Canad. J. Math. Vol. **63** (2), 2011 pp. 436–459
doi:10.4153/CJM-2010-083-5

Simplicial Complexes and Open Subsets of Non-Separable LF-Spaces

Kotaro Mine and Katsuro Sakai

Abstract. Let F be a non-separable LF-space homeomorphic to the direct sum $\sum_{n\in\mathbb{N}} \ell_2(\tau_n)$, where $\aleph_0 < \tau_1 < \tau_2 < \cdots$. It is proved that every open subset U of F is homeomorphic to the product $|K| \times F$ for some locally finite-dimensional simplicial complex K such that every vertex $v \in K^{(0)}$ has the star $\operatorname{St}(v, K)$ with $\operatorname{card} \operatorname{St}(v, K)^{(0)} < \tau = \sup \tau_n$ (and $\operatorname{card} K^{(0)} \le \tau$), and, conversely, if K is such a simplicial complex, then the product $|K| \times F$ can be embedded in F as an open set, where $|K|$ is the polyhedron of K with the metric topology.

1 Introduction

A locally convex topological linear space is called an *LF-space* if it is the strict inductive limit of Fréchet spaces.[1] A typical LF-space is the limit $\mathbb{R}^\infty$ of the Euclidean spaces $\mathbb{R} \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots$. Let $\ell_2(\tau)$ be Hilbert space with density $\operatorname{dens} \ell_2(\tau) = \tau$. According to the topological classification of LF-spaces ([10, Theorem 2.14] combined with [19, Theorem 6.1]), every LF-space F is homeomorphic to ($\approx$) one of the spaces $\mathbb{R}^\infty$, $\ell_2(\tau) \times \mathbb{R}^\infty$ or $\sum_{n\in\mathbb{N}} \ell_2(\tau_n)$, where $\tau = \operatorname{dens} F$ and $\tau_1 < \tau_2 < \cdots$ with $\sup \tau_i = \operatorname{dens} F$.

Given a space E (called a model space), a paracompact Hausdorff space M is called an *E-manifold* if it is locally homeomorphic to E, that is, each point of M has an open neighborhood that is homeomorphic to an open set in E. In the theory of manifolds modeled on an LF-space, one can also consider three cases by the topological classification of LF-spaces.

First of all, the theory of $\mathbb{R}^\infty$-manifolds has been well developed. The classification, the open embedding, and the triangulation theorems were established in [4] (cf. [3]), that is, two $\mathbb{R}^\infty$-manifolds are homeomorphic if they have the same homotopy type; every connected $\mathbb{R}^\infty$-manifold can be embedded into $\mathbb{R}^\infty$ as an open set, and every $\mathbb{R}^\infty$-manifold is homeomorphic to $|K| \times \mathbb{R}^\infty$ for some locally finite simplicial complex K. These results were derived from the stability theorem asserting that $M \times \mathbb{R}^\infty \approx M$ for every $\mathbb{R}^\infty$-manifold M. Later, a topological characterization of $\mathbb{R}^\infty$-manifolds was given in [17], where easy proofs of the above results were also given.

Concerning the second case, it was proved in [11] that every open subset of $\ell_2(\tau) \times \mathbb{R}^\infty$ is homeomorphic to the product of an $\ell_2(\tau)$-manifold and $\mathbb{R}^\infty$. As a

Received by the editors July 2, 2008.
Published electronically December 29, 2010.
AMS subject classification: **57N20**, 46A13, 46T05, 57N17, 57Q05, 57Q40.
Keywords: LF-space, open set, simplicial complex, metric topology, locally finite-dimensional, star, small box product, ANR, $\ell_2(\tau)$, $\ell_2(\tau)$-manifold, open embedding, $\sum_{i\in\mathbb{N}} \ell_2(\tau_i)$.

[1] A *Fréchet space* is a locally convex completely metrizable topological linear space.

consequence, we obtained the stability, classification, and triangulation theorems for open subsets of $\ell_2(\tau) \times \mathbb{R}^\infty$, where the triangulation theorem asserts that every open subset of $\ell_2(\tau) \times \mathbb{R}^\infty$ is homeomorphic to the product $|K| \times \ell_2(\tau) \times \mathbb{R}^\infty$ for some locally finite-dimensional simplicial complex[2] K, where $|K|$ is the polyhedron of K with the metric topology. Thus, if the open embedding theorem were established for $\ell_2(\tau) \times \mathbb{R}^\infty$-manifold, then the classification and triangulation theorems would be obtained for $\ell_2(\tau) \times \mathbb{R}^\infty$-manifolds. But, this is still open.

In this paper, we show that the stability and triangulation theorems are valid for open subsets of LF-spaces of the third type $\sum_{n\in\mathbb{N}} \ell_2(\tau_n)$, where $\aleph_0 < \tau_1 < \tau_2 < \cdots$. In the following, polyhedra are endowed with the metric topology instead of the Whitehead (weak) topology. The set of vertices of a simplicial complex K is denoted by $K^{(0)}$. For a simplex $\sigma \in K$, let $\sigma^{(0)}$ be the set of vertices of σ. The star $\mathrm{St}(v, K)$ at a vertex $v \in K^{(0)}$ in K is the subcomplex consisting of all faces of simplexes $\sigma \in K$ with $v \in \sigma^{(0)}$. Our first main result is the following triangation theorem for open subsets of LF-spaces.

Theorem 1.1 *Every open subset U of $\sum_{n\in\mathbb{N}} \ell_2(\tau_n)$ is homeomorphic to the product $|K| \times \sum_{n\in\mathbb{N}} \ell_2(\tau_n)$ for some locally finite-dimensional simplicial complex K such that each vertex $v \in K^{(0)}$ has the star $\mathrm{St}(v, K)$ with $\operatorname{card} \mathrm{St}(v, K)^{(0)} < \sup_{n\in\mathbb{N}} \tau_n$.*

Observe that $\sum_{n\in\mathbb{N}} \ell_2(\tau_n) \times \sum_{n\in\mathbb{N}} \ell_2(\tau_n) \approx \sum_{n\in\mathbb{N}} \ell_2(\tau_n)$. This is trivial, since $\sum_{n\in\mathbb{N}} \ell_2(\tau_n)$ is regarded as the small box product $\boxdot_{n\in\mathbb{N}} \ell_2(\tau_n)$ (see §1). As a corollary of Theorem 1.1, we have the following stability theorem.

Corollary 1.2 (Stability) *Every open subset U of $\sum_{n\in\mathbb{N}} \ell_2(\tau_n)$ is homeomorphic to $U \times \sum_{n\in\mathbb{N}} \ell_2(\tau_n)$.*

We can also prove the following converse of Theorem 1.1.

Theorem 1.3 *For a locally finite-dimensional simplicial complex K, if $\operatorname{card} K^{(0)} \leq \tau = \sup_{n\in\mathbb{N}} \tau_n$ and $\operatorname{card} \mathrm{St}(v, K)^{(0)} < \tau$ for each vertex $v \in K^{(0)}$, then the product $|K| \times \sum_{n\in\mathbb{N}} \ell_2(\tau_n)$ can be embedded in $\sum_{n\in\mathbb{N}} \ell_2(\tau_n)$ as an open set.*

Remark 1.4 The condition $\operatorname{card} \mathrm{St}(v, K)^{(0)} < \sup_{n\in\mathbb{N}} \tau_n$ is equivalent to the condition that $\operatorname{card} \mathrm{St}(v, K)^{(0)} \leq \tau_n$ for some $n \in \mathbb{N}$. Replacing the first condition with the latter, Theorems 1.1 and 1.3 are also valid in the case $\tau_1 \leq \tau_2 \leq \cdots$ (the same proof is available). When $\tau_n = \tau$ for sufficiently large $n \in \mathbb{N}$, we have $\sum_{n\in\mathbb{N}} \ell_2(\tau_n) \approx \ell_2(\tau) \times \mathbb{R}^\infty$, which is the case of the previous paper [11]. In this case, Theorem 1.1 is none other than [11, Corollary 3]. But this induces the Main Theorem of [11]. Indeed, $|K| \times \ell_2(\tau) \times \mathbb{R}^\infty \approx (|K| \times \ell_2(\tau)) \times \ell_2(\tau) \times \mathbb{R}^\infty$. Since $|K|$ is a completely metrizable ANR,[3] it follows from Toruńczyk's ANR Factor Theorem [18] (see Section 1) that $|K| \times \ell_2(\tau)$ is an $\ell_2(\tau)$-manifold. On the other hand, Theorem 1.3 in this case is trivial. Indeed, by Toruńczyk's ANR Factor Theorem, $|K| \times \ell_2(\tau)$ is an $\ell_2(\tau)$-manifold with density τ, which can be embedded into $\ell_2(\tau)$

[2] A simplicial complex K is *locally finite-dimensional* if each vartex v of K has the finite-dimensional star, that is, $\sup\{\dim \sigma \mid v \in \sigma \in K\} < \infty$.

[3] ANR = absolute neighborhood retract (for metrizable spaces); the local finite-dimensionality of K implies the complete metrizability of $|K|$ (cf. [9, Lemma 11.5]).

as an open set by Henderson's Open Embedding Theorem [5]. Thus, $|K|\times\ell_2(\tau)\times\mathbb{R}^\infty$ can be embedded into $\ell_2(\tau)\times\mathbb{R}^\infty$ as an open set.

A subcomplex L of a simplicial complex K is said to be *full* in K if every simplex $\sigma \in K$ with $\sigma^{(0)} \subset L^{(0)}$ belongs to L. For such a pair $L \subset K$, let $N(L,K) = \bigcup_{v\in L^{(0)}} |\operatorname{St}(v, \operatorname{Sd} K)|$, where $\operatorname{Sd} K$ is the barycentric subdivision of K. To prove Theorem 1.3, we need the following theorem, which is well known for locally finite simplicial complexes or the Whitehead topology but we are treating non-locally finite simplicial complexes endowed with the metric topology.

Theorem 1.5 *Let L be a full subcomplex of a locally finite-dimensional simplicial complex K. Then, the topological boundary* $\operatorname{bd}_{|K|} N(L,K)$ *of* $N(L,K)$ *in* $|K|$ *is bicollared in* $|K|$.

Here, it is said that a subset $A \subset X$ is *bicollared* in X if there exists an open embedding $k\colon A\times(-1,1)\to X$ such that $k(x,0)=x$ for every $x\in A$.

The Whitehead topology is preserved by subdivisions of K, but the metric topology is not. To prove Theorem 1.5, we have to use simplicial subdivisions preserving the metric topology. The barycentric subdivision is a typical one. In [7], D. W. Henderson called such a subdivision a *proper subdivision* and gave its characterization (see Theorem 5.2). Since it is not easy to check the condition even for derived subdivisions, we show the following characterization.

Theorem 1.6 *For a locally finite-dimensional simplicial complex K, a derived subdivision* K' *of K is proper if and only if* $K'^{(0)}$ *is discrete in* $|K|$. [4]

2 Preliminaries

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of topological spaces. The box product $\square_{i\in\mathbb{N}}X_i$ is the product $\prod_{i\in\mathbb{N}} X_i$ with the box topology, whose basis consists of sets $\prod_{i\in\mathbb{N}} U_i$, where each U_i is open in X_i. Given maps $f_i\colon X_i\to Y_i$, $i\in\mathbb{N}$, the box product $\square_{i\in\mathbb{N}} f_i\colon \square_{i\in\mathbb{N}}X_i\to \square_{i\in\mathbb{N}}Y_i$ is defined by $(\square_{n\in\mathbb{N}} f_i)(x) = (f_i(x_i))_{i\in\mathbb{N}}$ for each $x=(x_i)_{i\in\mathbb{N}}$. Then, $\square_{i\in\mathbb{N}} f_i$ is obviously continuous. In case every X_i is a pointed space with $*_i\in X_i$ the base point, the following subspace of $\square_{i\in\mathbb{N}}X_i$ is called the *small box product*:

$$\boxdot_{i\in\mathbb{N}}X_i = \left\{ (x_i)_{i\in\mathbb{N}} \in \square_{i\in\mathbb{N}}X_i \;\middle|\; x_i = *_i \text{ except for finitely many } i\in\mathbb{N}\right\}.$$

When each $f_i\colon X_i\to Y_i$ is a pointed map (*i.e.*, $f_i(*_i)=*_i$), we can define the map $\boxdot_{i\in\mathbb{N}} f_i\colon \boxdot_{i\in\mathbb{N}} X_i\to\boxdot_{i\in\mathbb{N}}Y_i$ as the restriction of $\square_{i\in\mathbb{N}} f_i$. For each $n\in\mathbb{N}$, we identify $\prod_{i=1}^n X_i$ with $\prod_{i=1}^n X_i = \prod_{i=1}^n X_i\times\{*_{n+1}\}\subset\prod_{i=1}^{n+1} X_i$. Then,

$$\boxdot_{i\in\mathbb{N}}X_i = \bigcup_{n\in\mathbb{N}}\prod_{i=1}^{n} X_i.$$

In case every X_i is the same space X, $\square_{i\in\mathbb{N}}X_i$ and $\boxdot_{i\in\mathbb{N}}X_i$ are denoted by $\square^{\mathbb{N}}X$ and $\boxdot^{\mathbb{N}}X$, respectively.

[4] Recently, Theorem 1.6 was proved in [12] for any subvidision of an arbitrary simplicial complex. In [12], a proper subdivision is renamed an *admissible subdivision*.

https://doi.org/10.4153/CJM-2010-083-5 Published online by Cambridge University Press

We always regard a topological linear space as a pointed space with 0 the base point. In case every X_i is a Fréchet space, the weak box product $\boxdot_{i\in\mathbb{N}} X_i$ is the strict inductive limit of the tower $X_1 \subset X_1 \times X_2 \subset X_1 \times X_2 \times X_3 \subset \cdots$, which is denoted by $\sum_{i\in\mathbb{N}} X_i$ in [10] (it is no other than the direct sum $\sum_{i\in\mathbb{N}} X_i$ in [21, §13-2] or the locally convex direct sum $\bigoplus_{i\in\mathbb{N}} X_i$ in [13, Example 5.10.6]).

One should take caution that the inductive limit in the category of topological linear spaces is different from the one in the category of topological spaces. In this paper, the latter is called the *direct limit*. The direct limit of a tower $X_1 \subset X_2 \subset \cdots$ is denoted by $\varinjlim X_n$.

