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# *FK* SPACES IN WHICH THE SEQUENCE OF COORDINATE VECTORS IS BOUNDED

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**1. Introduction.** The work presented in this paper was initially motivated by the following question of A. Wilansky: "Is there a smallest *FK*-space $E$ in which $\mathscr{E} = \{e_i : i = 1, 2, \ldots\}$ is bounded?" Here *FK*-space means a complete linear metric space of real or complex sequences $\mathbf{x} = (x_i)$ upon which the coordinate functionals $\mathbf{x} \to x_i$ are continuous for each $i$ (see [**10**, p. 202]), and $e_i = (\delta_{ij})_{j=1}^{\infty}$. An *FK*-space need not be locally convex, and therein lies the difficulty of the problem since it is easy to see that $l^1$ is the smallest locally convex *FK*-space. Theorem 3.4 gives a negative answer to Wilansky's question by showing that the intersection of all *FK*-spaces in which $\mathscr{E}$ is bounded is $\phi$, the space of all finitely non-zero sequences in which $\mathscr{E}$ is not bounded and which is not an *FK*-space.

The problem was treated by constructing a class of *FK*-spaces, $L(f)$, where $f$ is a modulus. (See Definition 3.1 and Theorem 3.2.) Each space of the type $L(f)$ is symmetric and has $\mathscr{E}$ as an absolute Schauder basis. There are only three sequentially complete barrelled locally convex spaces having symmetric absolute bases, namely $\phi$, $l^1$ (all $\mathbf{x}$ with $\sum_{n=1}^{\infty} |x_n| < \infty$) and $\omega$ (all sequences). (See Proposition 2.2 and Corollary 2.3.) However, there are uncountably many spaces $L(f)$. Spaces of the type $L(f)$ are a special case of the spaces structured by B. Gramsch in [**11**]. The idea of modulus was structured in 1953 by Nakano [**12**].

Symmetric sequence spaces, which are locally convex have been frequently investigated; see, for example [**1**; **2**; **5**; **7**; **8**]. From the point of view of local convexity, spaces of the type $L(f)$ are quite pathological. In fact, $L(f)$ will not be locally semiconvex unless it contains $\bigcap_{p>0} l^p$. (See Definition 2.4 and Theorem 2.5.) Nevertheless, these spaces are relatively easy to define and study.

For $0 < p < \infty$ let $l^p$ denote the set of all sequences $\mathbf{x}$ with $\sum_{n=1}^{\infty} |x_n|^p < \infty$. Then $l^p$ is an *FK*-space in which $\mathscr{E}$ is bounded and forms a symmetric absolute basis. The space $\bigcap_{p>0} l^p$ is also an *FK*-space in which $\mathscr{E}$ is bounded, and we note in Theorem 2.5 that it is the smallest such space which is locally semiconvex.

We now recall some definitions. A *basis* for a linear topological space $X$ is a sequence $(x_n)$ of vectors in $X$ such that each $x$ in $X$ has a unique expansion

$$x = \sum_{n=1}^{\infty} a_n x_n \tag{1}$$

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where each $a_n$ is a scalar. The basis $(x_n)$ is a *Schauder* basis if each linear functional $x_n'(x) = a_n$ is in $X'$, the topological dual space of $X$. When $X$ is a locally convex space the basis $(x_n)$ is called *absolute* if for each $x$ in $X$, $\sum_{n=1}^{\infty} |a_n| p(x_n) < \infty$ for each continuous seminorm on $X$. If $X$ is a linear metric space which is not locally convex $(x_n)$ is called *absolute* if there is an $(F)$-norm [**4**, p. 113] $|\ |$ such that $\sum_{n=1}^{\infty} |a_n||x_n| < \infty$. The space $E = \{(x_n'(x)) : x \in X\}$ is called the sequence space *associated* with the basis $(x_n)$ of $X$. For $E$ a sequence space

$$E^\alpha = \{(s_n) : \textstyle\sum_n |s_n t_n| < \infty \text{ for each } \mathbf{t} \in E\}$$
$$E^\beta = \{(s_n) : \textstyle\sum_n s_n t_n \text{ converges for each } \mathbf{t} \text{ in } E\}$$
$$E^\gamma = \{(s_n) : \sup_N |\textstyle\sum_{n=1}^{N} s_n t_n| < \infty \text{ for each } \mathbf{t} \text{ in } E\}.$$

