for all } x' \in U\}.$$

Further denote by $S_{m,x}$ the image of $S_{x'} \cap \Pi_{p \geq m} R_{p,x'}$ in $R_{m,x'}$ under the quotient map

$$\prod_{p \geq m} R_{p,x'} \to \Big(\prod_{p \geq m} R_{p,x'}\Big) \Big/ \Big(\prod_{p \geq m+1} R_{p,x'}\Big) \approx R_{m,x'}, \quad \text{for} \quad x' \in U, \quad m \in N$$

and let

$$S_m = \{\{g^\mu(x')\} \in I_m;\ g^\mu(x') \in S_{m,x'} \text{ for } |\mu| = m \text{ and } x' \in U\}.$$

The usual Euclidean product on R^l induces an inner product on the symmetric powers of R^l, and hence on $R_{m,x'}$ which is naturally isomorphic to the m-th symmetric power and we let

$$T_m = \{\{g^\mu(x')\} \in I_m;\ g^\mu(x') \text{ belongs to the orthogonal complement in } R_{m,x'} \text{ of } S_{m,x'} \text{ for } x' \in U\}.$$

The dimension of $S_{m,x'}$ is independent of $x' \in U$ by Proposition 1.

PROPOSITION 4. *$I_m \to S_m \times T_m$, given by orthogonal projection, is a homeomorphism.*

PROOF. For a function α on V we denote by $\alpha^{(m)}(x)$ the $m - x''$-jet of α at $x \in V$. We may assume that U is connected since for a suitable choice of V

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it has only finitely many components (Malgrange (1966)) and we need to show that orthogonal projection into S_m is continuous.

By coherence, given $x' \in U$ there are $n_m = \Sigma_{p \leqq m} \dim S_{p,x'}$ analytic functions defined on V and vanishing on X such that their $m - x''$-jets at $(x', \phi(x'))$ together span $\Pi_{p \leqq m} S_{p,x'}$ and by suitably choosing a collection A of n_m monomials of degrees less than or equal to m in the x'' it can be supposed that their A-components are independent at $(x', \phi(x'))$. The set of points in V where their A-components are dependent is of course an analytic subset of V and the intersection of this set with W is of dimension $< k$. Hence, by the descending chain condition for analytic germs, there are sets of analytic functions on V $\{\phi_{ij}, \psi_{ik}; j = 1, \cdots, n_{m-1}, k = 1, \cdots, n_m - n_{m-1}\}$ for i in a finite indexing set K, and collections A_i and B_i, $i \in K$, of n_m (respectively $n_m - n_{m-1}$) monomials in the x'' of degrees $\leqq m - 1$ (respectively m) such that $\bigcap_{i \in K}(W \cap Y_i) = \phi$, where $Y_i = Y_{i,1} \cup Y_{i,2}$ and

$$Y_{i,1} = \{x \in V;\ \text{the } A_i\text{-components of the } \phi_{ij}^{(m)}(x),\ j = 1, \cdots, n_{m-1}, \text{ are dependent}\}$$

$$Y_{i,2} = \{x \in V;\ \text{the } (A_i \cup B_i)\text{-components of the } \phi_{ij}^{(m)}(x) \text{ and } \psi_{ik}^{(m)}(x),\ j = 1, \cdots, n_{m-1}, k = 1, \cdots, n_m - n_{m-1}, \text{ are dependent}\}.$$

If the determinants of the matrices composed of the A_i (respectively $A_i \cup B_i$) components of the $\phi_{ij}^{(m)}(x)$ (respectively $\phi_{ij}^{(m)}(x)$ and $\psi_{ik}^{(m)}(x)$) are $\gamma_i(x)$ (respectively $\nu_i(x)$) for $x \in V$ then $Y_{i,1} = \gamma_i^{-1}(0)$ and $Y_{i,2} = \nu_i^{-1}(0)$.