For the reader's convenience, we shall list the fundamental results on $\ell_2(\tau)$-manifolds that will be used in our proof. In the proof of Theorem 1.1, we adopt the same strategy as the previous paper [11], but we now need Toruńczyk's ANR Factor Theorem [18].

Theorem 2.1 (ANR Factor) *For every completely metrizable ANR X with* $\operatorname{dens} X \leq \tau$*, the product $X \times \ell_2(\tau)$ is an $\ell_2(\tau)$-manifold. In case X is an AR, $X \times \ell_2(\tau) \approx \ell_2(\tau)$.*

For a locally finite-dimensional simplicial complex K with $\operatorname{card} K^{(0)} \leq \tau$, $|K| \times \ell_2(\tau)$ is an $\ell_2(\tau)$-manifold, where $|K|$ is the polyhedron with the metric topology. Indeed, $|K|$ is a completely metrizable ANR and $\operatorname{card} K^{(0)} \leq \tau$ implies $\operatorname{dens} |K| \leq \tau$.

A closed set A in X is called a *Z-set* (or a *strong Z-set*) if there are maps $f\colon X \to X \setminus A$ (or $A \cap \operatorname{cl} f(X) = \varnothing$) arbitrarily close to id. It is said that a subset $A \subset X$ is *collared* in X if there is an open embedding $k\colon A \times [0,1) \to X$ such that $k(x,0) = x$ for every $x \in A$. The following is well known (see the statement after [14, Corollary 4.4]).

Theorem 2.2 (Collaring) *If a Z-set in an $\ell_2(\tau)$-manifold M is also an $\ell_2(\tau)$-manifold then it is collared in M.*

Combining [20, Theorem B1] with the ANR Factor Theorem, we have the following.

Theorem 2.3 (Enlargement) *Let X be a completely metrizable ANR and A a strong Z-set in X. If $X \setminus A$ is an $\ell_2(\tau)$-manifold, then X is also an $\ell_2(\tau)$-manifold.*

We call an embedding $f\colon X \to Y$ a *Z-embedding* if $f(X)$ is a Z-set in Y. The following Z-Set Unknotting Theorem was established in [1].

Theorem 2.4 (Z-set Unknotting) *Let A be a Z-set in an $\ell_2(\tau)$-manifold M. If a Z-embedding $h\colon A \to M$ is homotopic to ($\simeq$)* id*, then h extends to a homeomorphism $\tilde{h}\colon M \to M$ that is isotopic to* id.

We also use the following version.

Corollary 2.5 *Let A be a Z-set in an $\ell_2(\tau)$-manifold M and $f\colon M \to N$ a homeomorphism from M onto another $\ell_2(\tau)$-manifold N. If a Z-embedding $g\colon A \to N$ is homotopic to the restriction $f|A$, then g extends to a homeomorphism $\tilde{g}\colon M \to M$ that is isotopic to f.*

The following is proved in [8] (cf. [6]).

Theorem 2.6 (Classification) *Let M and N be $\ell_2(\tau)$-manifolds. Every homotopy equivalence $f\colon M \to N$ is homotopic to a homeomorphism.*

The following result is due to the second author [15] (cf. [16]).

Theorem 2.7 *Let M be an $\ell_2(\tau)$-manifold with $\operatorname{dens} M = \tau$ and N a Z-set in M that is an $\ell_2(\tau)$-manifold and contains a strong deformation retract of M. Then, there is a closed embedding $h\colon M \to \ell_2(\tau)$ such that $h(N) = \operatorname{bd} h(M)$ is bicollared in $\ell_2(\tau)$.*

3 The First Step in the Proof of Theorem 1.1

In this section, we translate the argument in the previous paper [11] by replacing the intervals $[0,1)$ (resp. $[0,1]$) by the unit open (resp. closed) ball $\mathbb{B}_i$ (resp. $\overline{\mathbb{B}}_i$) of $\ell_2(\tau_i)$. For the sake of readers' convenience and completeness, we repeat the arguments. We also improve the notation.

In Theorem 1.1, $\sum_{i\in\mathbb{N}} \ell_2(\tau_i) = \boxdot_{i\in\mathbb{N}}\ell_2(\tau_i)$ can be replaced with $\boxdot_{i\in\mathbb{N}}\mathbb{B}_i$ because $\ell_2(\tau_i) \approx \mathbb{B}_i$. For each $s > 0$, let

$$s\mathbb{B}_i = \{x \in \ell_2(\tau_i) \mid \|x\| < s\} \text{ and } s\overline{\mathbb{B}}_i = \{x \in \ell_2(\tau_i) \mid \|x\| \leq s\}.$$

For a subset $N \subset \prod_{i=1}^{n} \mathbb{B}_i$ and a map $\alpha\colon N \to (0,1)$, we define

$$N(\alpha) = \{(x,y) \in N \times \mathbb{B}_{n+1} \mid \|y\| < \alpha(x)\} \subset \prod_{i=1}^{n+1} \mathbb{B}_i.$$

Let U be an open set in $\boxdot_{i\in\mathbb{N}}\mathbb{B}_i$. For each $n \in \mathbb{N}$, let $U_n = U \cap \prod_{i=1}^{n} \mathbb{B}_i$. Then, U_n is an $\ell_2(\tau_n)$-manifold and U_n is closed in U_{n+1}. For a sequence $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ of maps $\alpha_k\colon U_k \to (0,1)$ satisfying the condition $U_k(\alpha_k) \subset U_{k+1}$, we can inductively define

$$U_n(\alpha_n,\dots,\alpha_k) = U_n(\alpha_n,\dots,\alpha_{k-1})(\alpha_k) \subset U_k(\alpha_k) \subset U_{k+1} \text{ for each } k > n.$$

Then, $U_n(\alpha_n) \subset U_n(\alpha_n,\alpha_{n+1}) \subset U_n(\alpha_n,\alpha_{n+1},\alpha_{n+2}) \subset \cdots$. Let

$$U_n^\alpha = \bigcup_{k\geqslant n} U_n(\alpha_n,\dots,\alpha_k) \subset U.$$

Thus, we have a tower $U_1^\alpha \subset U_2^\alpha \subset U_3^\alpha \subset \cdots$ with $U = \bigcup_{n\in\mathbb{N}} U_n^\alpha$. If each U_n^α is open in U, then U is the direct limit of this tower, that is, $U = \varinjlim U_n^\alpha$.

Lemma 3.1 *There exists a sequence $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ of maps $\alpha_k\colon U_k \to (0,1)$ such that $U_k(\alpha_k) \subset U_{k+1}$ for every $k \in \mathbb{N}$ and each U_n^α is open in U, hence $U = \varinjlim U_n^\alpha$. Moreover, each $x \in U_k$ has a neighborhood V in U_k with $a_i > 0$, $i > k$ such that $\inf_{y\in V} \alpha_k(y) > 0$ and*

$$\inf\Big\{ \alpha_n(y) \;\Big|\; y \in V \times \prod_{i=k+1}^{n} a_i\overline{\mathbb{B}}_i \Big\} > 0 \text{ for every } n > k.$$

https://doi.org/10.4153/CJM-2010-083-5 Published online by Cambridge University Press

Proof For each $k \in \mathbb{N}$, let $\mathcal{V}_k = \{V_\lambda \mid \lambda \in \Lambda_k\}$ be a locally finite open cover of U_k with $a_{\lambda,i} \in (0,1]$, $i > k$, such that $\operatorname{cl} V_\lambda \times \boxdot_{i>k} a_{\lambda,i}\overline{\mathbb{B}}_i \subset U$ for each $k \in \mathbb{N}$ and $\lambda \in \Lambda_k$, where $\Lambda_k \cap \Lambda_{k'} = \varnothing$ if $k \neq k'$. Suppose that $\mathcal{W}_k = \{W_\lambda \mid \lambda \in \Lambda_k\}$ is an open cover of U_k that is a shrinking of $\mathcal{V}_k$, that is, $\operatorname{cl} W_\lambda \subset V_\lambda$ for each $\lambda \in \Lambda_k$. We define $\beta_k \colon U_k \to \mathbf{I}$ as follows:

$$\beta_k(x) = \max\left\{ \frac{a_{\lambda,k+1}}{2} \;\middle|\; x \in \operatorname{cl} W_\lambda \times \prod_{i=j+1}^{k} \frac{a_{\lambda,i}}{2}\overline{\mathbb{B}}_i,\ \lambda \in \Lambda_j,\ j \leq k \right\},$$

where $\operatorname{cl} W_\lambda \times \prod_{i=j+1}^{k} \frac{a_{\lambda,i}}{2}\overline{\mathbb{B}}_i = \operatorname{cl} W_\lambda$ if $j = k$. Observe

$$\left\{ (x,t) \in U_k \times \mathbf{I} \;\middle|\; t \leq \beta_k(x) \right\} = \bigcup_{j \leq k} \bigcup_{\lambda \in \Lambda_j} \operatorname{cl} W_\lambda \times \prod_{i=j+1}^{k} \frac{a_{\lambda,i}}{2}\overline{\mathbb{B}}_i \times \left[0, \frac{a_{\lambda,k+1}}{2}\right],$$

which is closed in $U_k \times \mathbf{I}$. This means that β_k is upper semi-continuous. Moreover, we can define a lower semi-continuous function $\gamma_k \colon U_k \to \mathbf{I}$ as follows:

$$\gamma_k(x) = \max\left\{ a_{\lambda,k+1} \;\middle|\; x \in V_\lambda \times \prod_{i=j+1}^{k} a_{\lambda,i}\mathbb{B}_i,\ \lambda \in \Lambda_j,\ j \leq k \right\}.$$

Since $\beta_k < \gamma_k$, there exists a continious map $\alpha_k \colon U_k \to (0,1)$ such that $\beta_k < \alpha_k < \gamma_k$. Then, $U_k(\alpha_k) \subset U_{k+1}$ for every $k \in \mathbb{N}$.

From the definition, it follows that

$$W_\lambda \times \prod_{i=k+1}^{n} \frac{a_{\lambda,i}}{2}\overline{\mathbb{B}}_i \times \frac{a_{\lambda,n+1}}{2}\overline{\mathbb{B}}_{n+1} \subset U_n(\alpha_n) \text{ for each } \lambda \in \Lambda_k \text{ and } n \geqslant k,$$

which implies $\inf_{y \in W_\lambda} \alpha_k(y) \geqslant a_{\lambda,k+1}/2 > 0$ and

$$\inf\left\{ \alpha_n(y) \;\middle|\; y \in W_\lambda \times \prod_{i=k+1}^{n} a_{\lambda,i}\overline{\mathbb{B}}_i \right\} \geqslant \frac{a_{\lambda,n+1}}{2} > 0 \text{ for every } n > k.$$

To show that each U_n^α is open in U, let $x \in U_n^\alpha$. Choose $k \geqslant n$ so that $x \in U_n(\alpha_n, \dots, \alpha_k) \subset U_{k+1}$. Since $U_{k+1} = \bigcup_{\lambda \in \Lambda_{k+1}} W_\lambda$, it follows that $x \in W_\lambda$ for some $\lambda \in \Lambda_{k+1}$. Let $G = W_\lambda \cap U_n(\alpha_n, \dots, \alpha_k)$. Then, $G \times \boxdot_{i>k+1} \frac{a_{\lambda,i}}{2}\mathbb{B}_i$ is a neighborhood of x in $\boxdot_{i \in \mathbb{N}}\mathbb{B}_i$. By induction on $m > k$, we shall show that

$$G \times \prod_{i=k+2}^{m+1} \frac{a_{\lambda,i}}{2}\overline{\mathbb{B}}_i \subset U_n(\alpha_n, \dots, \alpha_m).$$

Take an arbitrary point $y = (w, z_{k+2}, \dots, z_{m+1})$ from the left side in the above. By the inductive assumption, we have

$$y' = (w, z_{k+2}, \dots, z_m) \in G \times \prod_{i=k+2}^{m} \frac{a_{\lambda,i}}{2}\overline{\mathbb{B}}_i \subset U_n(\alpha_n, \dots, \alpha_{m-1}).$$

Since $\|z_{m+1}\| < a_{\lambda,m+1}/2 < \alpha_m(y')$, we have

$$y \in U_n(\alpha_n, \dots, \alpha_{m-1})(\alpha_m) = U_n(\alpha_n, \dots, \alpha_m).$$

Thus, it follows that

$$G \times \boxdot_{i>k+1} \frac{a_{\lambda,i}}{2}\overline{\mathbb{B}}_i = \bigcup_{m>k} \left(G \times \prod_{i=k+2}^{m+1} \frac{a_{\lambda,i}}{2}\overline{\mathbb{B}}_i \right) \subset \bigcup_{m>k} U_n(\alpha_n, \dots, \alpha_m) = U_n^\alpha.$$

Therefore, U_n^α is open in U. ■

Now, we shall construct a sequence $\Psi = (\psi_i)_{i\in\mathbb{N}}$ of open embeddings $\psi_i \colon U_i \times \mathbb{B}_{i+1} \to U_{i+1}$ so that $\psi_i(x, 0) = (x, 0)$ for every $x \in U_i$ and U is homeomorphic to the direct limit U_Ψ of the following open tower:

$$U_1 \times \boxdot_{i>1}\mathbb{B}_i \underset{\psi_1\times \mathrm{id}}{\subset} U_2 \times \boxdot_{i>2}\mathbb{B}_i \underset{\psi_2\times \mathrm{id}}{\subset} \cdots,$$

where $U_n \times \boxdot_{i>n}\mathbb{B}_i$ is regarded as an open set in $U_{n+1} \times \boxdot_{i>n+1}\mathbb{B}_i$ by the embedding

$$\psi_n \times \mathrm{id} : U_n \times \boxdot_{i>n}\mathbb{B}_i = U_n \times \mathbb{B}_{n+1} \times \boxdot_{i>n+1}\mathbb{B}_i \to U_{n+1} \times \boxdot_{i>n+1}\mathbb{B}_i.$$

Lemma 3.2 *There exists a sequence $\Psi = (\psi_n)_{n\in\mathbb{N}}$ of open embeddings $\psi_n \colon U_n \times \mathbb{B}_{n+1} \to U_{n+1}$ such that $U_\Psi \approx U$, $\psi_n(x, 0) = (x, 0)$ for every $x \in U_n$ and $\psi_n|U_n \times s\overline{\mathbb{B}}_{n+1} \colon U_n \times s\overline{\mathbb{B}}_{n+1} \to U_{n+1}$ is a closed embedding for each $s \in (0, 1)$.*

Proof Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be a sequence of maps $\alpha_n \colon U_n \to (0, 1)$ obtained in Lemma 3.1. Then, U_n^α is open in U and $U = \varinjlim U_n^\alpha$. The desired open embeddings $\psi_n \colon U_n \times \mathbb{B}_{n+1} \to U_{n+1}$, $n \in \mathbb{N}$, are defined by $\psi_n(x, y) = (x, \alpha_n(x)y)$. It is sufficient to show that $U_\Psi \approx U$ because the other conditions are easily observed.