A sequence space $E$ is *symmetric* if $(x_{\pi(n)})$ is in $E$ whenever $(x_n)$ is in $E$ and $\pi$ is a permutation on the integers. A sequence space $E$ is called *balanced* if $(a_n x_n)$ is in $E$ whenever $(x_n)$ is in $E$ and $|a_n| \leqq 1$ for each $n$. A basis is called *symmetric* if its associated sequence space is symmetric. A basis $(x_n)$ for a linear topological space $X$ is *equivalent* to a basis $(y_n)$ for a topological space $Y$ if there is a topological isomorphism $T$ from $X$ onto $Y$ with $Tx_\pi = y_n$ for each $n$. If two bases are equivalent they have the same associated sequence spaces. The converse to this is true if they are bases of complete metric linear spaces or Schauder bases of barrelled spaces [**3**; **8**].

## 2. Locally convex and locally semi-convex $FK$-spaces.

2.1 LEMMA. *If $E$ is a balanced symmetric locally convex space of sequences with basis $\mathscr{E}$ then either $E = \phi$ or $\mathscr{E}$ is bounded in $E$.*

*Proof.* Since $\mathscr{E}$ is a basis for $E$ the sequence $\sum_n (x'e_n)x_n$ converges for $\mathbf{x}$ in $E$ and $x' \in E'$. Thus $\{(x'e_i) : x' \in E'\} \subset E^\beta$. Since $E$ is balanced $E^\beta = E^\alpha$ and $E^\alpha$ is symmetric and perfect. By [**5**, Satz 2, §14] $E^\alpha = \phi$, $E^\alpha = \omega$ or $l^1 \subset E^\alpha \subset m$. If $E^\alpha = \omega$ then $E = \phi$. Otherwise $\sup_n |x'e_n| < \infty$ for each $x'$ in $E'$. Therefore $\mathscr{E}$ is weakly, hence strongly bounded in $E$.

2.2 PROPOSITION. *If $E$ is a symmetric sequentially complete barrelled locally convex space of sequences in which $\mathscr{E}$ is an absolute Schauder basis, then $E = \phi$, $E = l^1$ or $E = \omega$.*

*Proof.* Suppose $E$ is distinct from $\phi$ and $\omega$. Then $l^1 \subset E^\alpha \subset m$ since $E^\alpha$ is symmetric and perfect. But this implies that $l^1 \subset E^{\alpha\alpha} \subset m$. By Lemma 2.1, $\mathscr{E}$ is bounded in $E$ so that $l^1 \subset E$ because $\mathscr{E}$ is sequentially complete. Since $E \subset E^{\alpha\alpha}$ is barrelled and $\mathscr{E}$ is a basis for $E$ the inclusion from $E$ into $m$ is continuous. Thus the norm $||\mathbf{x}||_\infty = \sup_n |x_n|$ is continuous on $E$. But $\mathscr{E}$ is an absolute basis for $E$ so that $\mathbf{x}$ in $E$ implies $\sum_{i=1}^{\infty} ||x_n e_n||_\infty = \sum_{i=1}^{\infty} |x_n| < \infty$. Therefore $E \supset l^1$.

2.3 COROLLARY. *If $(x_n)$ is a symmetric absolute Schauder basis for a sequentially complete barrelled locally convex space then $(x_n)$ is equivalent to the usual basis for* $\phi$, $l^1$ or $\omega$.

2.4 *Definition.* A linear topological space $E$ is called *locally semi-convex* if it has a neighborhood base $\mathscr{V}$ at 0 such that

(1) for each $V \in \mathscr{V}$ there is $a > 0$ such that $aV + aV \subset V$.

