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Infinite-Dimensional Polyhedrality

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Abstract. This paper deals with generalizations of the notion of a polytope to infinite dimensions. The most general definition is the following: a bounded closed convex subset of a Banach space is called a *polytope* if each of its finite-dimensional affine sections is a (standard) polytope.

We study the relationships between eight known definitions of infinite-dimensional polyhedrality. We provide a complete isometric classification of them, which gives solutions to several open problems. An almost complete isomorphic classification is given as well (only one implication remains open).

Introduction

Convex polytopes in finite dimensions play an important role in many areas of mathematics. This was the main reason that many attempts have been made to give definitions of infinite-dimensional convex polytopes (*e.g.* [17], [19], [1], [14], [3] and others). Of course, each such definition has some "good" properties of finite-dimensional polytopes and it is equivalent to the standard definition in finite dimensions. Since each affine section of a (standard finite-dimensional) polytope is again a polytope, it is quite clear that the most general of such definitions should be the following: a bounded closed convex (BCC, for short) set is a convex polytope if each of its finite-dimensional sections is a (finite-dimensional) polytope. This definition was given by V. Klee [17] for the unit ball of a Banach space, and by M. I. Kadets [15] in a general setting. As another example of a definition of a polytope, we mention here the following one, which is an equivalent reformulation (*cf.* [6]) of the notion of "quasi-polyhedrality" due to D. Amir and F. Deutsch [1]: a BCC body $C \subset X$ is a polytope if, for every boundary point x of C and each boundary point $y \in C$ sufficiently near to x, the whole segment $[x, y]$ is contained in the boundary of C.

Infinite-dimensional polytopes have an important property that plays a significant role in our discussion. Let us start with finite dimensions. It is well known that any closed convex set in a finite-dimensional space has non-empty interior in its affine span. As simple examples show, this is not the case in infinite-dimensional spaces (think of an infinite-dimensional compact set). Nevertheless, the following important result holds.

Theorem 0.1 ([13], [9]) *Let $P \subset X$ be a Klee polytope in a separable Banach space X. Then P has non-empty interior in its affine span and this affine span is closed in X. If in addition the polytope P is symmetric (with respect to the origin) then the separability assumption may be omitted.*

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As we mentioned above, the most general reasonable definition of an infinite-dimensional polytope is that given by Klee. Thus Theorem 0.1 remains true if we substitute a Klee polytope by any other one.

Theorem 0.1 enables us to study so called *polyhedral spaces* (see below) instead of symmetric polytopes (and for simplicity we restrict ourselves to symmetric polytopes only) which is more convenient from many points of view. Indeed, let P be an infinite-dimensional symmetric polytope (according to any definition) and let $Y \subset X$ be the linear span of P. Then by Theorem 0.1, Y is a Banach space and P is the unit ball of Y in an equivalent norm. It is natural to call the space Y a polyhedral space (in the same sense as P is a polytope). Nowadays, "polyhedral space" standardly means a space whose unit ball is a polytope in the sense of Klee. We call these spaces (K)-polyhedral.

The aim of the present paper is to classify eight known definitions of polyhedrality of infinite-dimensional Banach spaces, from both isometric and isomorphic point of view. It turns out that, surprisingly enough, these definitions are linearly ordered by implication (this was proved mainly in [6]) but no two of them are equivalent. We provide here corresponding counterexamples, solving in this way several open problems from [6] and [5] (*cf.* Remark 1.5). Moreover, we prove that the five most general of the eight definitions are isomorphically equivalent in the sense that, for any two of these five definitions, if X satisfies one of them then X can be equivalently renormed to satisfy the other one.

It is easy to see that, if X is a finite-dimensional polyhedral space, each extreme point of the dual ball B_{X^*} defines a maximal face of the polytope B_X. So it is not surprising that several of the definitions of polyhedrality deal with the set $\operatorname{ext} B_{X^*}$ or, more precisely, with weak*-limit points of this set.

The paper is organized in the following way. Section 1 contains definitions of the eight notions of polyhedrality considered by us, together with an exact description of our main results. We collect in Section 2 some equivalent reformulations and remarks concerning the eight definitions, as well as some auxiliary facts. Sections 3 and 4 contain the proofs of all results (positive results and counterexamples, respectively) of the present paper. In Appendix (Section 5), we state two definitions from [5] and show briefly how they are related to our eight notions, in order to answer a question from [5] (see Remark 1.5).