For every $k \in \mathbb{N}$, we inductively define $\delta_{n,k} \colon U_n \times \prod_{i=n+1}^{n+k} \mathbb{B}_i \to \mathbb{B}_{n+k}$ as follows:

$$\delta_{n,k}(x, y_{n+1}, \dots, y_{n+k}) = \alpha_{n+k-1}\big(x, \delta_{n,1}(x, y_{n+1}), \dots, \delta_{n,k-1}(x, y_{n+1}, \dots, y_{n+k-1})\big)\, y_{n+k},$$

where $\delta_{n,1}(x, y) = \alpha_n(x)y$. Then, by induction, we have the following equation:

$$\delta_{n,k}\left(x, \frac{z_{n+1}}{\alpha_n(x)}, \dots, \frac{z_{n+k}}{\alpha_{n+k-1}(x, z_{n+1}, \dots, z_{n+k-1})}\right) = z_{n+k}. \tag{3.1}$$

Define $h_n \colon U_n \times \boxdot_{i>n}\mathbb{B}_i \to U_n^\alpha$ and $g_n \colon U_n^\alpha \to U_n \times \boxdot_{i>n}\mathbb{B}_i$ as follows:

$$h_n(x, y_{n+1}, y_{n+2}, \dots) = (x, \delta_{n,1}(x, y_{n+1}), \delta_{n,2}(x, y_{n+1}, y_{n+2}), \dots) \quad \text{and}$$

$$g_n(x, z_{n+1}, z_{n+2}, \dots) = \left(x, \frac{z_{n+1}}{\alpha_n(x)}, \frac{z_{n+2}}{\alpha_{n+1}(x, z_{n+1})}, \frac{z_{n+3}}{\alpha_{n+2}(x, z_{n+1}, z_{n+2})}, \dots\right).$$

As is easily observed, $g_n \circ h_n = \mathrm{id}$. By (3.1), $h_n \circ g_n = \mathrm{id}_{U_n^\alpha}$. Hence, g_n is a bijection with $h_n = g_n^{-1}$. Moreover, the following diagram commutes:

$$\begin{array}{ccc} U_n^\alpha & \subset & U_{n+1}^\alpha \\ {\scriptstyle g_n}\downarrow & & \downarrow{\scriptstyle g_{n+1}} \\ U_n \times \boxdot_{i>n}\mathbb{B}_i & \xrightarrow[\psi_n \times \mathrm{id}]{\subset} & U_{n+1} \times \boxdot_{i>n+1}\mathbb{B}_i. \end{array}$$

Indeed, let $(x, z_{n+1}, z_{n+2}, \dots) \in U_n^\alpha$. Then, $((x, z_{n+1}), z_{n+2}, \dots) \in U_{n+1}^\alpha$ and

$$\begin{aligned} (\psi_n \times \mathrm{id}) \circ g_n(x, z_{n+1}, z_{n+2}, \dots) &= (\psi_n \times \mathrm{id})\Big(x, \frac{z_{n+1}}{\alpha_n(x)}, \frac{z_{n+2}}{\alpha_{n+1}(x, z_{n+1})}, \dots \Big) \\ &= \Big(\psi_n\Big(x, \frac{z_{n+1}}{\alpha_n(x)} \Big), \frac{\varphi_{n+2}}{\alpha_{n+1}(x, z_{n+1})}, \dots \Big) \\ &= \Big((x, z_{n+1}), \frac{z_{n+2}}{\alpha_{n+1}(x, z_{n+1})}, \dots \Big) \\ &= g_{n+1}\big((x, z_{n+1}), z_{n+2}, \dots \big). \end{aligned}$$

It remains to show that each h_n and g_n are continuous, which means that g_n is a homeomorphism. Thus, we would have

$$U = \varinjlim U_n^\alpha \approx \varinjlim \big(U_n \times \boxdot_{i>n}\mathbb{B}_i \big) = U_\Psi.$$

To see the continuity of h_n at $x \in U_n \times \boxdot_{i>n}\mathbb{B}_i$, let V be a neighborhood of $h_n(x)$ in U_n^α. Then, x is contained in some $U_n \times \prod_{i=n+1}^{n+k} \mathbb{B}_i$, which implies that $h_n(x) \in U_n(\alpha_n, \dots, \alpha_{n+k-1})$. We can find a neighborhood V' in $h_n(x)$ in $U_n \times \prod_{i=n+1}^{n+k} \mathbb{B}_i$ and $0 < \varepsilon_i < 1$, $i \geqslant n+k+1$, such that $h_n(x) \in V' \times \boxdot_{i>n+k}\varepsilon_i\mathbb{B}_i \subset V$. Since $\delta_{n,1}, \dots, \delta_{n,k}$ are continuous, it follows that $h_n|U_n \times \prod_{i=n+1}^{n+k} \mathbb{B}_i$ is continuous, hence x has a neighborhood W in $U_n \times \prod_{i=n+1}^{n+k} \mathbb{B}_i$ such that $h_n(W) \subset V'$. Then, $W \times \boxdot_{i>n+k}\varepsilon_i\mathbb{B}_i$ is a neighborhood of x in $U_n \times \boxdot_{i>n}\mathbb{B}_i$ and

$$h_n\big(W \times \boxdot_{i>n+k}\varepsilon_i\mathbb{B}_i \big) \subset V' \times \boxdot_{i>n+k}\varepsilon_i\mathbb{B}_i \subset V,$$

which implies that h_n is continuous at x.

To see the continuity of g_n at $x \in U_n^\alpha$, let V be a neighborhood of $g_n(x)$ in $U_n \times \boxdot_{i>n}\mathbb{B}_i$. Then, $g_n(x)$ is contained in some $U_n \times \prod_{i=n+1}^{n+k} \mathbb{B}_i$. Put $m = n + k$ and choose an open set W in U_m and $\varepsilon_i > 0$, $i \geqslant m+1$, so that $g_n(x) \in W \times \boxdot_{i>m}\varepsilon_i\mathbb{B}_i \subset V$. Since $\alpha_n, \dots, \alpha_{m-1}$ are continuous, it follows that $g_n|U_n(\alpha_n, \dots, \alpha_{m-1})$ is continuous, hence we have a neighborhood W' of x in $U_n(\alpha_n, \dots, \alpha_{m-1}) \subset U_m$ such that $g_n(W' \times \{(0, 0, \dots)\}) \subset W \times \{(0, 0, \dots)\}$. By virtue of Lemma 3.1, it can be assumed that $\inf_{y \in W'} \alpha_m(y) > 0$ and

$$\inf\Big\{ \alpha_{m+l}(y) \;\Big|\; y \in W' \times \prod_{i=m+1}^{m+l} \varepsilon_i\mathbb{B}_i \Big\} > 0 \ \text{ for every } l \in \mathbb{N}.$$

https://doi.org/10.4153/CJM-2010-083-5 Published online by Cambridge University Press

We can find $0 < \delta_{m+l} \leq \varepsilon_{m+l}$, $l \in \mathbb{N}$, such that

$$y \in W', \ \|z_{m+1}\| < \delta_{m+1} \Rightarrow \frac{\|z_{m+1}\|}{\alpha_m(y)} < \varepsilon_{m+1} \quad \text{and}$$

$$(y, z_{m+1}, \ldots, z_{m+l}) \in W' \times \prod_{i=m+1}^{m+l} \varepsilon_i \mathbb{B}_i, \ \|z_{m+l+1}\| < \delta_{m+l+1}$$
$$\Rightarrow \frac{\|z_{m+l+1}\|}{\alpha_{m+l}(y, z_{n+k+1}, \ldots, z_{m+l})} < \varepsilon_{m+l+1}.$$

Then, we have $g_n\big(W' \times \boxdot_{i>m}\delta_i\mathbb{B}_i\big) \subset W \times \boxdot_{i>m}\varepsilon_i\mathbb{B}_i \subset V$, which implies that g_n is continuous at x. ■

4 The Second Step in the Proof of Theorem 1.1

To construct the simplical complex K in Theorem 1.1, for each $n \in \mathbb{N}$, let K_n be a locally finite-dimensional simplicial complex of the homotopy type of U_n and $\xi_n \colon U_n \to |K_n|$ a homotopy equivalence with a homotopy inverse $\eta_n \colon |K_n| \to U_n$. Moreover, take a subdivision $K'_n \lhd K_n$ and a simplicial approximation $\varphi_n \colon K'_n \to K_{n+1}$ of $\xi_{n+1} i_n \eta_n \colon |K_n| \to |K_{n+1}|$, where $i_n \colon U_n = U_n \times \{0\} \subset U_{n+1}$ is the inclusion.

The simplex with vertices $v_1, \ldots, v_n$ is denoted by $\langle v_1, \ldots, v_n \rangle$, where we allow the case $v_1, \ldots, v_n$ are not pairwise distinct. We give orders on $K_n^{(0)}$ and $K_n'^{(0)}$ such that the set $\sigma^{(0)}$ of vertices of each simplex σ is totally ordered. The simplicial mapping cylinder $Z(\varphi_n)$ of $\varphi_n \colon K'_n \to K_{n+1}$ can be defined as follows:

$$Z(\varphi_n) = K_{n+1} \cup \big\{ \langle \varphi_n(v_1), \ldots, \varphi_n(v_i), v_j, \ldots, v_k \rangle \bigm| v_1, \ldots, v_k \in K'_n,$$
$$v_1 < \cdots < v_k, \ 1 \leq i \leq j \leq k \big\}.$$

Then, K'_n and K_{n+1} are subcomplexes of $Z(\varphi_n)$ and $Z(\varphi_n)^{(0)} = K_n'^{(0)} \cup K_{n+1}^{(0)}$. We also define the triangulation $I(K_n, K'_n)$ of the product $|K_n| \times [2n-1, 2n]$ as follows:

$$I(K_n, K'_n) = (K_n \times \{2n-1\}) \cup (K'_n \times \{2n\})$$
$$\cup \big\{ \langle \sigma^{(0)} \times \{2n-1\} \cup \{v_m, \cdots, v_n\} \times \{2n\} \rangle \bigm| \sigma \in K'_n,$$
$$\langle v_1, \cdots, v_n \rangle \in K_n, \ v_1 < \cdots < v_n, \ \sigma \subset \langle v_1, \cdots, v_m \rangle \big\}.$$

Identifying K'_n and K_{n+1} in $Z(\varphi_n)$ with $K'_n \times \{2n\} \subset I(K_n, K'_n)$ and $K_{n+1} \times \{2n+1\} \subset I(K_{n+1}, K'_{n+1})$ respectively, we can obtain the simplicial complex

$$K = \bigcup_{n \in \mathbb{N}} (I(K_n, K'_n) \cup Z(\varphi_n)).$$

Take an increasing sequence $0 < c_1 < c_2 < \cdots$ with $\sup_{n \in \mathbb{N}} c_n = 1$ (*e.g.*, $c_n =$

https://doi.org/10.4153/CJM-2010-083-5 Published online by Cambridge University Press

$n/(n+1)$, $n \in \mathbb{N}$). For each $n \in \mathbb{N}$, let

$$N_n = (|K_n| \times [2n-1, 2n-\tfrac{1}{2})) \cup \bigcup_{i=1}^{n-1} (|I(K_i, K_i')| \cup |Z(\varphi_i)|),$$

$$\overline{N}_n = (|K_n| \times [2n-1, 2n-\tfrac{1}{2}]) \cup \bigcup_{i=1}^{n-1} (|I(K_i, K_i')| \cup |Z(\varphi_i)|),$$

$$M_n = N_n \times \prod_{i=1}^{n+1} c_n \mathbb{B}_i\,,\ \overline{M}_n = \overline{N}_n \times \prod_{i=1}^{n+1} c_n \overline{\mathbb{B}}_i \text{ and}$$

$$\partial M_n = \overline{M}_n \setminus M_n = \left(\theta_n(|K_n|) \times \prod_{i=1}^{n+1} c_n \overline{\mathbb{B}}_i\right) \cup (\overline{N}_n \times D_{n+1}),$$

where $\theta_n \colon |K_n| \to |K_n| \times \{2n - \frac{1}{2}\} \subset \overline{N}_n \setminus N_n \subset \overline{N}_n$ is the natural injection and

$$D_{n+1} = \prod_{i=1}^{n+1} c_n \overline{\mathbb{B}}_i \setminus \prod_{i=1}^{n+1} c_n \mathbb{B}_i = c_n \left(\prod_{i=1}^{n+1} \overline{\mathbb{B}}_i \setminus \prod_{i=1}^{n+1} \mathbb{B}_i \right).$$

It should be remarked that

$$\theta_{n+1}\varphi_n \simeq \theta_n \ \text{ in } \overline{N}_{n+1} \setminus N_n \text{ for each } n \in \mathbb{N}. \tag{4.1}$$

Lemma 4.1 *Every D_{n+1} is homeomorphic to $\ell_2(\tau_{n+1})$.*

Proof By induction, we shall show that

$$D'_{n+1} = c_n^{-1} D_{n+1} = \prod_{i=1}^{n+1} \overline{\mathbb{B}}_i \setminus \prod_{i=1}^{n+1} \mathbb{B}_i \approx \ell_2(\tau_{n+1}),$$

so we will have $D_{n+1} \approx \ell_2(\tau_{n+1})$. The unit sphere $\mathbb{S}_{n+1} = \overline{\mathbb{B}}_{n+1} \setminus \mathbb{B}_{n+1}$ and the unit closed ball $\overline{\mathbb{B}}_{n+1}$ of $\ell_2(\tau_{n+1})$ is homeomorphic to $\ell_2(\tau_{n+1})$. Then, $D'_1 = \mathbb{S}_1 \approx \ell_2(\tau_1)$. Assume that $D'_n \approx \ell_2(\tau_n)$. Observe

$$D'_{n+1} = (D'_n \times \overline{\mathbb{B}}_{n+1}) \cup (\overline{\mathbb{B}}_1 \times \cdots \times \overline{\mathbb{B}}_n \times \mathbb{S}_{n+1}) \quad \text{and}$$

$$(D'_n \times \overline{\mathbb{B}}_{n+1}) \cap (\overline{\mathbb{B}}_1 \times \cdots \times \overline{\mathbb{B}}_n \times \mathbb{S}_{n+1}) = D'_n \times \mathbb{S}_{n+1}.$$