2.5 THEOREM. *The smallest locally semi-convex FK-space in which E is bounded is* $\bigcap_{p>0} l^p$.

*Proof.* First observe that $\bigcap_{p>0} l^p$ is a locally semi-convex $FK$ space in which $\mathscr{E}$ is bounded. Thus it suffices to prove that every locally semi-convex $FK$ space $E$ in which $\mathscr{E}$ is bounded contains $\bigcap_{p>0} l^p$. Let $\mathscr{V}$ be a base of 0-neighborhoods in $E$ such that each $V$ in $\mathscr{V}$ has property (1). Since $E$ is a metric space we may assume $E$ has the form $V_1 \supset V_2 \supset \ldots$. We may also assume $E \subset l^1$ and $V_1$ is the unit ball of $l^1$ since $\mathscr{E}$ is bounded in $l$ . For each $n$ let

$$\mathscr{V}_n = \{aV_n : 0 < a \leqq 1\}.$$

Then $E$ with the uniformity derived from $\mathscr{V}_n$ is a locally bounded linear topological space [**4**, p. 159]. Thus there is a $p$-norm $|||\ |||_n$, for some $p_n > 0$ which determines the topology of $(E, \mathscr{V}_n)$ [**4**, p. 160–161]. Since $\{e_j\}$ is bounded in $\{E, \mathscr{V}_n\}$ we may assume $|||e_k|||_n \leqq 1$ for $k = 1, 2, \ldots$.

Given $(x_i)$ in $\bigcap_{p>0} l^p$ we show

$$\left\{\sum_{i=1}^{m} x_i e_i : m = 1, 2, \ldots\right\}$$

is a Cauchy sequence in $E$. Let $V_n$ in $\mathscr{V}$ be given and let $\delta$ be such that $x \in V_n$ for $|||x|||_n < \delta$. Let $M$ be such that $\sum_{i=M}^{\infty} |x_i|^{p_n} < \delta$. Then if $s$, $t > M$, $|||\sum_{i=s}^{t} x_i e_i||| < \delta$ so that $\sum_{i=s}^{t} x_i e_j$ is in $V_n$. Since the coordinate functionals are total on $E$, $\sum_{i=1}^{\infty} x_i e_i$ converges to $(x_i)$. Thus $(x_i)$ is in $E$.

## 3. Moduli and *FK*-spaces with absolute bases.

3.1 *Definition.* A real valued function $f$ defined on $[0, \infty)$ is called a modulus if it has the following properties:

(1) $f(x) \geqq 0$ for each $x$,
(2) $f(x) = 0$ if and only if $x = 0$,
(3) $f(x + y) \leqq f(x) + f(y)$,
(4) $f$ is increasing,
(5) $\lim_{x \downarrow 0} f(x) = 0$.

Because of (3), $|f(x) - f(y)| \leqq f(x - y)$ so that in view of (5), $f$ is continuous on $[0, \infty)$. If $f$ and $g$ are moduli then $f \circ g$, $af (a \geqq 0)$, $f/(1 + f)$ and $f + g$ are moduli.

3.2 THEOREM. (a) *If $f$ is a modulus then the space $L(f)$ of all sequences* $\mathbf{x}$ *such that*

$$|\mathbf{x}| = \sum_{n=1}^{\infty} f(|x_n|) < \infty \tag{6}$$

*is an FK-space with invariant metric $|\ |$ in which $\mathscr{E}$ is bounded.*

(b) *If $E$ is any FK-space in which $\mathscr{E}$ is bounded, then there is a modulus $f$ such that $L(f) \subset E$.*

*Proof.* (a) (i). $L(f)$ is a linear space. It is closed under addition because of (3) and the triangular inequality. It is closed under scalar multiplication because if $\mathbf{x} \in L(f)$ and $|a| < K$ where $K$ is an integer,

$$|a\mathbf{x}| = \sum_{n=1}^{\infty} f(|ax_n|) \leqq \sum_{n=1}^{\infty} Kf(x_n) < \infty. \tag{7}$$

(ii) $||$ satisfies the conditions (F1)–(F6) of [**4**, p. 163] and is thus an $(F)$-norm:

(F3). $|a\mathbf{x}| \leqq |\mathbf{x}|$ for $|a| \leqq 1$ since $f$ is increasing.