Let $Y = \bigcap_i Y_i$, $D' = \delta \times V''$ and let $\pi: V' \times V'' \to V'$ denote the projection. If $H, J \subseteq V$ and p is a positive integer let $S(H, J, p) = \{x \in V; d(x, H) < d(x, J)^p\}$. By the separation property of the $Y_i \cup D'$ relative to their intersection $Y \cup D'$ and similarly for $Y \cup D'$ and $W \cup D'$ there are p' and p such that

$$(5.1)\qquad \begin{aligned} \bigcap_i S(Y_i \cup D', Y \cup D', p') &= \phi \\ &= S(Y \cup D', D', p) \cap S(W \cup D', D', p). \end{aligned}$$

On $\sim Y_i$ given $h(x) \in E(V)$ there are unique $h'_{it}(x)$ such that $h^{(m)}(x) - \Sigma h'_{it}(x)\phi_{it}^{(m)}(x)$, where $h^{(m)}(x)$ denotes the $m - x''$-jet of $h(x)$ at x, has A_i-component zero and each $\gamma_i h'_{it}$ has an everywhere analytic extension to V. In particular for $h = \psi_{ij}$ denote the corresponding h'_{it} by μ_{ijt}. The $\mu_{ijt}(x)$ are multipliers for $I(D', V)$ outside $S(Y \cup D', D', p)$ and $S(Y_i \cup D', Y \cup D', p')$ (see Malgrange (1966) page 59). Let

$$\beta_{ij}(x) = \psi_{ij}^{(m)}(x) - \sum_t \lambda_{ijt}(x')\phi_{it}^{(m)}(x), \quad \text{where} \quad \lambda_{ijt}(x') = \mu_{ijt}(x', \phi(x')).$$

λ_{ijt} is a multiplier for $I(\delta, Z)$ by (5.1) on $\sim U_i$ where $U_i = \pi(S(Y_i \cup D', Y \cup D', p) \cap W)$. If $\alpha_{ij}(x') = \beta_{ij}(x', \phi(x'))$ then $\{\alpha_{ij}(x'); j = 1, \cdots\}$ spans $S_{m,x'}$ at each point $x' \in \sim U_i$. The $\alpha_{ij}(x')$ are also multipliers on $\sim U_i$ for $I(\delta, Z)$. When the standard monomial generators of degree m (that is fields $\{g^{\mu}\}$ such that $g^{\mu}(x') = 0$ except for one λ of degree m and $g^{\lambda}(x') = 1$) are projected orthogonally onto S_m and the answer expressed in terms of the α_{ij} the coefficients are multipliers for $I(\delta, Z)$ on $U - U_i$. As $\bigcup_i (U - U_i) = U$ orthogonal projection of I_m to S_m is continuous.

COROLLARY 1. *I is homeomorphic to* $S \oplus T$ *where* $T = \Pi_{m=0}^{\infty} T_m$.

PROOF. S is a closed subspace of I (by Malgrange's result) and obviously T is also. $\sigma_m : I \to T_m$ is defined inductively such that $\tau_m(x) = [(\sigma_0, \cdots, \sigma_m)(x) - x]$ is in S modulo $\Pi_{p>m} I_p$. If $z = (\tau_m(x) - y)$ belongs to $\Pi_{p>m} I_p$ for some y in S then $\sigma_{m+1}(x)$ is the projection of z modulo $\Pi_{p>m+1} I_p$ into T_{m+1}. $\sigma = (\sigma_0, \sigma_1, \cdots): I \to T$ is continuous and $(\sigma(x) - x)$ belongs to S by Malgrange's result.

COROLLARY 2. *There is a continuous map* $E': S \to I(V - Q, V) \cap J$ *which is a right inverse for the restriction, where J denotes the ideal of* C^{∞} *functions vanishing on X.*

PROOF. By the note to Proposition 2 there are functions χ_i' on V with values between 0 and 1 satisfying (in the notation of Proposition 4)

$$\chi_i'(x) = 1 \quad \text{on} \quad S(Y_i \cup D', Y \cup D', q)$$
$$= 0 \quad \text{outside} \quad S(Y_i \cup D', Y \cup D', p')$$

for some $q > p'$ and which are multipliers for $I(Y \cup D', V)$. Outside $S(Y \cup D', D', p)$ and so on $S(W \cup D', D', p)$ they are multipliers for $I(D', V)$. Let $\{\chi_i\}$ be the partition of unity on $V - (Y \cup D')$ corresponding to the set $\{1 - \chi_i'\}$.