By the ANR Factor Theorem, we have

$$D'_n \times \overline{\mathbb{B}}_{n+1} \approx \overline{\mathbb{B}}_1 \times \cdots \times \overline{\mathbb{B}}_n \times \mathbb{S}_{n+1} \approx D'_n \times \mathbb{S}_{n+1} \approx \ell_2(\tau_{n+1}).$$

As is easily observed, $D'_n \times \mathbb{S}_{n+1}$ is a Z-set in both $D'_n \times \overline{\mathbb{B}}_{n+1}$ and $\overline{\mathbb{B}}_1 \times \cdots \times \overline{\mathbb{B}}_n \times \mathbb{S}_{n+1}$. Since $\ell_2(\tau_{n+1}) \times (-1,1) \approx \ell_2(\tau_{n+1}) \times [0,1) \approx \ell_2(\tau_{n+1})$, it is easy to obtain $D'_{n+1} \approx \ell_2(\tau_{n+1})$ by the Z-set Unknotting Theorem. ∎

Lemma 4.2 *Each M_n, $\overline{M}_n$, and ∂M_n is an $\ell_2(\tau_{n+1})$-manifold, and ∂M_n is a Z-set in $\overline{M}_n$.*

https://doi.org/10.4153/CJM-2010-083-5 Published online by Cambridge University Press

Proof Since $\overline{\mathbb{B}}_{n+1} \approx D_{n+1} \approx \ell_2(\tau_{n+1})$, the following are $\ell_2(\tau_{n+1})$-manifolds by the ANR Factor Theorem:

$$M_n,\ \overline{M}_n,\ \theta_n(|K_n|) \times \prod_{i=1}^{n+1} c_n\overline{\mathbb{B}}_i,\ \overline{N}_n \times D_{n+1} \text{ and}$$

$$\left(\theta_n(|K_n|) \times \prod_{i=1}^{n+1} c_n\overline{\mathbb{B}}_i\right) \cap (\overline{N}_n \times D_{n+1}) = \theta_n(|K_n|) \times D_{n+1}.$$

The last one in the above is a Z-set in the both $\theta_n(|K_n|) \times \prod_{i=1}^{n+1} c_n\overline{\mathbb{B}}_i$ and $\overline{N}_n \times D_{n+1}$, so it is collared in them by the Collaring Theorem. Then, ∂M_n is an $\ell_2(\tau_{n+1})$-manifold because

$$\partial M_n = \left(\theta_n(|K_n|) \times \prod_{i=1}^{n+1} c_n\overline{\mathbb{B}}_i\right) \cup (\overline{N}_n \times D_{n+1}).$$

Observe that $\theta_n(|K_n|) \times \prod_{i=1}^{n+1} c_n\overline{\mathbb{B}}_i$ and $\overline{N}_n \times D_{n+1}$ are Z-sets in $\overline{M}_n = \overline{N}_n \times \prod_{i=1}^{n+1} c_n\overline{\mathbb{B}}_i$. Thus, ∂M_n is a Z-set in $\overline{M}_n$. ∎

We also consider the following sets:

$$\begin{aligned}
\bar{\partial} M_n &= (\partial M_n \times c_n\overline{\mathbb{B}}_{n+2}) \cup (\overline{M}_n \times c_n\mathbb{S}_{n+2}) \\
&= \left(\theta_n(|K_n|) \times \prod_{i=1}^{n+2} c_n\overline{\mathbb{B}}_i\right) \cup \left(\overline{N}_n \times \left(\prod_{i=1}^{n+2} c_n\overline{\mathbb{B}}_i \setminus \prod_{i=1}^{n+2} c_n\mathbb{B}_i\right)\right), \\
L_{n+1} &= \overline{M}_{n+1} \setminus (M_n \times c_n\mathbb{B}_{n+2}) \\
&= \left((\overline{N}_{n+1} \setminus N_n) \times \prod_{i=1}^{n+2} c_{n+1}\overline{\mathbb{B}}_i\right) \cup \left(\overline{N}_{n+1} \times \left(\prod_{i=1}^{n+2} c_{n+1}\overline{\mathbb{B}}_i \setminus \prod_{i=1}^{n+2} c_n\mathbb{B}_i\right)\right).
\end{aligned}$$

Then, we can write $\overline{M}_{n+1} = (\overline{M}_n \times c_n\overline{\mathbb{B}}_{n+2}) \cup L_{n+1}$ and $\bar{\partial} M_n = (\overline{M}_n \times c_n\overline{\mathbb{B}}_{n+2}) \cap L_{n+1}$, where $\bar{\partial} M_n$ is the topological boundary of both L_{n+1} and $M_n \times c_n\mathbb{B}_{n+2}$ in $\overline{M}_{n+1}$.

Lemma 4.3 *Each $\bar{\partial} M_n$ and L_{n+1} is an $\ell_2(\tau_{n+2})$-manifold, and $\bar{\partial} M_n$ is a Z-set in L_{n+1}.*

Proof The following two sets are bicollared in $\overline{M}_{n+1}$:

$$\theta_n(|K_n|) \times \prod_{i=1}^{n+2} c_{n+1}\overline{\mathbb{B}}_i \text{ and } \overline{N}_{n+1} \times \left(\prod_{i=1}^{n+2} c_n\overline{\mathbb{B}}_i \setminus \prod_{i=1}^{n+2} c_n\mathbb{B}_i\right).$$

Then, it is easy to construct a homeomorphism $f\colon \overline{M}_{n+1} \to \overline{M}_{n+1}$ arbitrarily close to id such that

$$\left(\overline{N}_n \times \prod_{i=1}^{n+2} c_{n+1}\overline{\mathbb{B}}_i\right) \cap \operatorname{cl} f\left((\overline{N}_{n+1} \setminus \overline{N}_n) \times \prod_{i=1}^{n+2} c_{n+1}\overline{\mathbb{B}}_i\right) = \varnothing \text{ and}$$

$$\left(\overline{N}_{n+1} \times \prod_{i=1}^{n+2} c_n\overline{\mathbb{B}}_i\right) \cap \operatorname{cl} f\left(\overline{N}_{n+1} \times \left(\prod_{i=1}^{n+2} c_{n+1}\overline{\mathbb{B}}_i \setminus \prod_{i=1}^{n+2} c_n\mathbb{B}_i\right)\right) = \varnothing,$$

https://doi.org/10.4153/CJM-2010-083-5 Published online by Cambridge University Press

which implies $(M_n \times c_n\overline{\mathbb{B}}_{n+2}) \cap \mathrm{cl}_{\overline{M}_{n+1}} f(L_{n+1}) = \varnothing$. Since $\bar{\partial} M_n \subset M_n \times c_n\overline{\mathbb{B}}_{n+2}$, we have a map $f|L_{n+1}\colon L_{n+1} \to L_{n+1}$ arbitrarily close to id such that $\bar{\partial} M_n \cap \mathrm{cl}\, f(L_{n+1}) = \varnothing$. Hence, $\bar{\partial} M_n$ is a strong Z-set in L_{n+1}. Observe that $L_{n+1} \setminus \bar{\partial} M_n = \overline{M}_{n+1} \setminus (\overline{M}_n \times c_n\overline{\mathbb{B}}_{n+2})$. Since $\overline{M}_{n+1}$ is an $\ell_2(\tau_{n+2})$-manifold by Lemma 4.2, it follows from the Enlargement Theorem that L_{n+1} is an $\ell_2(\tau_{n+2})$-manifold.

By the ANR Factor Theorem, $\partial M_n \times c_n\overline{\mathbb{B}}_{n+2}$ and $\overline{M}_n \times c_n\mathbb{S}_{n+2}$ are $\ell_2(\tau_{n+2})$-manifolds because $\overline{\mathbb{B}}_{n+2} \approx \mathbb{S}_{n+2} \approx \ell_2(\tau_{n+2})$. Observe that

$$(\partial M_n \times c_n\overline{\mathbb{B}}_{n+2}) \cap (\overline{M}_n \times c_n\mathbb{S}_{n+2}) = \partial M_n \times c_n\mathbb{S}_{n+2},$$

which is also an $\ell_2(\tau_{n+2})$-manifold by the ANR Factor Theorem. Since $\partial M_n \times c_n\mathbb{S}_{n+2}$ is a Z-set in both $\partial M_n \times c_n\overline{\mathbb{B}}_{n+2}$ and $\overline{M}_n \times c_n\mathbb{S}_{n+2}$, it is collared in them. Then it follows that $\bar{\partial} M_n$ is an $\ell_2(\tau_{n+2})$-manifold. ■

For each $n \in \mathbb{N}$, let $j_n\colon \overline{N}_n \to \overline{N}_n \times \{v_{n+1}\} \subset \overline{M}_n$ be the natural injection, where

$$v_{n+1} = (c_n e_1, \dots, c_n e_{n+1}) \in D_{n+1} = \prod_{i=1}^{n+1} c_n\overline{\mathbb{B}}_i \setminus \prod_{i=1}^{n+1} c_n\mathbb{B}_i$$

and each $e_i \in \mathbb{S}_i$ is a fixed point. It should be remarked that

$$j_n\theta_n(|K_n|) = \theta_n(|K_n|) \times \{v_{n+1}\} = |K_n| \times \{2n - \tfrac{1}{2}\} \times \{v_{n+1}\} \subset \partial M_n.$$

Since $j_{n+1}(\overline{N}_{n+1} \setminus N_n) \subset L_{n+1}$, the following follows from (4.1):

$$j_{n+1}\theta_{n+1}\varphi_n \simeq j_{n+1}\theta_n \text{ in } L_{n+1} \text{ for every } n \in \mathbb{N}. \tag{4.2}$$

Lemma 4.4 *For each $n \in \mathbb{N}$, there exists a retraction $r_n\colon \overline{M}_n \to j_n\theta_n(|K_n|)$ such that*

$$r_n \simeq \mathrm{id} \text{ rel. } j_n\theta_n(|K_n|) \text{ in } \overline{M}_n \text{ and } r_n|\partial M_n \simeq \mathrm{id} \text{ rel. } j_n\theta_n(|K_n|) \text{ in } \partial M_n,$$

where the latter homotopy is obtained as the restriction of the former, hence

$$(r_n \times \mathrm{id})|\bar{\partial} M_n \simeq \mathrm{id} \text{ rel. } j_n\theta_n(|K_n|) \times c_n\overline{\mathbb{B}}_{n+2} \text{ in } \bar{\partial} M_n.$$

Moreover, r_{n+1} satisfies that $r_{n+1}|L_{n+1} \simeq \mathrm{id}$ rel. $j_{n+1}\theta_{n+1}(|K_{n+1}|)$ in L_{n+1}, which is obtained by restricting $r_{n+1} \simeq \mathrm{id}$ rel. $j_{n+1}\theta_{n+1}(|K_{n+1}|)$ in $\overline{M}_{n+1}$.

Proof Observe that $\theta_n(|K_n|) = |K_n| \times \{2n - \frac{1}{2}\}$ is a strong deformation retract of $\overline{N}_n$, D_{n+1} is a strong deformation retract of $\prod_{i=1}^{n+1} c_n\overline{\mathbb{B}}_i$, and $\{v_{n+1}\}$ is a strong deformation retract of D_{n+1}. It is easy to construct a deformation $h\colon \overline{M}_n \times \mathbf{I} \to \overline{M}_n$ such that $h(\partial M_n \times \mathbf{I}) \subset \partial M_n$, $h_1\colon \overline{M}_n \to j_n\theta_n(|K_n|)$ is a retraction and $h_t|j_n\theta_n(|K_n|) = \mathrm{id}$ for every $t \in \mathbf{I}$. Then, $r_n = h_1$ is the desired retraction.

In case $n > 1$, $\theta_n(|K_n|)$ is a strong deformation retract of $\overline{N}_n \setminus N_{n-1}$ and $\overline{N}_n \setminus N_{n-1}$ is a strong deformation retract of $\overline{N}_n$. Then, we can construct h so as to satisfy $h(L_n \times \mathbf{I}) \subset L_n$. ■

By Lemma 3.2, we have a sequence $\Psi = (\psi_i)_{i\in\mathbb{N}}$ of open embeddings $\psi_i \colon U_i \times \mathbb{B}_{i+1} \to U_{i+1}$ such that U is homeomorphic to the direct limit U_Ψ of the open tower

$$U_1 \times \boxdot_{i>1}\mathbb{B}_i \underset{\psi_1\times\mathrm{id}}{\subset} U_2 \times \boxdot_{i>2}\mathbb{B}_i \underset{\psi_2\times\mathrm{id}}{\subset} \cdots .$$

Theorem 1.1 is reduced to the following.

Lemma 4.5 $|K| \times \boxdot_{i\in\mathbb{N}}\mathbb{B}_i \approx U_\Psi$.