(F5). Suppose $|\mathbf{x}_n| \to 0$ and $a$ is a scalar. If $|a| < K$ where $K$ is an integer then $|a\mathbf{x}_n| < K|\mathbf{x}_n|$ for each $n$ so that $|a\mathbf{x}_n| \to 0$.

(F6). Suppose $a_n \to 0$ and $\mathbf{x}$ is in $L(f)$. For arbitrary $\epsilon > 0$ let $K$ be such that $\sum_{n=K+1}^{\infty} f(|x|) < \epsilon/2$. Then $g(t) = \sum_{n=1}^{K} f(|t\mathbf{x}_n|)$ is continuous at 0 so there is $1 > \delta > 0$ such that $|g(t)| < \epsilon/2$ for $0 < t < \delta$. Let $N$ be such that $|a_n| < \delta$ for $n > N$. Then for $n > N$,

$$\begin{aligned}|a_n\mathbf{x}| &= \sum_{k=1}^{K} f(|a_n x_k|) + \sum_{k=K+1}^{\infty} f(|a_n x_k|) \\ &\leqq \epsilon/2 + \sum_{k=K+1}^{\omega} f(|x_k|) \\ &< \epsilon.\end{aligned}$$

(iii) The coordinate functionals $E_i(\mathbf{x}) = x_i$ are continuous. For $\epsilon > 0$ let $\delta = f(\epsilon)$. Then if $|f(|x|) < \delta$, $|x| < \epsilon$ since $f$ is increasing. Consequently $|\mathbf{x}| < \delta$ implies $|x_n| < \epsilon$ for each $n$.

(iv) $L(f)$ is complete. Let $(\mathbf{x}^{(n)})$ be a Cauchy sequence in $L(f)$. By (iii), $(x_i^{(n)} : n = 1, 2, \ldots)$ is a Cauchy sequence for each $i$. Let $\mathbf{x}$ be the pointwise limit of $(\mathbf{x}^{(n)})$. If $K$ is such that $|\mathbf{x}^{(m)} - \mathbf{x}^{(n)}| < \epsilon$ for $m, n > K$ then

$$\sum_{i=1}^{N} f(|x_i^{(m)} - x_i^{(n)}|) < \epsilon$$

for each $N$. Therefore

$$\begin{aligned}\lim_n \sum_{i=1}^{N} f(|x_i^{(m)} - x_i^{(n)}| &= \sum_{i=1}^{N} f(|x_i^{(m)} - x_i|) \\ &\leqq \epsilon\end{aligned}$$

for each $N$ so that $\mathbf{x}$ is in $L(f)$ and $|\mathbf{x} - \mathbf{x}^{(m)}| < \epsilon$ for $m > K$.

(v) $\mathscr{E}$ is bounded in $L(f)$ since $|ae_i| = |ae_j| = f(|a|)$ for each $a$ and each pair $i, j$. Thus if $ae_1 \in \{\mathbf{x} : |\mathbf{x}| < \epsilon\}$ we have $a \in \mathscr{E}\ \{\mathbf{x} : |\mathbf{x}| < \epsilon\}$.

(b) (vi). Let $|\ |$ be an $(F)$-norm which determines the $FK$ topology on $E$ [**4**, p. 163]. Define a real valued function $f$ on $[0, \infty)$ by

$$f(t) = \sup_n |te_n|.$$

Since $\epsilon$ is bounded in $E$, $f(t) < \infty$ for each $t$. Obviously $f$ satisfies (1), (2), (3) and (4). To show $f$ satisfies (5) let $\epsilon > 0$ be given. Since $\epsilon$ is bounded in $E$ there is $\delta > 0$ such that $a \in \{\mathbf{x} : |\mathbf{x}| < \epsilon\}$ for $|a| < \delta$. Thus $|f(x)| < \epsilon$ for $0 \leqq x < \delta$.