From the proof of Proposition 4 the derivatives of $\beta_{ij}(x)$ of order $n = 0, 1, \cdots, m-1$, are bounded by an expression of the form

$$\|x'' - \phi(x')\|^{m-n} d(x, D')^{-N(m)} \text{ on } V'' \times (U - V_i), \quad \text{where}$$

(5.2) $$V_i = \pi(S(Y_i \cup D', Y \cup D', q) \cap W),$$
and if $n \geqq m$ by $d(x, D')^{-N(n)}$ for suitable integers $N(n)$.

Let f be in $S \cap \Pi_{p \geqq m} I_m$ (that is an x''-Taylor field on W vanishing to order $m-1$, formally vanishing on the germ of X along W and flat along D') and with support in $\sim V_i$. Then there are unique $f_j(x') \in I(\delta, Z)$ vanishing outside V_i such that $f - \Sigma_j \alpha_{ij} f_j$ vanishes to order m. In the notation of the proof of

Proposition 2 ($x = x'$, $y = x''$ etcetera) by (5.2), since on Z, $d(x', \delta)^{-1}$ is maximized by some power of g^{-1} which is a multiplier for $I(\delta, Z)$, $\alpha_m(x)$ can again be chosen to depend continuously on the f_j such that

$$G_m(x, y) = \sum_j B\left(\frac{\|y\|}{g(x)\alpha_m(x)}\right) \beta_{ij}(x, y + \phi(x)) f_j(x)$$

satisfies $\|G_m\|_{m-1} < 1/2^m |K|$. Generally, when f has unrestricted support, $f(x')$ can be expressed as a sum $\Sigma_i \chi_i(x', \phi(x'))f(x')$, and then G_m can be found for the i-th function in the sum and summing gives G_m for f satisfying $\|G_m\|_{m-1} < 1/2^m$. $G_m(x', x'' - \phi(x'))$ is in J. Finally if h is in S then $E'(h) = \Sigma_{M=0}^{\infty} G_M$ where the G_M are defined inductively such that the Taylor field along $U \times 0$ of $h - \Sigma_{M<m} G_M$ is in $\Pi_{M \geqq m} I_M$. G_m is then constructed as above with $f = h - \Sigma_{M<m} G_M$ along $U \times 0$.

PROPOSITION 5. *There is a continuous $E: C(X \cap V) \to C(V)$ which is a right inverse for the restriction.*

PROOF. As in Proposition 3, E is constructed inductively over $Y_k = \bigcup_{i \leq k} X_i$. Suppose that $E_{k-1}: E(Y_{k-1}) \to E(V)$ is such that (a): $(r'r''E_{k-1}(g) - g)$ belongs to $S(Y_{k-1})$ for all g in $E(Y_{k-1})$, where $r'': E(V) \to E(Y_k)$ and $r': E(Y_k) \to E(Y_{k-1})$ are the restrictions and $S(Y_{k-1})$ denotes the set of restrictions to Y_{k-1} of the set J of functions vanishing on X, and (b): $E_{k-1}(g)$ depends only on g modulo $S(Y_{k-1})$. We will now find a similar E_k.

The last corollary gives (as in the proof of Proposition 3) $E': S(Y_{k-1}) \to J$, a right inverse for the restriction. Let $\pi: I(Y_{k-1}, Y_k) \to T$ be the projection given by the first corollary (where T is a complement for $S(Y_k) \cap I(Y_{k-1}, Y_k)$) and let $E: I(Y_{k-1}, Y_k) \to E(V)$ be the original extension. If g is a Taylor field on Y_k let $E_k(g) = E_{k-1}(r'(g)) + E(\pi(h - r''E'r'(h)))$, where $h = g - r''E_{k-1}r'(g)$. [Compare with Proposition 3: the "correcting" term $r''E'r'(h)$ is needed since E_{k-1} only extends $r'(g)$ modulo $S(Y_{k-1})$.]