1 The Eight Definitions. Description of Main Results

Throughout the paper, X denotes a real Banach space with closed unit ball B_X, open unit ball B_X^0 and unit sphere S_X, and X^* is the dual of X. By a *subspace* of X we mean a closed linear subspace of X. We shall use the following notations.

The *density character* $\operatorname{dens} X$ of X is the smallest cardinality of a dense subset of X. By $\operatorname{ext} C$ we denote the *set of the extreme points* of a convex set C. For $x \in S_X$, $D(x)$ is the image of x by the (multivalued) *duality mapping*, *i.e.*,

$$D(x) = \{f \in S(X^*) : f(x) = 1\}.$$

Observe that $\operatorname{ext} D(x) = D(x) \cap \operatorname{ext} B_{X^*}$ by the Krein-Milman theorem.

If A is a set in X^*, then A' denotes the set of all w^*-*limit points* (called also w^*-accumulation points or w^*-cluster points) of A:

$$A' = \left\{ f \in X^* : f \in w^*\text{-cl}(A \setminus \{f\}) \right\}.$$

Moreover, for $x \in X$ we set $\langle A, x\rangle := \{f(x) : f \in A\}$.

Recall that a finite-dimensional space X is *polyhedral* if B_X is a polytope. The following definition collects the eight generalizations of polyhedrality to infinite-dimensional spaces, considered in the present paper. In parentheses we indicate references to papers in which they appeared for the first time, in this or equivalent form. The properties (II) and (III) are formulated for the first time here.

Definition 1.1 Let us consider the following properties of X:

- (I) $(\text{ext}\, B_{X^*})' \subset \{0\}$ (Maserick [19]);
- (II) $(\text{ext}\, B_{X^*})' \subset rB_{X^*}$ for some $0 < r < 1$;
- (III) $(\text{ext}\, B_{X^*})' \subset B^0_{X^*}$;
- (IV) $f(x) < 1$ whenever $x \in S_X$ and $f \in (\text{ext}\, B_{X^*})'$ (Gleit and McGuigan [14]);
- (V) $\sup\{f(x) : f \in \text{ext}\, B_{X^*} \setminus D(x)\} < 1$ for each $x \in S_X$ (Brosowski and Deutsch [3]);
- (VI) every $x \in S_X$ has a neighborhood V such that, for each $y \in V \cap S_X$, the segment $[x, y]$ lies entirely in S_X (Amir and Deutsch [1]);
- (VII) the set $M_v := \{x \in S_X : \max\langle D(x), v\rangle \le 0\}$ is open in S_X for each direction $v \in S_X$ (Durier and Michelot [4]);
- (K) the unit ball of every finite-dimensional subspace of X is a polytope (Klee [17]).

For $j \in \{\text{I, II, III, IV, V, VI, VII, K}\}$, X will be called (j)-*polyhedral* if it satisfies the property (j).

By an *isomorphically* (j)-*polyhedral space* we mean a Banach space that admits an equivalent (j)-polyhedral norm.

We refer the reader to Section 2 for basic information about these notions.

Theorem 1.2 (Isometric Classification) *For a Banach space X, the following relationships among the properties from Definition 1.1 hold.*

- (a) (I) $\Rightarrow$ (II) $\Rightarrow$ (III) $\Rightarrow$ (IV) $\Rightarrow$ (V) $\Rightarrow$ (VI) $\Rightarrow$ (VII) $\Rightarrow$ (K).
- (b) *None of the implications in* (a) *can be reversed.*
- (c) *X is* (I)*-polyhedral if and only if X is isometric to a subspace of $c_0(\Gamma)$ where* $\text{card}\, \Gamma = \text{dens}\, X$.

For (a) above and a counterexample showing that (V) $\not\Rightarrow$ (IV) see [6] (where various other related results can be found). The remaining counterexamples are contained in Examples 4.1–4.6

Note that for finite-dimensional X, all eight properties are equivalent to polyhedrality (*i.e.*, to the fact that B_X is a polytope). Indeed, if such X is polyhedral then it satisfies (I) since $\operatorname{ext} B_{X^*}$ is finite, and if X is (K)-polyhedral it is obviously polyhedral; the rest follows from Theorem 1.2(a).