Proof Here, we regard U_Ψ as the direct limit of the open tower

$$U_1 \times \boxdot_{i>1}c_1\mathbb{B}_i \underset{\psi_1\times\mathrm{id}}{\subset} U_2 \times \boxdot_{i>2}c_2\mathbb{B}_i \underset{\psi_2\times\mathrm{id}}{\subset} \cdots .$$

For each $n \in \mathbb{N}$, let

$$M_n^\infty = N_n \times \boxdot_{i\in\mathbb{N}}c_n\mathbb{B}_i = M_n \times \boxdot_{i>n+1}c_n\mathbb{B}_i.$$

Then, $M_1^\infty \subset M_2^\infty \subset \cdots$ are open in $|K| \times \boxdot_{i\in\mathbb{N}}\mathbb{B}_i$ and $\bigcup_{n\in\mathbb{N}} M_n^\infty = |K| \times \boxdot_{i\in\mathbb{N}}\mathbb{B}_i$. To show that $|K| \times \boxdot_{i\in\mathbb{N}}\mathbb{B}_i \approx U_\Psi$, we may construct homeomorphisms $h_n \colon M_n^\infty \to U_n \times \boxdot_{i>n}c_n\mathbb{B}_i$, $n \in \mathbb{N}$, so that the following diagram commutes:

$$\begin{array}{ccc} M_n^\infty & \xrightarrow{\subset} & M_{n+1}^\infty \\ {\scriptstyle h_n}\downarrow & & \downarrow{\scriptstyle h_{n+1}} \\ U_n \times \boxdot_{i>n}c_n\mathbb{B}_i & \xrightarrow[\psi_n\times\mathrm{id}]{\subset} & U_{n+1} \times \boxdot_{i>n+1}c_{n+1}\mathbb{B}_i. \end{array}$$

Note that $M_n \times c_n\mathbb{B}_{n+2} \subset M_{n+1}$. If we could construct homeomorphisms $f_n \colon M_n \to U_n \times c_n\mathbb{B}_{n+1}$, $n \in \mathbb{N}$, so that the following commutes:

$$\begin{array}{ccc} M_n \times c_n\mathbb{B}_{n+2} & \xrightarrow{\subset} & M_{n+1} \\ {\scriptstyle f_n\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ U_n \times c_n\mathbb{B}_{n+1} \times c_n\mathbb{B}_{n+2} & \xrightarrow[\psi\times\mathrm{id}]{} & U_{n+1} \times c_{n+1}\mathbb{B}_{n+2}, \end{array}$$

then the desired homeomorphism h_n could be defined as follows:

$$h_n = f_n \times \mathrm{id} : M_n^\infty = M_n \times \boxdot_{i>n+1}c_n\mathbb{B}_i \to U_n \times \boxdot_{i>n}c_n\mathbb{B}_i.$$

By Lemma 4.4, we have a retraction $r_n \colon \overline{M}_n \to j_n\theta_n(|K_n|)$ such that $r_n \simeq \mathrm{id}$ rel. $j_n\theta_n(|K_n|)$ in $\overline{M}_n$ and $r_n|\partial M_n \simeq \mathrm{id}$ rel. $j_n\theta_n(|K_n|)$ in ∂M_n, hence both r_n and $r_n|\partial M_n \colon \partial M_n \to j_n\theta_n(|K_n|)$ are homotopy equivalences. Let

$$i_n^* \colon U_n \to U_n \times \{c_n e_{n+1}\} \subset U_n \times c_n\mathbb{S}_{n+1} \subset U_n \times c_n\overline{\mathbb{B}}_{n+1}$$

be the natural injection. Recall $\eta_n \colon |K_n| \to U_n$ is a homotopy equivalence. Then, the map $q_n = i_n^*\eta_n(j_n\theta_n)^{-1}r_n \colon \overline{M}_n \to U_n \times c_n\overline{\mathbb{B}}_{n+1}$ is a homotopy equivalence. Moreover,

https://doi.org/10.4153/CJM-2010-083-5 Published online by Cambridge University Press

$\psi_n i_n^* \simeq i_n$ in U_{n+1}, where $i_n : U_n = U_n \times \{0\} \subset U_{n+1}$ is the inclusion. Since $\mathbb{S}_{n+1}$ is an AR, the restriction $q_n|\partial M_n \colon \partial M_n \to U_n \times c_n\mathbb{S}_{n+1}$ is also a homotopy equivalence.

We shall construct homeomorphisms $\bar{f}_n \colon \overline{M}_n \to U_n \times c_n\overline{\mathbb{B}}_{n+1}$, $n \in \mathbb{N}$, so that $\bar{f}_n \simeq q_n$, $\bar{f}_n(\partial M_n) = U_n \times c_n\mathbb{S}_{n+1}$ (*i.e.*, $\bar{f}_n(M_n) = U_n \times c_n\mathbb{B}_{n+1}$) and the following diagram commutes:

$$\begin{array}{ccc} \overline{M}_n \times c_n\overline{\mathbb{B}}_{n+2} & \xrightarrow{\ \subset\ } & \overline{M}_{n+1} \\ {\scriptstyle \bar{f}_n \times \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \bar{f}_{n+1}} \\ U_n \times c_n\overline{\mathbb{B}}_{n+1} \times c_n\overline{\mathbb{B}}_{n+2} & \xrightarrow[\psi_n \times \mathrm{id}]{} & U_{n+1} \times c_{n+1}\overline{\mathbb{B}}_{n+2}. \end{array}$$

Then, $f_n = \bar{f}_n|M_n$, $n \in \mathbb{N}$, are the desired homeomorphisms.

First, by the Classification Theorem, we have two homeomorphisms

$$f \colon \overline{M}_1 \to U_1 \times \overline{\mathbb{B}}_2 \quad \text{and} \quad f' \colon \partial M_1 \to U_1 \times \mathbb{S}_2$$

such that $f \simeq q_1$ and $f' \simeq q_1|\partial M_1$. Since $f' \simeq f|\partial M_1$ in $U_1 \times \overline{\mathbb{B}}_2$, we can apply the Z-set Unknotting Theorem to extend f' to a homeomorphism $\bar{f}_1 \colon \overline{M}_1 \to U_1 \times \overline{\mathbb{B}}_2$ that is isotopic to f, hence $\bar{f}_1 \simeq q_1$.

Now, assume that $\bar{f}_n$ has been obtained. Let

$$\begin{aligned} F_n &= (\psi_n \bar{f}_n \times \mathrm{id})(\bar{\partial} M_n) \\ &= (\psi_n \bar{f}_n(\partial M_n) \times c_n\overline{\mathbb{B}}_{n+2}) \cup (\psi_n \bar{f}_n(\overline{M}_n) \times c_n\mathbb{S}_{n+2}) \\ &= (\psi_n(U_n \times c_n\mathbb{S}_{n+1}) \times c_n\overline{\mathbb{B}}_{n+2}) \cup (\psi_n(U_n \times c_n\overline{\mathbb{B}}_{n+1}) \times c_n\mathbb{S}_{n+2}) \ \text{ and} \\ W_{n+1} &= (U_{n+1} \times c_{n+1}\overline{\mathbb{B}}_{n+2}) \setminus (\psi_n \bar{f}_n \times \mathrm{id})(M_n \times c_n\mathbb{B}_{n+2}) \\ &= (U_{n+1} \times c_{n+1}\overline{\mathbb{B}}_{n+2}) \setminus (\psi_n(U_n \times c_n\mathbb{B}_{n+1}) \times c_n\mathbb{B}_{n+2}). \end{aligned}$$

Then, we have

$$\begin{aligned} U_{n+1} \times c_{n+1}\overline{\mathbb{B}}_{n+2} &= (\psi_n \bar{f}_n \times \mathrm{id})(\overline{M}_n \times c_n\overline{\mathbb{B}}_{n+2}) \cup W_{n+1}, \\ F_n &= (\psi_n \bar{f}_n \times \mathrm{id})(\overline{M}_n \times c_n\overline{\mathbb{B}}_{n+2}) \cap W_{n+1} \end{aligned}$$

and the homeomorphism

$$g = (\psi_n \times \mathrm{id})(\bar{f}_n \times \mathrm{id})|\bar{\partial} M_n = (\psi_n \bar{f}_n \times \mathrm{id})|\bar{\partial} M_n \colon \bar{\partial} M_n \to F_n.$$

Hence, F_n is an $\ell_2(\tau_{n+2})$-manifold. Similarly to Lemma 4.3, it can be shown that W_{n+1} is also an $\ell_2(\tau_{n+2})$-manifold and F_n is a Z-set in W_{n+1}. Recall

$$\bar{\partial} M_n = (\partial M_n \times c_n\overline{\mathbb{B}}_{n+2}) \cup (\overline{M}_n \times c_n\mathbb{S}_{n+2}) \subset \overline{M}_n \times c_n\overline{\mathbb{B}}_{n+2} \subset \overline{M}_{n+1}.$$

Note that $c_n\mathbb{S}_{n+2}$ is a strong deformation retract of $c_n\overline{\mathbb{B}}_{n+2}$ and $\{c_n e_{n+2}\}$ is a strong deformation retract of $c_n\mathbb{S}_{n+2}$. Let $c \colon c_{n+1}\mathbb{B}_{n+2} \to \{c_n e_{n+2}\}$ be the constant map. Since $\bar{f}_n \simeq q_n$, it follows that

$$g \simeq (\psi_n \bar{f}_n \times c)|\bar{\partial} M_n \simeq (\psi_n q_n \times c)|\bar{\partial} M_n = (\psi_n i_n^* \times c)(\eta_n(j_n\theta_n)^{-1} r_n \times \mathrm{id})|\bar{\partial} M_n \ \text{ in } F_n.$$

https://doi.org/10.4153/CJM-2010-083-5 Published online by Cambridge University Press

In addition to the natural injection $i^*_{n+1}\colon U_{n+1} \to U_{n+1} \times \{c_{n+1}e_{n+2}\} \subset W_{n+1}$, consider the natural injection $i'_{n+1}\colon U_{n+1} \to U_{n+1} \times \{c_n e_{n+2}\} \subset W_{n+1}$. Then we have

$$\psi_n i^*_n \times c \simeq i_n \times c = i'_{n+1} i_n \operatorname{pr}_{U_n} \simeq i^*_{n+1} i_n \operatorname{pr}_{U_n} \quad \text{in } W_{n+1}.$$

Since $\eta_{n+1}\xi_{n+1} \simeq \mathrm{id}_{U_{n+1}}$ and φ_n is a simplicial approximation of $\xi_{n+1} i_n \eta_n$, it follows that

$$\begin{aligned} g &\simeq i^*_{n+1} i_n \operatorname{pr}_{U_n}(\eta_n \theta_n^{-1} j_n^{-1} r_n \times \mathrm{id})|\bar{\partial} M_n \\ &\simeq i^*_{n+1}\eta_{n+1}\xi_{n+1} i_n \eta_n \theta_n^{-1} j_n^{-1} r_n \operatorname{pr}_{\overline{M}_n} |\bar{\partial} M_n \\ &\simeq i^*_{n+1}\eta_{n+1}\varphi_n \theta_n^{-1} j_n^{-1} r_n \operatorname{pr}_{\overline{M}_n} |\bar{\partial} M_n \quad \text{in } W_{n+1}. \end{aligned}$$

Note that $r_{n+1}|L_{n+1}\colon L_{n+1} \to j_{n+1}\theta_{n+1}(|K_{n+1}|)$ is a retraction. By (4.2), we have

$$\begin{aligned} \varphi_n \theta_n^{-1} j_n^{-1} &= \theta_{n+1}^{-1} j_{n+1}^{-1} r_{n+1} j_{n+1} \theta_{n+1} \varphi_n \theta_n^{-1} j_n^{-1} \\ &\simeq \theta_{n+1}^{-1} j_{n+1}^{-1} r_{n+1} j_{n+1} \theta_n \theta_n^{-1} j_n^{-1} = (j_{n+1}\theta_{n+1})^{-1} r_{n+1} j_{n+1} j_n^{-1} \quad \text{in } |K_{n+1}|. \end{aligned}$$

Then it follows that

$$\begin{aligned} g &\simeq i^*_{n+1}\eta_{n+1}(j_{n+1}\theta_{n+1})^{-1} r_{n+1} j_{n+1} j_n^{-1} r_n \operatorname{pr}_{\overline{M}_n} |\bar{\partial} M_n \\ &= i^*_{n+1}\eta_{n+1}(j_{n+1}\theta_{n+1})^{-1} r_{n+1} j_{n+1} \operatorname{pr}_{\overline{N}_n}(r_n \times \mathrm{id})|\bar{\partial} M_n \quad \text{in } W_{n+1}. \end{aligned}$$

Since $\overline{M}_n \times c_n\mathbb{S}_{n+2}$ is a strong deformation retract of $\bar{\partial} M_n$ and

$$j_{n+1} \operatorname{pr}_{\overline{N}_n} |\overline{M}_n \times c_n \mathbb{S}_{n+2} \simeq \mathrm{id} \ \text{ in } \overline{M}_n \times c_n\mathbb{S}_{n+2},$$

we have $j_{n+1} \operatorname{pr}_{\overline{N}_n} |\bar{\partial} M_n \simeq \mathrm{id}$ in $\bar{\partial} M_n$. On the other hand, due to Lemma 4.4, $(r_n \times \mathrm{id})|\bar{\partial} M_n \simeq \mathrm{id}$ in $\bar{\partial} M_n$. Thus, we have

$$g \simeq i^*_{n+1}\eta_{n+1}(j_{n+1}\theta_{n+1})^{-1} r_{n+1}|\bar{\partial} M_n = q_{n+1}|\bar{\partial} M_n \quad \text{in } W_{n+1}.$$

Recall that $q_{n+1}|\partial M_{n+1}\colon \partial M_{n+1} \to U_{n+1} \times c_{n+1}\mathbb{S}_{n+2}$ is a homotopy equivalence. By the Classification Theorem, we have a homeomorphism $g'\colon \partial M_{n+1} \to U_{n+1} \times c_{n+1}\mathbb{S}_{n+2}$ such that $g' \simeq q_{n+1}|\partial M_{n+1}$. On the other hand, due to Lemma 4.4,

$$r_{n+1}|L_{n+1} \simeq \mathrm{id} \ \text{ rel. } j_{n+1}\theta_{n+1}(|K_{n+1}|) \text{ in } L_{n+1},$$

hence $r_{n+1}|L_{n+1}\colon L_{n+1} \to j_{n+1}\theta_{n+1}(|K_{n+1}|)$ is a homotopy equivalence. Then it follows that $q_{n+1}|L_{n+1}\colon L_{n+1} \to W_{n+1}$ is also a homotopy equivalence. By the Classification Theorem, we have a homeomorphism $g''\colon L_{n+1} \to W_{n+1}$ such that $g'' \simeq q_{n+1}|L_{n+1}$. Note that $\bar{\partial} M_n$ and ∂M_{n+1} are disjoint Z-sets in the $\ell_2(\tau_{n+2})$-manifold L_{n+1}, and F_n and $U_{n+1} \times c_{n+1}\mathbb{S}_{n+2}$ are disjoint Z-sets in the $\ell_2(\tau_{n+2})$-manifold W_{n+1}. Since $g \simeq q_{n+1}|\bar{\partial} M_n \simeq g''|\bar{\partial} M_n$ and $g' \simeq q_{n+1}|\partial M_{n+1} \simeq g''|\partial M_{n+1}$, we can apply the Z-set Unknotting Theorem to obtain a homeomorphism $f\colon L_{n+1} \to W_{n+1}$ such that f

https://doi.org/10.4153/CJM-2010-083-5 Published online by Cambridge University Press

is isotopic to g'', $f|\bar{\partial}M_n = g$ and $f|\partial M_{n+1} = g'$. Then f can be extended to a homeomophism

$$\bar{f}_{n+1} \colon \overline{M}_{n+1} \to U_{n+1} \times c_{n+1}\mathbb{B}_{n+2} \text{ by } \bar{f}_{n+1}|\overline{M}_n = (\psi_n \times \mathrm{id})(\bar{f}_n \times \mathrm{id}).$$

Recall that $r_{n+1} \simeq \mathrm{id}$ in $\overline{M}_{n+1}$ and

$$r_{n+1}(\overline{M}_{n+1}) = j_{n+1}\theta_{n+1}(|K_{n+1}|) \subset \partial M_{n+1}.$$

It follows that $\bar{f}_{n+1} \simeq \bar{f}_{n+1}r_{n+1} = g'r_{n+1} \simeq q_{n+1}r_{n+1} \simeq q_{n+1}$. This completes the proof. ■

5 Proofs of Theorems 1.5 and 1.6

In this section, we shall prove Theorems 1.5 and 1.6. For each point $x \in |K|$, let $(\beta_v^K(x))_{v\in K^{(0)}} \in \mathbf{I}^{K^{(0)}}$ be the barycentric coordinate, that is, $\sum_{v\in K^{(0)}} \beta_v^K(x) = 1$ and $\{v \in K^{(0)} \mid \beta_v^K(x) > 0\}$ is the set of vertices of the carrier of x, which is the simplex of K containing x as an interior point. Then we can write $x = \sum_{v\in K^{(0)}} \beta_v^K(x)v$. The *open star* at $v \in K^{(0)}$ is defined by

$$O(v, K) = \left\{ x \in |K| \;\middle|\; \beta_v^K(x) > 0 \right\}.$$

The metric ρ_K for the polyhedron $|K|$ is defined as follows:

$$\rho_K(x, y) = \sum_{v\in K^{(0)}} \left|\beta_v^K(x) - \beta_v^K(y)\right|.$$

A simplicial subdivision K' of K is *proper*[5] if and only if the metric $\rho_{K'}$ is admissible for $|K|$.