THEOREM 2. *If X is a coherent analytic subset of R^n satisfying F everywhere there exists a continuous $E: C(X) \to C(R^n)$ which is a right inverse for the restriction.*

(There is also a relative version to Theorem 2.) [Added in revision: the assumption of coherence in Theorem 2 is not needed. The assumption of condition F can also be removed (see G. Wells (1977)), so that infinitesimal stability implies stability in general for proper smooth maps of closed real semianalytic subsets of R^m into a manifold.] Mather's proof (1969) that infinitesimal stability implies stability now goes over immediately using Theorem 2 to proper smooth mappings of X into a manifold (for further details see Wells (preprint)).

6. G invariant smooth functions

The general reference to the next few paragraphs is Schwarz (1975). Let G be a compact lie group acting orthogonally on R^n and let $\phi: R^n \to R^N$ be a polynomial map, with homogeneous polynomial coordinate functions, into a trivial G-space R^N such that $\phi^* P(R^N) = P_G(R^n)$, where $P(R^N)$ is the ring of polynomials on R^N and $P_G(R^n)$ is the ring of G invariant polynomials on R^n. The existence of ϕ is given by Hilbert's invariant theory — see Dieudonné (1970). $\phi(R^n)$ is a semialgebraic set which it will turn out has a stratification with homogeneous strata. Let H be the stabilizer at $x \in R^n$ with orbit $V = Gx$ and let W be the set of points in orbits containing points with stabilizer H; W is a locally closed submanifold of R^n. Let E be the subspace of the tangent space at x of R^n perpendicular to V. H maps V into itself fixing x and so maps E to itself. We will identify tangent spaces at points in Euclidean space with affine subspaces in the usual way. If $E = E_1 \oplus E_2$ (orthogonal decomposition), where E_2 is a linear subspace of E fixed by H, then by taking sufficiently small (closed) balls $B_i \subset E_i$ at the origins, $i = 1, 2$, we can suppose that B_2 is a subset of $W \cap B$, where $B = B_1 \times B_2$, and $G(B)$ is a tubular neighbourhood over $G(B_2)$ with transverse cells of the form $g(B_1 \times x)$ for $g \in G$, $x \in B_2$. If E_2 is chosen as large as possible and B sufficiently small then $B_2 = W \cap B$.

As is well known G invariant functions on $G(B)$ can be identified with H invariant functions on B. By choosing B sufficiently small the restrictions ϕ'_i to $B \subset E$ of the functions ϕ_i generate the H invariant analytic functions of E restricted to B (Schwarz). Let $(x_1, \cdots, x_q)$, respectively $(y_1, \cdots, y_m)$, be linear coordinates on E_1, respectively E_2, let $\psi_1(x), \cdots, \psi_p(x)$ be a minimal generating set consisting of homogeneous polynomials for $P_H(E_1)$, and let P_i, $i = 1, 2, \cdots$ be monomials in the ψ_j which are homogeneous in the x such that the $P_i(\psi_j(x))$ of degree k, $k = 0, 1, 2, \cdots$ form a linear basis for the space of H invariant polynomials of degree k on E_1. $\{y_1, \cdots, y_m, \psi_1, \cdots, \psi_p\}$ is a minimal generating set for $P_H(E)$ and $\{y^\alpha P_i(\psi_j);\ \text{degree } (P_i(\psi_j(x))) + |\alpha| = k\}$ is a linear basis for the homogeneous polynomials in $P_H(E)$ of degree k. The ϕ'_i can be expressed uniquely in terms of the $y^\alpha P_i(\psi)$ and hence in the form $\phi'_i(x, y) = F_i(y, \psi(x))$ where the $F_i(y, z)$, $y \in R^m$, $z \in R^p$, are polynomials and F is an embedding on $B' = B_2 \times B_3 \subseteq R^m \times R^p$ into R^N if B_2 is sufficiently small and for a choice of ball B_3 at 0 in R^p (Schwarz). Then also

$$\phi' | (\phi'')^{-1}(B') = F \circ \phi'', \tag{6.1}$$

where ϕ'' is given by $(x, y) \to (y, \psi(x))$. By choosing sufficiently small balls (and by the existence of tubular neighbourhoods) there is an embedding