Let us remark that the properties from Definition 1.1 are hereditary to closed subspaces. It follows easily from the fact that, if Y is a closed subspace of X, then each $g \in \operatorname{ext} B_{Y^*}$ is the restriction of some $f \in \operatorname{ext} B_{X^*}$. (Indeed, by the Krein-Milman theorem the set of all norm-preserving extensions of g has extreme points; take f as one of them.)

The next theorem describes the isometric classification of our eight properties in the class of all spaces satisfying a geometric condition (Δ). This condition implies that each point of the unit sphere is contained only in finitely many solid faces of B_X (*cf.* Theorem 2.1 for this notion).

Definition 1.3 We shall say that X satisfies the property (Δ) if, for each $x \in S_X$, the set $\operatorname{ext} D(x)$ is finite.

Theorem 1.4 *Let X be a Banach space satisfying the property* (Δ)*. Then the following relationships among the properties from Definition 1.1 hold.*

(a) (I) $\Rightarrow$ (II) $\Rightarrow$ (III) $\Rightarrow$ (IV) $\Leftrightarrow$ (V) $\Rightarrow$ (VI) $\Leftrightarrow$ (VII) $\Leftrightarrow$ (K).
(b) *None of the simple implications ($\Rightarrow$) can be reversed.*

For (a) in the above theorem, see Theorem 1.2(a), Observation 3.5 and Theorem 3.6. Moreover, since (IV) implies (Δ) (Observation 3.5), the first three implications cannot be reversed by Theorem 1.2. Now, Example 4.4 completes the proof of (b).

Remark 1.5 Our results give solutions to some open problems from [5],[6].

(a) In [6, p. 872], the authors ask whether some of the implications (V) $\Rightarrow$ (VI) $\Rightarrow$ (VII) $\Rightarrow$ (K) (in our notation) could be reversed, and mention in particular the implication (K) $\Rightarrow$ (VI) as "an old open problem" (*cf.* also [5, p. 631]). Our Theorem 1.2(b) answers all these questions in negative.

(b) In [5], two properties (lm) and (PH) are considered (p. 634) and the following question is posed (p. 642): Does (PH) imply (lm)? As we show in Appendix of the present paper (Section 5), it is not difficult to prove that

- X satisfies (PH) iff it is (VI)-polyhedral and satisfies (Δ);
- X satisfies (lm) iff it is (V)-polyhedral and satisfies (Δ).

By Theorem 1.4(b), (PH) does not imply (lm).

As for polyhedrality definitions from the *isomorphic* point of view, we restrict ourselves to separable Banach spaces.

For simplicity, we shall use the following notation for "isomorphic implication" in the next theorem. If (i), (j) are two of the eight properties from Definition 1.1, the symbol $(i) \to (j)$ means that isomorphic (i)-polyhedrality implies isomorphic (j)-polyhedrality. (Recall that X is called isomorphically (j)-*polyhedral* if it is isomorphic to a (j)-polyhedral space.)

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Theorem 1.6 (Isomorphic Classification) *Let X be a separable infinite dimensional Banach space. Then the following isomorphic relations hold for the properties from Definition 1.1.*

(a) (I) $\leftrightarrow$ (II) $\rightarrow$ (III) $\rightarrow$ (IV) $\leftrightarrow$ (V) $\leftrightarrow$ (VI) $\leftrightarrow$ (VII) $\leftrightarrow$ (K).
(b) (III) $\not\rightarrow$ (II).

In the above theorem, (a) follows from Theorems 3.2, 3.3 and 1.2(a); for (b) see Example 4.2. It remains an *open problem* whether the isomorphic implication (IV) $\rightarrow$ (III) holds for separable Banach spaces, or equivalently, whether each separable (K)-polyhedral space can be renormed to satisfy (III).

The next theorem, containing three isomorphic characterizations, follows from Theorems 3.3, 3.4 and 3.2(b). Recall that *boundary* for X is a set $B \subset S_{X^*}$ such that, for each $x \in X$ there exists $f \in B$ with $f(x) = \|x\|$.