(vii) $L(f) \subset E$. If $\mathbf{x}$ is in $L(f)$ then $(\sum_{i=1}^{n} x_i e_i)$ is a Cauchy sequence in $E$ since

$$\left| \sum_{i=m}^{n} x_i e_i \right| \leqq \sum_{i=m}^{n} f(|x_i|).$$

Thus $\sum_{i=1}^{\infty} x_i e_i$ converges, necessarily to $\mathbf{x}$, because $\mathrm{E}_j(\sum_{i=1}^{\infty} x_i e_i) = x_j$ for each $j$.

3.3 Theorem. *For every modulus f the following assertions are valid:*

(a) *$L(f)$ is a balanced symmetric FK-space for which $\mathscr{E}$ is an absolute basis;*

(b) *$L(f) \subseteq l^1$;*

(c) *$L(f)^\alpha = L(f)^\beta = L(f)^\gamma = \{(x'e_i) : x' \in L(f)'\} = m$.*

*Proof.* We omit the proof of (a) since most of it is part of the proof of Theorem 3.2.

(b) Suppose there were a sequence $\mathbf{x}$ in $L(f)$ but not in $l^1$. Then we could find an increasing sequence of indices $(k_n)$ such that $\sum_{i=k_{n-1}}^{k_n-1} |x_i| \geqq 1$. Then we have

$$f(1) \leqq f\left(\sum_{i=k_{n-1}}^{k_n-1} |x_i|\right) \leqq \sum_{i=k_{n-1}}^{k_n-1} f(|xi|)$$

for each $n$. But since $\sum_{j=1}^{\infty} f(|x_j|) < \infty$,

$$\lim_n \sum_{i=k_{n-1}}^{k_n-1} f(|x_i|) = 0$$

which implies $f(1) = 0$. This contradicts (2) of Definition 3.1.

(c) Since $L(f)$ is normal, $L(f)^\alpha = L(f)^\beta = L(f)^\gamma$. Since $L(f)$ is an *FK*-space $L(f)^\alpha \subset \{(x'e_i) : x' \in L(f)'\}$ [**10**, p. 205, Problem 1] while since $\epsilon$ is an unconditional basis for $L(f)$ $\{(x'e_i) : x' \in L(f)'\} \subset L(f)^\alpha$.

Since $L(f)$ is symmetric $L(f)^\alpha$ is perfect and symmetric so by [**5**, Satz 2, §14], $L(f)^\alpha = \phi$, $\omega$ or $l^1 \subset L(f) \subset m$. Since $L(f) \neq \phi$, $L(f)^\alpha \neq \omega$ but since $l^1 \subset L(f)$, $L(f)^\alpha \subset m$; consequently $L(f)^\alpha = m$.

3.4 Theorem. (a) *The intersection of all FK-spaces in which $\mathscr{E}$ is bounded is $\phi$.*

(b) *The intersection of all symmetric FK-spaces is $\phi$.*

(c) *The intersection of all FK-spaces $L(f)$ where f is a modulus is $\phi$.*

*Proof.* In view of Theorems 3.2 and 3.3 (a) it suffices to prove (c).

Let $\mathbf{x} = (x_n)$ be a sequence which is not in $\phi$. We may assume that $(x_n)$ is in $l^1$ since $l^1$ is of the form $L(f)$ for $f(x) = x$. Let $\hat{\mathbf{x}} = (\hat{x}_n)$ be the sequence consisting of all numbers $\{|x_n| : x_n \neq 0\}$ is descending order with repetitions allowed.

Define $g$ on $[0, \infty)$ by

$$g(x) = \left[ \sum_{n=1}^{\infty} \frac{|x|/\hat{x}_n}{n^2(1 + |x/\hat{x}_n|)} \right]^{\frac{1}{2}}.$$

Then $g$ is a modulus and $g(\hat{x}_n) \geqq 1/\sqrt{2}n$ for each $n$ so that $\hat{\mathbf{x}}$ is not in $L(g)$. Since $L(g)$ is symmetric and balanced, $\mathbf{x}$ cannot be in $L(g)$.

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