$F': B_2 \times B_3 \times B_4 \to R^N$, where B_4 is a ball at 0 in R^{N-m-p}, extending $F = F'|B_2 \times B_3 \times 0$ and such that $F'(B_2 \times B_3 \times B_4) \cap \phi(R^n) = F(B_2 \times B_3)$. By changing coordinates on R^p if necessary we can suppose that $\psi_1(x) = \Sigma x_i^2$. Let ψ_i have degree k_i and let $\alpha: R_+^p(z) \to R^p(z')$ be given by $z_i' = z_i / z_1^{\frac{1}{2}(k_i - 1)}$, where $R_+^p(z) = \{z = (z_1, \cdots, z_p) \in R^p; z_1 > 0\}$. $\alpha\psi$ maps lines through the origin in R^q to lines through the origin in R^p. If $B^q \subset R^q$ is the unit ball at the origin then there are constants $A_i > 0$ $i = 1, \cdots, p$ such that $|\psi_i(x)| < A_i \psi_1^{\frac{1}{2}k_i}(x)$ for $x \in B^q - \{0\}$ so that $\psi(B^q) \subseteq Y = \{z; z_1 \geqq 0, |z_i| \leqq A_i z_1^{\frac{1}{2}k_i}\}$. Clearly $\alpha(Y) = \{z'; z_1' > 0, |z_i'| \leqq A_i z_1'\}$; we denote $\alpha(Y)$ by Y'.

α induces a homeomorphism $\alpha^*: I(R^p - Y', R^p) \to I(R^p - Y, R^p)$ because $(\alpha^{-1})^*$ is obviously continuous and so too is α^* since if $f \in I(R^p - Y', R^p)$ each $\partial^\beta f / \partial z_\beta'(z')$ tends to 0 faster than any power of z_1' so that $\partial^\beta f / \partial z_\beta'(\alpha(z))$ tends to 0 as $z_1 = (z_1')^2$ tends to 0 faster than any power of z_1 and hence by the formula for $\partial^\beta / \partial z_\beta (f \circ \alpha)$ the result follows. The following diagram of G invariant maps commutes.

$$\begin{array}{ccccc} R^q & \xrightarrow{\psi} & R_+^p & \xrightarrow{\alpha} & R^p \\ h_1 \uparrow & & & & \uparrow h_2 \\ S^{q-1} \times [0, \infty) & & \xrightarrow{\lambda} & & S^{p-1} \times [0, \infty) \end{array}$$

where S^{q-1} and S^{p-1} are the unit spheres in R^q and R^p respectivley, G acts trivially on R^p and $[0, \infty)$, h_2 is the mapping given by "polar coordinates" on R^p and h_1 is chosen such that λ is a product map and level preserving with respect to projections onto $[0, \infty)$. h_2 and h_1 (using a similar argument to that above for α) induce homeomorphisms

$$h_1^*: I_H(0, R^q) \stackrel{\approx}{\to} I_H(S^{q-1} \times 0, S^{q-1} \times [0, \infty)) \quad \text{and}$$

$$h_2^*: I(0, R^p) \stackrel{\approx}{\to} I(S^{p-1} \times 0, S^{p-1} \times [0, \infty)).$$

The restrictions $I(R^p - Y, R^p) \to I(0, \psi(B^q))$ and $I(R^p - Y', R^p) \to I(0, \alpha\psi(B^q))$ are both onto (Malgrange).

LEMMA 1. *Let* $r, s, t, u, v \in R^+$ *with* $u < s < t < v$. *If there is a continuous* $K': I_H(S^{q-1} \times \{s, t\}, S^{q-1} \times [s, t]) \to E(S^{p-1} \times [u, v])$, *a right inverse for* λ^*, *then there exists a continuous*

$$K: I_H(S^{q-1} \times \{0, r\}, S^{q-1} \times [0, r]) \to I(S^{p-1} \times \{0, r\}, S^{p-1} \times [0, r])$$

which is a right inverse for λ^* *and such that if the image of* K' *is contained in* $I(h_2^{-1}(R^p - Y'), S^{p-1} \times [0, \infty))$ *then so too is the image of* K.