Theorem 1.7 *Let X be a separable infinite-dimensional Banach space.*

(a) *X is isomorphically* (K)*-polyhedral if and only if X admits an equivalent renorming with a countable boundary.*
(b) *X is isomorphically* (III)*-polyhedral if and only if X admits an equivalent renorming whose dual unit ball has only countably many extreme points.*
(c) *X is isomorphically* (II)*-polyhedral if and only if X is isomorphic to a subspace of c_0.*

2 Remarks on the Eight Definitions. Auxiliary Results

Recall that a set $N \subset S_{X^*}$ is said to be 1-*norming* if, for each $x \in X$,

$$\|x\| = \sup_{f \in N} f(x). \tag{1}$$

By the Hahn-Banach theorem, a set $N \subset S_{X^*}$ is 1-norming if and only if $B_{X^*} = w^*\text{-cl}\,\mathrm{co}N$.

A *boundary* is a 1-norming set $N \subset S_{X^*}$ such that the least upper bound in (1) is in fact a maximum for each $x \in X$. The set ext B_{X^*} is easily seen to be a boundary by the Krein-Milman theorem, hence it is 1-norming.

2.1 (I)-Polyhedrality

Maserick [19] defined a notion of a convex polytope in a geometric way using half-spaces. Reformulated for B_X, his definition reads as follows:

(M) there exists a 1-norming set $N \subset S_{X^*}$ such that, for each $x \in X$, the set $\{f \in N : f(x) > 1\}$ is finite.

Claim (M) *is equivalent to* (I)*-polyhedrality.*

Proof If X is (I)-polyhedral, then (by an easy w^*-compactness argument), for each $x \in X$, the set $\{f \in \operatorname{ext} B_{X^*} : f(x) > 1\}$ is finite. Hence X satisfies (M) with $N = \operatorname{ext} B_{X^*}$. Now, let X satisfy (M). Since $B_{X^*} = w^*\text{-cl}\,\operatorname{co} N$, Milman's theorem ("converse" of the Krein-Milman theorem) implies that $\operatorname{ext} B_{X^*} \subset w^*\text{-cl} N$. Consequently, $(\operatorname{ext} B_{X^*})' \subset N'$. Moreover, $N' \subset \{0\}$ since, for all $x \in X$ and $\varepsilon > 0$, the set

$$\{f \in N : |f(x)| > \varepsilon\} = \{f \in N : f(x/\varepsilon) > 1\} \cup \{f \in N : f(-x/\varepsilon) > 1\}$$

is finite. ■

2.2 (II)- and (III)-Polyhedrality

These two properties were introduced by the authors as natural intermediates between (I)-polyhedrality and (IV)-polyhedrality. They admit significant isomorphic characterizations (*cf.* Theorem 3.2(b) and Theorem 3.4).

2.3 (IV)-Polyhedrality

(IV)-polyhedrality was introduced by Gleit and McGuigan [14], who showed that it is sufficient for (K)-polyhedrality. Moreover, they proved that (IV)- and (K)-polyhedrality coincide for Lindenstrauss spaces.

(IV)-polyhedrality has the following obvious reformulation:

(IV$'$) $(\operatorname{ext} B_{X^*})' \cap D(x) = \varnothing$ for each $x \in S_X$.

It follows from Theorem 1 in [6] and [5, p. 634] that (IV)-polyhedrality is equivalent to the following property:

X has the property (Δ) and
$\sup\{f(x) : f \in \operatorname{ext} B_{X^*} \setminus D(x)\} < 1$ for each $x \in S_X$.

2.4 (V)-Polyhedrality

This property appears in [3, Lemma 2.2].

Claim *X is* (V)*-polyhedral if and only if it satisfies the property*

$$\big(\operatorname{ext} B_{X^*} \setminus D(x)\big)' \cap D(x) = \varnothing \quad \textit{for every } x \in S_X.$$

Proof It suffices to consider the following chain of equivalent statements for every fixed $x \in S_X$:

- $\sup\{f(x) : f \in \operatorname{ext} B_{X^*} \setminus D(x)\} = 1$;
- there exists a sequence $\{f_n\} \subset \operatorname{ext} B_{X^*} \setminus D(x)$ with $f_n(x) \to 1$;
- there exists a net $\{f_\alpha\} \subset \operatorname{ext} B_{X^*} \setminus D(x)$, w^*-converging to some $h \in B_{X^*}$, such that $f_\alpha(x) \to 1$;
- there exists $h \in D(x)$ belonging to the set $\big(\operatorname{ext} B_{X^*} \setminus D(x)\big)'$. ■