Remark 5.1 Identifying x with $(\beta_v^K(x))_{v\in K^{(0)}} \in \ell_1(K^{(0)})$, we can regard $|K|$ as a subspace of the Banach space $\ell_1(K^{(0)})$. Then, the metric ρ_K is induced by the norm of $\ell_1(K^{(0)})$.

The following characterization was established by D. W. Henderson.

Theorem 5.2 ([7, Lemma V.5]) *A simplicial subdivision K' of K is proper if and only if the open star $O(v, K')$ at each vertex $v \in K'^{(0)}$ is open in $|K|$.*

For each $x \in |K|$, let $\sigma_x \in K$ be the carrier of x and define $O(x, K) = \bigcap_{v\in\sigma_x^{(0)}} O(v, K)$. Then, $O(x, K)$ is open in $|K|$ with $\mathrm{cl}_{|K|}\, O(x, K) = |\operatorname{St}(\sigma_x, K)|$, where $\operatorname{St}(\sigma, K)$ is the star at $\sigma \in K$, which is the subcomplex of K defined as follows:

$$\operatorname{St}(\sigma, K) = \left\{ \sigma' \in K \;\middle|\; \exists \sigma'' \in K \text{ such that } \sigma, \sigma' \leq \sigma'' \right\}.^{6}$$

For $0 < t < 1$, we can define $\varphi_t^x \colon |\operatorname{St}(\sigma_x, K)| \to |\operatorname{St}(\sigma_x, K)|$ by

$$\varphi_t^x(y) = (1 - t)x + ty \quad \text{for } y \in |\operatorname{St}(\sigma_x, K)|.$$

[5]Or *admissible* (cf. Footnote 4).

[6]The notation $\sigma \leq \sigma'$ means that σ is a face of σ'.

Lemma 5.3 *For each $x \in |K|$ and $0 < t \leq 1$, the image $\varphi_t^x(|\operatorname{St}(\sigma_x, K)|)$ is closed in $|K|$, and $\varphi_t^x(O(x, K))$ is open in $|K|$.*

Proof Regarding $|K| \subset \ell_1(K^{(0)})$ as in Remark 5.1 above, φ_t^x extends to the homeomorphism $\tilde{\varphi}_t^x \colon \ell_1(K^{(0)}) \to \ell_1(K^{(0)})$. Hence, φ_t^x is a closed embedding, and the restriction $\varphi_t^x|O(x, K) \colon O(x, K) \to O(x, K)$ is an open embedding. ■

For $A \subset |K|$, let $C(A, K) = \{\sigma \in K \mid \sigma \cap A = \varnothing\}$. Then $C(A, K)$ is a subcomplex of K. In case $A = \{x\}$, we write $C(\{x\}, K) = C(x, K)$. Then, $O(x, K) = |K| \setminus |C(x, K)|$. Observe that $K = \operatorname{St}(\sigma, K) \cup C(\sigma^\circ, K)$ for each simplex $\sigma \in K$, where σ° is the interior of σ. In particular, $K = \operatorname{St}(v, K) \cup C(v, K)$ for each vertex $v \in K^{(0)}$. Note that $K \neq \operatorname{St}(\sigma, K) \cup C(\sigma, K)$ in general.

Let $V \subset |K|$ such that $O(v, K) \cap O(v', K) = \varnothing$ if $v \neq v' \in V$. For each $v \in V$ and $\sigma \in \operatorname{St}(\sigma_v, K) \cap C(v, K)$, let $v\sigma$ be the simplex spanned by $\{v\} \cup \sigma^{(0)}$, that is, $(v\sigma)^{(0)} = \{v\} \cup \sigma^{(0)}$. Then, we can define the simplicial subdivision K_V of K as follows:

$$K_V = C(V, K) \cup \left\{ v\sigma \;\middle|\; v \in V,\ \sigma \in \operatorname{St}(\sigma_v, K) \cap C(v, K) \right\}.$$

Observe that $K_V^{(0)} = V \cup K^{(0)}$, $C(V, K_V) = C(V, K)$, and $O(v, K_V) = O(v, K)$ for each $v \in V$. When $V = \{w\}$, we write $K_{\{w\}} = K_w$. The operation $K \to K_w$ is a starring K at w. A subdivision obtained by finite starrings is known as a *stellar subdivision* (cf. [2]).

Lemma 5.4 *For each $w \in |K| \setminus K^{(0)}$, K_w is a proper subdivision of K.*

Proof Let $\sigma_w \in K$ be the carrier of w. Observe that $O(v, K_w) = O(v, K)$, $v \in K^{(0)} \setminus \sigma_w^{(0)}$, and $O(w, K_w) = O(w, K) = \bigcap_{v \in \sigma_w^{(0)}} O(v, K)$ are open in $|K|$. To apply Theorem 5.2, it remains to show that $O(v, K_V)$ is open in $|K|$ for each $v \in \sigma_w^{(0)}$. Since $O(v, K_w) = (\beta_v^{K_w})^{-1}((0, 1])$, it suffices to prove the continuity of $\beta_v^{K_w} \colon |K| \to \mathbf{I}$ for each $v \in \sigma_w^{(0)}$.

By using the barycentric coordinate with respect to K_w, each point $x \in |K|$ can be written

$$x = \beta_w^{K_w}(x) w + \sum_{u \in K^{(0)}} \beta_u^{K_w}(x) u.$$

Since $\beta_v^K(v) = 1$ and $\beta_v^K(u) = 0$ for each $u \in K^{(0)} \setminus \{v\}$, it follows that

$$\beta_v^K(x) = \beta_w^{K_w}(x) \beta_v^K(w) + \beta_v^{K_w}(x),$$

hence $\beta_v^{K_w}(x) = \beta_v^K(x) - \beta_w^{K_w}(x) \beta_v^K(w)$. Since $\beta_v^K \colon |K| \to \mathbf{I}$ is continuous, it is enough to show that $\beta_w^{K_w} \colon |K| \to \mathbf{I}$ is continuous.

We shall show that $\beta_w^{K_w} \colon |K| \to \mathbf{I}$ is lower semi-continuous. For each $t \in [0, 1)$,

$$(\beta_w^{K_w})^{-1}((t, 1]) = \varphi_{1-t}^w(O(w, K)) = \left\{ tw + (1-t)z \;\middle|\; z \in O(w, K) \right\},$$

which is open in $|K|$ by Lemma 5.3. Indeed, let $y \in (\beta_w^{K_w})^{-1}((t, 1])$. If $\beta_w^{K_w}(y) = 1$, then $y = w \in \varphi_{1-t}^w(O(w, K))$. When $\beta_w^{K_w}(y) < 1$, we have

$$y^* = \sum_{v \in K^{(0)}} \frac{\beta_v^{K_w}(y)}{1 - \beta_w^{K_w}(y)} v \in \sigma_y \subset |\operatorname{St}(\sigma_w, K)|. \tag{5.1}$$

Observe that $\beta_w^{K_w}(y^*) = 0$ and $y = \beta_w^{K_w}(y)w + (1 - \beta_w^{K_w}(y))y^*$. Since $\beta_w^{K_w}(y) > t$, we have

$$z = \frac{\beta_w^{K_w}(y) - t}{1 - t} w + \frac{1 - \beta_w^{K_w}(y)}{1 - t} y^* \in (\beta_w^{K_w})^{-1}((0, 1]) = O(w, K).$$

Then it follows that

$$\begin{aligned} tw + (1 - t)z &= tw + (\beta_w^{K_w}(y) - t)w + (1 - \beta_w^{K_w}(y))y^* \\ &= \beta_w^{K_w}(y)w + (1 - \beta_w^{K_w}(y))y^* = y, \end{aligned}$$

hence $y = \varphi_{1-t}^w(z) \in \varphi_{1-t}^w(O(w, K))$. Conversely,

$$\begin{aligned} &z \in O(w, K) = (\beta_w^{K_w})^{-1}((0, 1]) \Longrightarrow \\ &\quad \beta_w^{K_w}(\varphi_{1-t}^w(z)) = \beta_w^{K_w}(tw + (1 - t)z) = t + (1 - t)\beta_w^{K_w}(z) > t, \end{aligned}$$

which means $\varphi_{1-t}^w(O(w, K)) \subset (\beta_w^{K_w})^{-1}((t, 1])$.

Next, we shall show that $\beta_w^{K_w} \colon |K| \to \mathbf{I}$ is upper semi-continuous. Note that $(\beta_w^{K_w})^{-1}(1) = \{w\}$ is closed in $|K|$. For each $t \in (0, 1)$,

$$(\beta_w^{K_w})^{-1}([t, 1]) = \varphi_{1-t}^w(|\operatorname{St}(\sigma_w, K)|) = \left\{ tw + (1 - t)z \;\middle|\; z \in |\operatorname{St}(\sigma_w, K)| \right\},$$

which is closed in $|K|$ by Lemma 5.3. Indeed, let $y \in (\beta_w^{K_w})^{-1}([t, 1])$. If $\beta_w^{K_w}(y) = 1$, then $y = w \in \varphi_{1-t}^w(|\operatorname{St}(\sigma_w, K_w)|)$. When $\beta_w^{K_w}(y) < 1$, take the same y^* as (5.1) in the above. Then, since $\beta_w^{K_w}(y) \geqslant t$, we have

$$z = \frac{\beta_w^{K_w}(y) - t}{1 - t} w + \frac{1 - \beta_w^{K_w}(y)}{1 - t} y^* \in |\operatorname{St}(\sigma_w, K)|.$$

Similarly to the above, it follows that

$$y = tw + (1 - t)z = \varphi_{1-t}^w(z) \in \varphi_{1-t}^w(|\operatorname{St}(\sigma_w, K_w)|).$$

The inclusion $\varphi_{1-t}^w(|\operatorname{St}(\sigma_w, K)|) \subset (\beta_w^{K_w})^{-1}([t, 1])$ follows from the following implication:

$$z \in |\operatorname{St}(w, K)| \Rightarrow \beta_w^{K_w}(\varphi_{1-t}^w(z)) = \beta_w^{K_w}(tw + (1 - t)z) = t + (1 - t)\beta_w^{K_w}(z) \geqslant t.$$

This completes the proof. ■

Lemma 5.5 *Let V be a discrete set in $|K|$ such that $O(v, K) \cap O(v', K) = \varnothing$ if $v \neq v' \in V$. If $\dim K = n < \infty$, then K_V is a proper subdivision of K.*

Proof Due to Theorem 5.2, the proof is reduced to show that $O(v, K_V)$ is open in $|K|$ for every $v \in K_V^{(0)} = V \cup K^{(0)}$. If $v \in V$, then $O(v, K_V) = O(v, K)$ is open in $|K|$. When $v \in K^{(0)}$, since $O(v, K_V) = (\beta_v^{K_V})^{-1}((0, 1])$, it suffices to show the continuity of $\beta_v^{K_V} : |K| \to \mathbf{I}$.

https://doi.org/10.4153/CJM-2010-083-5 Published online by Cambridge University Press

Note that each $x \in |K|$ can be written as follows:

$$x = \sum_{w \in V} \beta_w^{K_V}(x) w + \sum_{u \in K^{(0)}} \beta_u^{K_V}(x) u,$$

where $\beta_{w_x}^{K_V}(x) > 0$ at most one $w_x \in V$. Then we have

$$\beta_v^K(x) = \beta_{w_x}^{K_V}(x)\beta_v^K(w_x) + \beta_v^{K_V}(x).$$

Thus, the following holds:

$$\forall v \in K^{(0)},\ \beta_v^{K_V}(x) = \beta_v^K(x) - \beta_{w_x}^{K_V}(x)\beta_v^K(w_x). \tag{5.2}$$

Now let $v \in K^{(0)}$ be fixed. To see the continuity of $\beta_v^{K_V}$ at $x \in |C(V, K)|$, for any $\varepsilon > 0$, let

$$0 < \delta = \frac{\varepsilon \operatorname{dist}(\sigma_x, V)}{4n} < \frac{\varepsilon}{2}.$$

Then we shall show the following:

$$y \in O(x, K),\ \rho_K(x, y) < \delta \Rightarrow |\beta_v^{K_V}(y) - \beta_v^{K_V}(x)| < \varepsilon.$$