PROOF. It is sufficient to show that K' can be modified to map into $I(S^{p-1} \times \{s, t\}, S^{p-1} \times [s, t])$ which is isomorphic to $I(S^{p-1} \times \{0, r\}, S^{p-1} \times [0, r])$

under a linear homeomorphism of $[s,t]$ with $[0,r]$. In Section 4 a continuous extension $A: E(S^{p-1}\times s)\to E(S^{p-1}\times I_1)$ was described (where I_1 is a small interval containing s) which has the property that if $\beta\in E(S^{p-1}\times s)$ arises from a function which is zero on the product of a set $X\subset S^{p-1}$ with one of the closed subintervals of I_1 into which s divides I_1 then $A(\beta)$ is zero on $X\times I_1$. $X=\lambda(S^{q-1}\times s)$ here. If $K'(\gamma)$, $\gamma\in I_H(S^{q-1}\times\{s,t\}, S^{q-1}\times[s,t])$, restricts to $\beta\in E(S^{p-1}\times s)$ then subtracting $A(\beta)$ from $F(\gamma)$ (and similarly for t) gives F of the required form. As $h_2^{-1}(Y'-\{0\})\approx[0,\infty)\times(h_2^{-1}(Y')\cap(S^{p-1}\times 1))$ the rest of the lemma follows automatically.

If $U\subset R^n$ is a submanifold with boundary (and possibly corners) of dimenison n then it is well known (see Mather (1969)) that the restriction $E(R^n)\to E(U)$ splits continuously on the right (for convenience we say that $E(U)$ embeds in $E(R^n)$). If U is G invariant then using the averaging process for functions on R^n $[E(R^n)\to E_G(R^n)$, $f(x)\to\int_G f(g(x))dg$, where dg denotes the normalized Haar measure on $G]$ it follows that $E_G(U)$ embeds in $E_G(R^n)$ also.

COROLLARY. *If there is a continuous right inverse for* ψ^*,

$$K'': I_H(\partial\overline{(B^q-B_1^q)}, \overline{(B^q-B_1^q)})\to E(R^p),$$

where B^q *is a ball at* 0 *and* B_1^q *is a sufficiently small ball at* 0 *in* R^q, *then the conclusion of Lemma* 1 *holds.*

PROOF. If B_1^q is sufficiently small $(B^q-B_1^q)-\partial B^q$ contains a set of the form $h_1(S^{q-1}\times[s,t])$. Since $I_H(h_1(S^{q-1}\times\{s,t\}), h_1(S^{q-1}\times[s,t]))$ embeds in $I_H(\partial\overline{(B^q-B_1^q)}, \overline{(B^q-B_1^q)})$ and since by multiplying by a smooth bump function which is zero in a suitable neighbourhood of 0 in R^p and outside Y it can be assumed that K'' maps into $I(R^p-Y, R^p)$ the corollary follows immediately (taking account of the remarks preceding the lemma). Note that Lemma 1 and its corollary have a parametrized form, where all spaces are multiplied by a fixed smooth manifold M. $\psi: R^q\to R^p$ is replaced by $(\psi\times id): R^q\times M\to R^p\times M$ etcetera and the deduction is $K: I_H(S^{q-1}\times\{0,r\}\times M, S^{q-1}\times[0,r]\times M)\to$ etcetera. The proof is the same.

LEMMA 2. *There exists a continuous* $J: E_H(0\times M)\to E(R^p\times M)$, *where* 0 *is the zero of* R^q, *which is a right inverse for* $(\psi\times id)^*$.

PROOF. Any $\beta\in E_H(0\times M)$ is of the form $\{f_i(y)P_i(\psi_j(x))\}$ where the $f_i(y)$ are in $C^\infty(M)$ and are uniquely determined. Let $J'(\beta)$ be the element $\{f_i(y)P_i(z_j)\}$ of $E(0'\times M)$ where $0'$ is the zero of R^p. J is the composition of J' with the extension $E(0'\times M)\to E(R^p\times M)$ of §4.

COROLLARY. *Under the hypothesis of the corollary to Lemma 1 there is a continuous map from* $I_H(h_1(S^{q-1}\times r), h_1(S^{q-1}\times[0,r]))$ *to* $I(\alpha^{-1}h_2(S^{p-1}\times r), \alpha^{-1}h_2(S^{p-1}\times[0,r]))$ *which is a right inverse for* ψ^*.