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2.5 (VI)-Polyhedrality

Amir and Deutsch [1] defined the following notion: X is said to be *quasi-polyhedral* if

$$x \notin \overline{K(x) \cap S_X} \text{ for any } x \in S_X,$$

where $K(x) = \{v \in X : f(v) < 1 \text{ for all } f \in D(x)\}$. It was proved in [6] that the following statements are equivalent:

(i) X is (VI)-polyhedral;
(ii) every $x \in S_X$ has a neighborhood V such that $D(y) \subset D(x)$ holds for each $y \in V \cap S_X$;
(iii) X is quasi-polyhedral.

(Note that [6, Theorem 3], contains many other characterizations of (VI)-polyhedrality.)

2.6 (VII)-Polyhedrality

Durier and Michelot considered the following property in [4, Section 3, pp. 517–519]:

the set $U_v := \{x \in S_X : f(v) < 0 \text{ for some } f \in D(x)\}$
is closed for each direction $v \in S_X$.

This property is just a reformulation of (VII)-polyhedrality since the set U_v is the complement of $-M_v$ in S_X.

It is easy to see that a point $x \in S_X$ belongs to U_v if and only if the half-line $\{x - tv : t \geq 0\}$ intersects B_X only at x and is not tangent to B_X. Roughly speaking, U_v is the set of all points of S_X that can be hit from outside of B_X in a non-tangent way by shooting in the direction v.

2.7 (K)-Polyhedrality

(K)-polyhedrality, which was introduced by V. Klee [17], seems to be the most natural and simplest generalization of finite-dimensional polyhedrality. This is what now is standardly meant by "polyhedral space". It follows from [16, Theorem 4.7], that X is (K)-polyhedral if and only if each of its *two-dimensional* subspaces is polyhedral.

Let us remark that (K)-polyhedrality admits an equivalent formulation using the finest locally convex topology on X. See [6] for further details.

Let us collect some basic properties of (K)-polyhedral spaces in the following theorem. Recall that a set $B \subset S_{X^*}$ is a *boundary* for X if for every $x \in S_X$ there exists $f \in B$ such that $f(x) = 1$.

Theorem 2.1 *Every* (K)*-polyhedral Banach space X has the following properties.*

(a) *X admits a boundary B_0 such that, for each $f \in B_0$, $f^{-1}(1) \cap S_X$ is a solid face,* i.e., *its relative interior in the hyperplane $f^{-1}(1)$ is nonempty.*

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(b) *X is c_0-saturated (*i.e.*, every infinite-dimensional subspace of X contains c_0 almost isometrically). In particular, X contains no infinite-dimensional subspace isomorphic to a dual space.*

(c) *B_{X^*} is the norm-closed convex hull of* ext B_{X^*}*, and every subspace Y of X has the same density character as Y^* (hence X is an Asplund space).*

(a) and (b) were proved in [8] (see also [10]; for shorter and simpler proofs of (a) see [11] or [21]), (c) is contained in [21] and [11].

Let us conclude this section with a few examples:

- for each set Γ, the space $c_0(\Gamma)$ is (K)-polyhedral (it is easily seen to be even (I)-polyhedral);
- the space c is not (K)-polyhedral (this is an easy exercise), but it is isomorphically (K)-polyhedral (since it is isomorphic to c_0);
- for each ordinal $\alpha > 0$, the space $C[1, \alpha]$ is isomorphically (K)-polyhedral [7];
- no infinite-dimensional $L_p(\mu)$ space ($1 \le p \le \infty$) is isomorphically (K)-polyhedral (it is dual if $p > 1$, and it contains the dual space ℓ_1 if $p = 1$).