Let $\sigma_x, \sigma_y \in K$ be the carriers of x and y respectively. Since $x \in |C(V, K)|$, we have $\sigma_x \cap V = \varnothing$, which implies $\beta_v^{K_V}(x) = \beta_v^K(x)$. If $\sigma_y \cap V = \varnothing$, then $\beta_v^{K_V}(y) = \beta_v^K(y)$ and hence

$$|\beta_v^{K_V}(y) - \beta_v^{K_V}(x)| = |\beta_v^K(y) - \beta_v^K(x)| \leq \rho_K(x, y) < \delta < \varepsilon.$$

In case $\sigma_y \cap V \neq \varnothing$, we have $w_y \in V$ such that $\beta_{w_y}^{K_V}(y) > 0$, which implies that the carrier $\sigma_{w_y} \in K$ of w_y is a face of σ_y. Then it follows from (5.2) that

$$\begin{aligned}|\beta_v^{K_V}(y) - \beta_v^{K_V}(x)| &= |\beta_v^K(y) - \beta_{w_y}^{K_V}(y)\beta_v^K(w_y) - \beta_v^K(x)| \\ &\leq |\beta_v^K(y) - \beta_v^K(x)| + \beta_{w_y}^{K_V}(y)\beta_v^K(w_y) \leq \rho_K(x, y) + \beta_{w_y}^{K_V}(y).\end{aligned}$$

Since $\rho_K(x, y) < \varepsilon/2$, it remains to show that $\beta_{w_y}^{K_V}(y) < \varepsilon/2$. We can take $z \in \sigma_x$ such that $\beta_u^K(z) \geqslant \beta_u^K(w_y)$ for each $u \in \sigma_x^{(0)}$. Observe that

$$\begin{aligned}\rho_K(z, w_y) &= \sum_{u \in \sigma_x^{(0)}} \left(\beta_u^K(z) - \beta_u^K(w_y)\right) + \sum_{u \in \sigma_{w_y}^{(0)} \setminus \sigma_x^{(0)}} \beta_u^K(w_y) \\ &= 1 - \sum_{u \in \sigma_x^{(0)}} \beta_u^K(w_y) + \sum_{u \in \sigma_{w_y}^{(0)} \setminus \sigma_x^{(0)}} \beta_u^K(w_y) = 2 \sum_{u \in \sigma_{w_y}^{(0)} \setminus \sigma_x^{(0)}} \beta_u^K(w_y).\end{aligned}$$

Since $\operatorname{dist}(\sigma_x, V) \leq \rho_K(z, w_y)$, we have

$$\sum_{u \in \sigma_{w_y}^{(0)} \setminus \sigma_x^{(0)}} \beta_u^K(w_y) \geqslant \frac{\operatorname{dist}(\sigma_x, V)}{2}.$$

https://doi.org/10.4153/CJM-2010-083-5 Published online by Cambridge University Press

Hence, $\beta_u^K(w_y) > \operatorname{dist}(\sigma_x, V)/2n$ for some $u \in \sigma_{w_y}^{(0)} \setminus \sigma_x^{(0)}$ because $\dim \sigma_{w_y} \leq n$. By virtue of (5.2), we have

$$\beta_{w_y}^{K_V}(y) = \frac{\beta_u^K(y) - \beta_u^{K_V}(y)}{\beta_u^K(w_y)} \leq \frac{\beta_u^K(y)}{\beta_u^K(w_y)} \leq \frac{\rho_K(x,y)}{\beta_u^K(w_y)} < \frac{2n\delta}{\operatorname{dist}(\sigma_x, V)} < \frac{\varepsilon}{2}.$$

Finally, it remains to show the continuity of $\beta_v^{K_V}$ at $x \in |K| \setminus |C(V,K)|$. Note that $\sigma_x \cap V \neq \varnothing$, which is the singleton $\{w_x\}$. For every $y \in O(x,K)$, σ_x is a face of σ_y, hence $\sigma_y \cap V = \{w_x\}$. Then it follows from (5.2) that

$$\beta_v^{K_V}(y) = \beta_v^K(y) - \beta_{w_x}^{K_V}(y)\beta_v^K(w_x) \text{ for every } y \in O(x,K).$$

Observe that $\beta_{w_x}^{K_V}|O(x,K) = \beta_{w_x}^{K_{w_x}}|O(x,K)$, which is continuous as saw in the proof of Lemma 5.4. Since β_v^K is continuous, $\beta_v^{K_V}|O(x,K)$ is also continuous. ■

Lemma 5.6 *Let K be a finite-dimensional simplicial complex. Then, a derived subdivision K' of K is proper if $K'^{(0)}$ is discrete in $|K|$.*

Proof Let $\dim K = n$ and $K'^{(0)} = K^{(0)} \cup \{v_\sigma \mid \sigma \in K \setminus K^{(0)}\}$, where each v_σ is an interior point of σ. For each $i = 1, \dots, n$, let $V_i = \{v_\sigma \mid \sigma \in K^{(i)} \setminus K^{(i-1)}\}$. By downward induction, we define subdivisions K_i of K as follows: $K_i = (K_{i+1})_{V_i}$, where $K_{n+1} = K$. Note that V_i is discrete in $|K|$ by the assumption and $O(v, K_{i+1}) \cap O(v', K_{i+1})$ if $v \neq v' \in V_i$. By using Lemma 5.5 inductively, we can see that each K_i is a proper subdivision. Observe that $K' = K_1$. Thus, we have the result. ■

Now, Theorem 1.6 easily follows from Lemma 5.6.

Proof of Theorem 1.6 Since $K'^{(0)}$ is discrete in $|K'|$, it suffices to show the "if" part. For each vertex $v \in K'^{(0)}$, let $\sigma_v \in K$ be the carrier of v and $u \in \sigma_v^{(0)}$. Then, $O(v,K') \subset O(v,K) \subset |\operatorname{St}(u,K)|$. Let L' be the subdivision of $L = \operatorname{St}(u,K)$ induced from K'. Since $\dim L < \infty$ and $L'^{(0)}$ is discrete in $|L|$, it follows from Lemma 5.6 that L' is a proper subdivision of L. Thus, $O(v,K') = O(v,L')$ is open in $|L| = |\operatorname{St}(u,K)|$, and therefore in $O(u,K)$. Since $O(u,K)$ is open in $|K|$, it follows that $O(v,K')$ is open in $|K|$. Then, K' is a proper subdivision of K by Theorem 5.2. ■

By using Theorem 1.6, Theorem 1.5 can be proved in the standard way.

Proof of Theorem 1.5 Let $f\colon |K| \to \mathbf{I}$ be the simplicial map defined by $f(L^{(0)}) = 0$ and $f(K^{(0)} \setminus L^{(0)}) = 1$. Then, $f^{-1}(0) = |L|$ and $N(L,K) \subset f^{-1}([0,1))$. Moreover, $f^{-1}(\frac{1}{2})$ is bicollared in $|K|$. In fact, for each $0 < t < t' < 1$, there is a homeomorphism $h\colon f^{-1}(t) \times \mathbf{I} \to f^{-1}([t,t'])$ such that $h(f^{-1}(t) \times \{0\}) = f^{-1}(t)$ and $h(f^{-1}(t) \times \{1\}) = f^{-1}(t')$. This can be shown as follows: let $(\beta_v(x))_{v \in K^{(0)}}$ be the barycentric coordinate for $x \in |K|$, that is, $x = \sum_{v \in K^{(0)}} \beta_v(x)v$. Note that $x \in f^{-1}(t)$ if and only if $\sum_{v \in L^{(0)}} \beta_v(x) = 1 - t$ and $\sum_{v \in K^{(0)} \setminus L^{(0)}} \beta_v(x) = t$. Then, h can be defined as follows:

$$h(x,s) = \sum_{v \in L^{(0)}} \frac{1 - ((1-s)t + st')}{1-t}\beta_v(x)v + \sum_{v \in K^{(0)} \setminus L^{(0)}} \frac{(1-s)t + st'}{t}\beta_v(x)v.$$

Thus, to see that $\operatorname{bd} N(L,K)$ is bicollared in $|K|$, it is sufficient to construct a homeomorphism $\varphi\colon |K| \to |K|$ such that $\varphi(\operatorname{bd}_{|K|} N(L,K)) = f^{-1}(\frac{1}{2})$.

Now, let

$$S = \left\{ \sigma \in K \;\middle|\; \sigma^{(0)} \cap L^{(0)} \neq \varnothing,\ \sigma^{(0)} \setminus L^{(0)} \neq \varnothing \right\}.$$

For each $\sigma \in S$, let σ_0 and σ_1 be the faces of σ spanned by $\sigma^{(0)} \cap L^{(0)}$ and $\sigma^{(0)} \setminus L^{(0)}$ respectively, and define $v_\sigma = \frac{1}{2}\hat{\sigma}_0 + \frac{1}{2}\hat{\sigma}_1 \in f^{-1}(\frac{1}{2}) \cap \sigma^\circ$. By using the barycenters $\hat{\sigma}$ of $\sigma \in K \setminus S$ and the points v_σ for $\sigma \in S$, we define the derived subdivision K'. Then, $K'^{(0)}$ is discrete in $|K|$. Indeed, $\operatorname{Sd} K^{(0)} \setminus \{\hat{\sigma} \mid \sigma \in S\}$ is discrete in $|K|$. Note that $f^{-1}(\frac{1}{2})$ is closed in $|K|$. Then, it suffices to see that $\{v_\sigma \mid \sigma \in S\}$ is discrete in $f^{-1}(\frac{1}{2})$. For each $x \in f^{-1}(\frac{1}{2})$, let $\sigma_x \in K$ be the carrier of x. Then $O(\hat{\sigma}_x, \operatorname{Sd} K)$ is a neighborhood of x in $|K|$. If $v_\sigma \in O(\hat{\sigma}_x, \operatorname{Sd} K)$, then σ_x is a proper face of σ. Since $\beta^K_v(x) = 0$ and $\beta^K_v(v_\sigma) \geqslant 1/2(\dim\sigma + 1)$ for every $v \in \sigma^{(0)} \setminus \sigma_x^{(0)}$, it follows that

$$\begin{aligned}\rho_K(x, v_\sigma) &\geqslant \frac{\dim\sigma - \dim\sigma_x}{2(\dim\sigma + 1)} = \frac{1}{2} - \frac{\dim\sigma_x + 1}{2(\dim\sigma + 1)}\\ &\geqslant \frac{1}{2} - \frac{\dim\sigma_x + 1}{2(\dim\sigma_x + 2)} = \frac{1}{2(\dim\sigma_x + 2)}.\end{aligned}$$

Thus, x has a neighborhood in $|K|$ that meets $\{v_\sigma \mid \sigma \in S\}$ at most one point.

By Theorem 1.6, the metric topology for $|K'|$ coincides with the one for $|K|$, that is, $|K'| = |K|$ as topological spaces. Recall that $|\operatorname{Sd} K| = |K|$ as topological spaces. Then the desired homeomorphism is obtained as the simplicial isomorphism $\varphi\colon |\operatorname{Sd} K| \to |K'|$ defined by $\varphi(\hat{\sigma}) = \hat{\sigma}$ for $\sigma \in K \setminus S$ and $\varphi(\hat{\sigma}) = v_\sigma$ for $\sigma \in S$. ■

6 Proof of Theorem 1.3

In this section, we shall prove Theorem 1.3. Replacing each $\ell_2(\tau_i)$ with the unit open ball $\mathbb{B}_i$, we construct an open embedding of $|K| \times \boxdot_{i\in\mathbb{N}}\mathbb{B}_i$ into $\boxdot_{i\in\mathbb{N}}\mathbb{B}_i$.

Lemma 6.1 *There exists a tower $P_1 \subset P_2 \subset \cdots$ of polyhedra in $|K|$ such that $\bigcup_{n\in\mathbb{N}} P_n = |K|$, each P_n is triangulated by a subcomplex of the n-th barycentric subdivision $\operatorname{Sd}^n K$, $\operatorname{dens} P_n \leq \tau_n$, $P_n \subset \operatorname{int}_{|K|} P_{n+1}$, and $\operatorname{bd}_{|K|} P_n$ is a bicollared in $|K|$, hence it is a Z-set in both P_n and $|K| \setminus \operatorname{int}_{|K|} P_n$.*

Proof Since $\operatorname{card} K^{(0)} \leq \tau = \sup_{n\in\mathbb{N}} \tau_n$, we can write $K^{(0)} = \bigcup_{n\in\mathbb{N}} V_n$, where $V_1 \subset V_2 \subset \cdots$ and $\operatorname{card} V_n \leq \tau_n$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $V'_n = \{v \in V_n \mid \operatorname{card} \operatorname{St}(v,K) \leq \tau_n\}$. Since $\operatorname{card}\operatorname{St}(v,K) < \tau$ for each $v \in K^{(0)}$, we have $K^{(0)} = \bigcup_{n\in\mathbb{N}} V'_n$. Moreover, let $V^*_n = (\operatorname{Sd}^{n-1} K)^{(0)} \cap \bigcup_{v\in V'_n} O(v,K)$. Then $\operatorname{St}(v, \operatorname{Sd}^n K)^{(0)} \subset V^*_{n+1}$ for each $v \in V^*_n$. Indeed, $v \in O(v',K)$ for some $v' \in V'_n$. Since $v \in (\operatorname{Sd}^{n-1} K)^{(0)}$, it follows that $|\operatorname{St}(v, \operatorname{Sd}^n K)| \subset O(v',K)$, hence we have $\operatorname{St}(v, \operatorname{Sd}^n K)^{(0)} \subset (\operatorname{Sd}^n K)^{(0)} \cap O(v',K) \subset V^*_{n+1}$.

For each $n \in \mathbb{N}$, we define a polyhedron

$$P_n = \bigcup_{v\in V^*_n} |\operatorname{St}(v, \operatorname{Sd}^n K)| \subset \bigcup_{v\in V_n} |\operatorname{St}(v,K)|,$$

that is, $P_n = |N(L_n, \mathrm{Sd}^{n-1} K)|$, where L_n is the full subcomplex of $\mathrm{Sd}^{n-1} K$ with $L_n^{(0)} = V_n^* \ (\subset (\mathrm{Sd}^{n-1} K)^{(0)})$. It follows from Theorem 1.5 that $\mathrm{bd}\, P_n$ is bicollared in $|K|$. Observe that $\mathrm{dens}\, P_n \leq \tau_n$ and $P_n \subset \bigcup_{v \in V_{n+1}^*} O(v, \mathrm{Sd}^{n+1} K) \subset \mathrm{int}_{|K|} P_{n+1}$. Moreover, $|K| = \bigcup_{n \in \mathbb{N}} P_n$. Indeed, each $x \in |K|$ is contained in $O(v, K)$ for some $v \in K^{(0)}$. Choose $n \in \mathbb{N}$ so that $v \in V_n'$ and $x \in |\mathrm{St}(v', \mathrm{Sd}^n K)| \subset O(v, K)$ for some $v' \in (\mathrm{Sd}^{n-1} K)^{(0)}$. Then $v' \in V_n^*$, which implies $x \in P_n$. ■

It should be remarked that the local finite-dimensionality of K implies the complete metrizability of $|K|$ and the barycentric subdivision is a proper subdivision, that is, it does not change the topology on $|K|$. Thus, in Lemma 6.1 above, each P_n is a completely metrizable ANR.