PROOF. The corollary to Lemma 1 gives a right inverse for

$$\psi^*: I(\alpha^{-1}h_2(S^{p-1}\times\{0,r\}), \alpha^{-1}h_2(S^{p-1}\times[0,r])\to I_H(h_1(S^{q-1}\times\{0,r\}), h_1(S^{q-1}\times[0,r])$$

and piecing together with J gives the result.
Note that this corollary again holds in parametrized form.

LEMMA 3. *Let* W' *be a* G *invariant submanifold with boundary of* R^n *of dimension* n *and suppose that there is a continuous* $E': E_G(W')\to E(R^n)$ *which is a right inverse for* ϕ^* *and that* W' *contains all points of* B *except those in a sufficiently small neighbourhood of* $0\times B_2$. *Then there exists a continuous right inverse for* $\phi^*: E(R^N)\to E_G(W)$ *where* W *is a* G*-invariant submanifold with boundary of dimension contained in the interior of* $G(B)\cup W'$ *which can be chosen so that* $G(B)\cup W'-W$ *is as small as required.*

PROOF. By the last corollary and note (with $M=B_2$) for W sufficiently large there are subbundles T and T' of the ball bundles $B_1\times B_2$ and $B_2\times B_3$ over B_2 with fibres balls of the same dimension such that $g(T)\cup W' = G(B)\cup W'$, $G(\partial T-(B_1\times\partial B_2))\subseteq \text{int } W$, $\phi'(\partial T)\subseteq F(\partial T')$ and there is a right inverse for $\phi^*: I(F(\partial T'), F(T'))\to I_G(G(\partial T), G(T))$ [recall formula (6.11)]. Since $I(\partial T', T')$ embeds in $I(\partial(T'\times B_4), T'\times B_4)$ there is a right inverse E'' for $\phi^*: I(R^N - F'(T'\times B_4), R^N)\to I_G(G(\partial T), G(T))$.

Let V be a G-invariant submanifold with boundary of W' of dimension n contained in int W' with $W'-V$ as small as required and $\partial W'$ intersecting $G(\partial T)$ transversally within $G(\text{int } B_1\times\partial B_2)$. $I_G(\partial V\cap\overline{V-T}, \overline{V-T})$ embeds in $I_G(\partial V, V)$ which embeds in $E_G(W')$ so that E' gives a right inverse for $\phi^*: E(R^N)\to I_G(\partial V\cap\overline{V-T}, \overline{V-T})$ and piecing together with E'' gives a right inverse for $\phi^*: E(R^N)\to I_G(\partial(V\cup G(T)), V\cup G(T))$. If W is a manifold with boundary of dimension n contained in $\text{int}(V\cup G(T))$ then $E_G(W)$ embeds in $I_G(\partial(V\cup G(T)), V\cup G(T))$ and the result follows.

PROPOSITION 6. *There exists a* G *invariant neighbourhood* V *of* 0 *in* R^n *and a continuous* $E: E_G(V)\to C(R^N)$, *a right inverse for* ϕ^*.

PROOF. We can build V from pieces $G(B)$ starting with points of minimal stabilizers and apply Lemma 3 at each step: there are of course only finitely many orbit types and the quotient by G of the set of points of a given orbit type with stabilizer of type H can be triangulated sufficiently finely over any compact set (using, say, a linear triangulation of the open subset of $(R^n)_H$ of points with stabilizer H) in order to use maps such as F on neighbourhoods of the pieces.

THEOREM 3. *There exists a continuous* $E: C_G(R^n) \to C(R^N)$ *such that* $\phi^* E = 1$.

PROOF. In earlier notation (with $G = H$, $\phi = \psi$, $p = N$, $q = n$, etc.) let M_r be a manifold with boundary of dimension p obtained, say, by thickening $h_2(S^{p-1} \times [0, r]) \cap Y'$ along its corners, containing $\alpha^{-1}(\alpha\psi(R^q) \cap h_2(S^{p-1} \times [0, r]))$ with ∂M_r intersecting this last set in $\alpha^{-1}(\alpha\psi(R^q) \cap h_2(S^{p-1} \times r))$. E given by the proposition gives (as in Lemma 1)

$$E: I_G(h_1(S^{n-1} \times r), h_1(S^{n-1} \times [0, r])) \to I(\partial M_r, M_r) \quad \text{and}$$

$$E: I_G(h_1(S^{n-1} \times \{s, t\}, h_1(S^{n-1} \times [s, t]))) \to I(\partial M_s \cup \partial M_t, \overline{M_t - M_s})$$

for some $r > 0$ and $0 < s < t$, where we have taken M_t containing M_s in its interior (as we can obviously suppose). Splitting $[0, \infty)$ into $[0, r] \cup [s_1, t_1] \cup [s_2, t_2] \cup \cdots$ where $0 < s_1 < r < s_2 < t_1 < s_3 < t_2 \cdots$ and using a partition of unity gives the result.

COROLLARY. *There is a continuous* $C(\phi(R^n)) \to C(R^N)$, *a right inverse for the restriction.*

PROOF. Follows from the theorem and the identification $C(\phi(R^n)) \approx C_G(R^n)$ (Schwarz).

Added in revision: Theorem 3 has also been proved by Mather (unpublished). The proof given here follows the lines laid down by Schwarz more closely.

7. G-stability

Suppose that G, a compact lie group, acts linearly on R^n and R^p and let $f: R^n \to R^p$ be a proper G invariant smooth map. $P\theta_G(R^n)$ and $P\theta_G(f)$, the spaces of polynomial G invariant vector fields on R^n and along f respectively, are finitely generated over $P_G(R^n)$, and $P\theta_G(R^p)$ is finitely generated over $P_G(R^p)$ (the proof given in Dieudonné (1970) that $P_G^+(R^n)$ is finitely generated over $P_G(R^n)$ goes through for the module case, and the generators are again homogeneous). Let $\{v_i\}$ be a finite generating set for $P\theta_G(R^n)$ over $P_G(R^n)$. Then $\Sigma_i C(R^n)v_i$ is closed in $C(R^n)$, by a result of Malgrange (1966) generalized by Tougeron (1972), and so $\Sigma_i C_G(R^n)v_i = (\Sigma_i C(R^n)v_i) \cap \theta_G(R^n)$ is closed in $\theta_G(R^n)$ and, since elements of $\theta_G(R^n)$ can be approximated arbitrarily closely over compact sets by polynomial vector fields,

$\Sigma_i C_G(R^n)v_i$ is also dense in $\theta_G(R^n)$ and equality follows. Similarly $\theta_G(R^p)$ and $\theta_G(f)$ are also finitely generated over $C_G(R^p)$ and $C_G(R^n)$ respectively.

These remarks, together with Theorem 3, allow Mather's proof that infinitesimal C^∞ stability implies C^∞ stability for a proper map in the G trivial case to go through in the G category for the map f. The converse result follows immediately from the same methods used for the G trivial case (by considering jet spaces — see Mather (1970)).

THEOREM 4. *Let $f: R^n \to R^p$ be a proper G invariant map between linear G spaces, where G is a compact lie group. Then f is C^∞ G-infinitesimally stable (that is $wf(\theta_G(R^p)) + tf(\theta_G(R^n)) = \theta_G(f)$, in Mather's notation, where wf and tf are induced by composition with f and the derivative of f) if and only if f is C^∞ G-stable (that is if g is a sufficiently close G invariant map to f there exist invariant smooth diffeomorphisms $h: R^n \to R^n$ and $h': R^p \to R^p$ such that $g = h'fh^{-1}$).*

COROLLARY. *The theorem holds if R^n and R^p are replaced by C^∞ G manifolds of finite orbit type — (see Schwarz: such a manifold has a G invariant embedding in some R^n and then the usual method (Mather* (1968) *for reducing questions about the manifold to those about R^n holds).*

[Added in revision: this corollary (at least when the source manifold is compact) has been announced by V. Poenaru (1975).]

8. Conclusion

Transversality theorems (and the use of jet spaces) used to give the result that stability implies infinitesimal stability can also be proved when the target is a manifold and the source a set of the type of Section 5 (the author will consider this question elsewhere). It is a simple exercise to show that $\phi(R^n)$, where ϕ is a Hilbert map, is coherent (clearly it has everywhere irreducible germs and by an argument of Malgrange, given in the preprint cited in the references, the result follows).

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