Similarly to the second step of the proof of Theorem 1.1, take an increasing sequence $0 < c_1 < c_2 < \cdots$ with $\sup_{n \in \mathbb{N}} c_n = 1$. Now, for each $n \in \mathbb{N}$, we define

$$M_n = \mathrm{int}_{|K|} P_n \times \prod_{i=1}^n c_n \mathbb{B}_i, \ \overline{M}_n = P_n \times \prod_{i=1}^n c_n \overline{\mathbb{B}}_i \ \text{ and}$$

$$\partial M_n = \overline{M}_n \setminus M_n = \Big(\mathrm{bd}_{|K|} P_n \times \prod_{i=1}^n c_n \overline{\mathbb{B}}_i \Big) \cup (P_n \times D_n),$$

where D_n is the same as in Section 3, that is, $D_n = \prod_{i=1}^n c_n \overline{\mathbb{B}}_i \setminus \prod_{i=1}^n c_n \mathbb{B}_i$. Moreover, let

$$L_{n+1} = M_{n+1} \setminus (M_n \times c_n \mathbb{B}_{n+1}) \ \text{ and } \ \bar{\partial} M_n = (\partial M_n \times c_n \overline{\mathbb{B}}_{n+1}) \cup (\overline{M}_n \times c_n \mathbb{S}_{n+1}).$$

Then it should be noted that

$$\overline{M}_{n+1} = L_{n+1} \cup (\overline{M}_n \times c_n \mathbb{B}_{n+1}) \ \text{ and } \ \bar{\partial} M_n = L_{n+1} \cap (\overline{M}_n \times c_n \overline{\mathbb{B}}_{n+1}).$$

Lemma 6.2 *Each M_n, $\overline{M}_n$, and ∂M_n is an $\ell_2(\tau_n)$-manifold with density τ_n, and ∂M_n is a Z-set in $\overline{M}_n$ that contains a strong deformation retract of $\overline{M}_n$.*

Proof Except for the last statement, the proof is the same as Lemma 4.2. Since D_n and $\prod_{i=1}^n c_n \overline{\mathbb{B}}_i$ are AR's (cf. Lemma 4.1), D_n is a strong deformation retract of $\prod_{i=1}^n c_n \overline{\mathbb{B}}_i$, hence $P_n \times D_n$ is a strong deformation retract of $\overline{M}_n = P_n \times \prod_{i=1}^n c_n \overline{\mathbb{B}}_i$. ■

Lemma 6.3 *Each $\bar{\partial} M_n$ and L_{n+1} is an $\ell_2(\tau_{n+1})$-manifold with density τ_{n+1}, both ∂M_{n+1} and $\bar{\partial} M_n$ are Z-sets in L_{n+1} and ∂M_{n+1} contains a strong deformation retract of L_{n+1}.*

Proof Because of the similarlity with Lemma 4.3, we shall show that ∂M_{n+1} is a Z-set in L_{n+1} and it contains a strong deformation retract of L_{n+1}. Since ∂M_{n+1} is a Z-set in $\overline{M}_{n+1}$ with $\partial M_{n+1} \subset \overline{M}_{n+1} \setminus \overline{M}_n$ and $\overline{M}_{n+1} \setminus \overline{M}_n$ is open in both $\overline{M}_{n+1}$ and L_{n+1}, it follows that ∂M_{n+1} is a Z-set in L_{n+1}. As we saw in the proof of Lemma 6.2, $D_n = \prod_{i=1}^n c_n \overline{\mathbb{B}}_i \setminus \prod_{i=1}^n c_n \mathbb{B}_i$ is a strong deformation retract of $\prod_{i=1}^n c_n \overline{\mathbb{B}}_i$, hence $\prod_{i=1}^n c_{n+1} \overline{\mathbb{B}}_i \setminus \prod_{i=1}^n c_n \mathbb{B}_i$ is a strong deformation retract of $\prod_{i=1}^n c_{n+1} \overline{\mathbb{B}}_i$. Moreover, $D_{n+1} = \prod_{i=1}^n c_{n+1} \overline{\mathbb{B}}_i \setminus \prod_{i=1}^n c_{n+1} \mathbb{B}_i$ is also a strong deformation retract of $\prod_{i=1}^n c_{n+1} \overline{\mathbb{B}}_i \setminus \prod_{i=1}^n c_n \mathbb{B}_i$. Then it easily follows that $P_{n+1} \times D_{n+1}$ is a strong deformation retract of L_{n+1}. ■

https://doi.org/10.4153/CJM-2010-083-5 Published online by Cambridge University Press

Now, we can complete the proof of Theorem 1.3

Proof of Theorem 1.3 Observe that $M_1 \times \boxdot_{i>1} c_n \mathbb{B}_i \subset M_2 \times \boxdot_{i>2} c_{n+1} \mathbb{B}_i \subset \cdots$ are open in $|K| \times \boxdot_{i\in\mathbb{N}} \mathbb{B}_i$ and

$$|K| \times \boxdot_{i\in\mathbb{N}} \mathbb{B}_i = \bigcup_{n\in\mathbb{N}} (M_n \times \boxdot_{i>n} c_n \mathbb{B}_i).$$

We shall inductively define closed embeddings $g_n \colon \overline{M}_n \to \prod_{i=1}^n \mathbb{B}_i$, $n \in \mathbb{N}$, such that

$$g_n(\partial M_n) = \operatorname{bd} g_n(\overline{M}_n) \text{ and } g_{n+1}|\overline{M}_n \times c_n \overline{\mathbb{B}}_{n+1} = g_n \times \mathrm{id}\,.$$

Now we have the following commutative diagram of open embeddings

$$\begin{array}{ccc} M_n \times \boxdot_{i>n} c_n \mathbb{B}_i & \subset & M_{n+1} \times \boxdot_{i>n+1} c_{n+1} \mathbb{B}_i \\ {\scriptstyle g_n \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle g_{n+1} \times \mathrm{id}} \\ \boxdot_{i\in\mathbb{N}} \mathbb{B}_i & = & \boxdot_{i\in\mathbb{N}} \mathbb{B}_i. \end{array}$$

This induces the open embedding $g \colon |K| \times \boxdot_{i\in\mathbb{N}} \mathbb{B}_i \to \boxdot_{i\in\mathbb{N}} \mathbb{B}_i$.

By Lemma 6.2, we can apply Theorem 2.7 to obtain an embedding $g_1 \colon \overline{M}_1 \to \mathbb{B}_1$ such that $g_1(\partial M_1) = \operatorname{bd} g_1(\overline{M}_1)$ is bicollared in $\mathbb{B}_1$. Now, assuming that $g_1, \dots, g_n$ have been obtained, we shall construct g_n. Let $E = \prod_{i=1}^{n+1} \mathbb{B}_i \setminus (g_n(M_n) \times c_n \mathbb{B}_{n+1})$. Observe that $g_n(\partial M_n) \times \mathbb{B}_{n+1}$ and $\prod_{i=1}^n \mathbb{B}_i \times c_n \mathbb{S}_{n+1}$ are bicollared in $\prod_{i=1}^{n+1} \mathbb{B}_i$. Then, similarly to the proof of Lemma 4.3, we can see that $(g_n \times \mathrm{id})(\bar\partial M_n)$ is a strong Z-set in E and hence E is an $\ell_2(\tau_{n+1})$-manifold. Since $c_n \mathbb{S}_{n+1}$ is a strong deformation retract of both $c_n \overline{\mathbb{B}}_{n+1}$ and $\overline{\mathbb{B}}_{n+1} \setminus c_n \mathbb{B}_{n+1}$, it is easy to see that $\prod_{i=1}^n \mathbb{B}_i \times c_n \mathbb{S}_{n+1}$ is a strong deformation retract of E. Since $\mathbb{S}_{n+1}$ is contractible, so is E, hence $E \approx \ell_2(\tau_{n+1})$ by the Classification Theorem.

By Theorem 2.7 and Lemma 6.3, we have an embedding $g' \colon L_{n+1} \to E$ such that $g'(\partial M_{n+1}) = \operatorname{bd}_E g'(L_{n+1})$ is bicollared in E. Note that $g'(\bar\partial M_n)$ is a Z-set in E because it is closed in E and a Z-set in the open set $g'(L_{n+1} \setminus \partial M_{n+1}) \subset E$. By using the Z-set Unknotting Theorem, we have a homeomorphism $g'' \colon E \to E$ such that $(g_n \times \mathrm{id})|\bar\partial M_n = g'' g' |\bar\partial M_n$. Then, we can define an embedding

$$g_{n+1} \colon \overline{M}_{n+1} \to \prod_{i=1}^{n+1} \mathbb{B}_i \text{ by } g_{n+1}|\overline{M}_n \times c_n \overline{\mathbb{B}}_i \text{ and } g_{n+1}|E = g'' g' |E.$$

Since $g_{n+1}(L_{n+1} \setminus \partial M_{n+1}) = g'' g'(L_{n+1} \setminus \partial M_{n+1})$ is open in E, we have an open set W in $\prod_{i=1}^{n+1} \mathbb{B}_i$ such that $g_{n+1}(L_{n+1} \setminus \partial M_{n+1}) = W \cap E$. Since $g_{n+1}(M_n \times c_n \mathbb{B}_n) = g_n(M_n) \times c_n \mathbb{B}_n$ is open in $\prod_{i=1}^{n+1} \mathbb{B}_i$, it follows that $g_{n+1}(M_{n+1}) = g_{n+1}(M_n \times c_n \mathbb{B}_n) \cup W$ is open in $\prod_{i=1}^{n+1} \mathbb{B}_i$. Hence, we have $g_{n+1}(\partial M_{n+1}) = \operatorname{bd}_E g_{n+1}(M_{n+1})$. This completes the induction. Then we have the result. ■

References

[1] R. D. Anderson and J. D. McCharen, *On extending homeomorphisms to Fréchet manifolds.* Proc. Amer. Math. Soc. **25**(1970), 283–289.

[2] L. C. Glaser, *Geometrical combinatorial topology. I.* Van Nostrand Reinhold Mathematical Studies, 27, Van Nostrand Reinhold Co., London, 1970.

[3] R.E. Heisey, *Stability, classification, open embeddings, and triangulation of* $\mathbb{R}^\infty$*-manifolds.* In: Proceeding of the International Conference on Geometric Topology, Polish Scientific Publishers, Warsaw, 1980, pp. 193–196.

[4] ———, *Manifolds modelled on the direct limit of lines.* Pacific J. Math. **102**(1982), no. 1, 47–54.

[5] D. W. Henderson, *Infinite-dimensional manifolds are open subsets of Hilbert space.* Topology **9**(1970), 25–33. doi:10.1016/0040-9383(70)90046-7

[6] ———, *Corrections and extensions of two papers about infinite-dimensional manifolds.* General Topology and Appl. **1**(1971), 321–327. doi:10.1016/0016-660X(71)90004-3

[7] ———, *Z-sets in ANR's.* Trans. Amer. Math. Soc. **213**(1975), 205–216.

[8] D. W. Henderson and R. M. Schori, *Topological classification of infinite dimensional manifolds by homotopy type.* Bull. Amer. Math. Soc. **76**(1970), 121–124. doi:10.1090/S0002-9904-1970-12392-8

[9] S.-T. Hu, *Theory of retracts.* Wayne State University Press, Detroit, MI, 1965.

[10] P. Mankiewicz, *On topological, Lipschitz, and uniform classification of LF-spaces.* Studia Math. **52**(1974), 109–142.

[11] K. Mine and K. Sakai, *Open subsets of LF-spaces.* Bull. Pol. Acad. Sci. Math. **56**(2008), no. 1, 25–37.

[12] ———, *Subdivision of simplicial complexes preserving the metric topology.* Canad. Math. Bull., to appear.

[13] L. Narici and E. Beckenstein, *Topological vector spaces.* Monographs and Textbooks in Pure and Applied Mathematics, 95, Marcel Dekker, Inc., New York, 1985. doi:10.4064/ba56-1-4

[14] K. Sakai, *Embeddings of infinite-dimensional pairs and remarks on stability and deficiency.* J. Math. Soc. **29**(1977), no. 2, 261–280. doi:10.2969/jmsj/02920261

[15] ———, *An embedding theorem of infinite-dimensional manifold pairs in the model space.* Fund. Math. **100**(1978), 83–87.

[16] ———, *Boundaries and complements of infinite-dimensional manifolds in the model space.* Topology Appl. **15**(1983), no. 1, 79–91. doi:10.1016/0166-8641(83)90050-0

[17] ———, *On* $\mathbb{R}^\infty$*-manifolds and* Q^∞*-manifolds.* Topology Appl. **18**(1984), no. 1, 69–79. doi:10.1016/0166-8641(84)90032-4

[18] H. Toruńczyk, *Absolute retracts as factors of normed linear spaces.* Fund. Math. **86**(1974), 53–67.

[19] ———, *Characterizing Hilbert space topology.* Fund. Math. **111**(1981), no. 3, 247–262.

[20] ———, *A correction of two papers concerning Hilbert manifolds: "Concerning locally homotopy negligible sets and characterization of* l_2*-manifolds" [Fund. Math.* ***101****(1978), no. 2, 93–110] and "Characterizing Hilbert space topology" [ibid.* ***111****(1981), no. 3, 247–262].* Fund. Math. **125**(1985), no. 1, 89–93.

[21] A. Wilansky, *Modern methods in topological vector spaces.* McGraw-Hill, New York, 1978.

Institute of Mathematics, University of Tsukuba, Tsukuba, 305-8571, Japan
e-mail: (Mine) pen@math.tsukuba.ac.jp
(Sakai) sakaiktr@sakura.cc.tsukuba.ac.jp