3 Positive Results

Proposition 3.1 *For every infinite-dimensional* (III)*-polyhedral Banach space X, the cardinality of the set* ext B_{X^*} *is equal to the density character of X.*

Proof (III) easily implies that the set ext B_{X^*} consists of $\|\cdot\|$-isolated points. Then, by Theorem 2.1,

$$\operatorname{card}(\operatorname{ext} B_{X^*}) = \operatorname{dens}(\operatorname{ext} B_{X^*}) \le \operatorname{dens} S_{X^*} = \operatorname{dens} X^* = \operatorname{dens} X$$

and $B_{X^*} = \overline{\operatorname{conv}}^{\|\cdot\|} \operatorname{ext} B_{X^*}$. Consequently, the set of all rational convex combinations of points of ext B_{X^*} is dense in B_{X^*} and has the same cardinality as ext B_{X^*}. Thus we can complete the proof: $\operatorname{dens} X = \operatorname{dens} X^* = \operatorname{dens} B_{X^*} \le \operatorname{card}(\operatorname{ext} B_{X^*})$. ∎

Theorem 3.2 *Let X be an infinite-dimensional Banach space.*

(a) *X is* (I)*-polyhedral if and only if X is isometric to a subspace of $c_0(\Gamma)$ where* $\operatorname{card}(\Gamma) = \operatorname{dens}(X)$.

(b) *A separable space X is isomorphically* (II)*-polyhedral if and only if X is isomorphic to a subspace of c_0.*

Proof (a) Let X be (I)-polyhedral. Choosing one representant from each pair of mutually opposite elements of ext B_{X^*}, we get a set Γ such that $\Gamma \cup (-\Gamma) = \operatorname{ext} B_{X^*}$ and $\Gamma \cap (-\Gamma) = \varnothing$. Then, for each $x \in X$, the function

$$\gamma \mapsto \gamma(x) \quad (\gamma \in \Gamma)$$

belongs to $c_0(\Gamma)$ by (I). In this way, we have defined a linear mapping from X into $c_0(\Gamma)$, which is easily checked to be an isometry. Moreover, $\operatorname{card}(\Gamma) = \operatorname{card}(\operatorname{ext} B_{X^*}) = \operatorname{dens}(S_X)$ by Proposition 3.1 (and Theorem 1.2(a)).

On the other hand, any $c_0(\Gamma)$ space, and hence also each its closed infinite-dimensional subspace, is (I)-polyhedral (see the examples after Theorem 2.1).

(b) Since any subspace of c_0 is (I)-polyhedral by (a) (and hence also (II)-polyhedral), it is sufficient to prove that any separable (II)-polyhedral space X can be renormed to be (I)-polyhedral. By Proposition 3.1, the set $\operatorname{ext} B_{X^*}$ is countable. Let us write $\operatorname{ext} B_{X^*} = \{\pm h_n\}_1^\infty$. The w^*-topology on B_{X^*} is metrizable by a metric d. Since $(\operatorname{ext} B_{X^*})'$ is w^*-compact, for each n there exists $g_n \in (\operatorname{ext} B_{X^*})'$ such that $d(h_n, g_n) = d\text{-}\operatorname{dist}\big(h_n, (\operatorname{ext} B_{X^*})'\big)$. Put

$$f_n = h_n - g_n \quad (n \in \mathbb{N}) \text{ and } C := w^*\text{-cl}\operatorname{co}\{\pm f_n\}_1^\infty.$$

Let $x \in X$ be an arbitrary point. We have

$$|||x||| := \sup_n |f_n(x)| \le \sup_n \big[\,\|h_n\| + \|g_n\|\,\big] \cdot \|x\| \le (2-\delta)\|x\|.$$

On the other hand, there exists $k \in \mathbb{N}$ such that $|h_k(x)| = \|x\|$. Therefore

$$|||x||| \ge |f_k(x)| \ge |h_k(x)| - |g_k(x)| \ge \delta\|x\|.$$

Consequently, $|||\cdot|||$ is an equivalent norm on X whose dual unit ball is C.

It remains to show that $(\operatorname{ext} C)' = \{0\}$. By Milman's theorem,

$$\operatorname{ext} C \subset w^*\text{-cl}\{\pm f_n\}_1^\infty.$$

Hence $(\operatorname{ext} C)' \subset (\{\pm f_n\}_1^\infty)'$. We claim that $(\{f_n\}_1^\infty)' = \{0\}$. Indeed, by w^*-compactness, it suffices to show that, if $h_{n_i} \xrightarrow{w^*} h_0$ and $g_{n_i} \xrightarrow{w^*} g_0$ (as $i \to \infty$), then $h_0 = g_0$; and this is easy: $d(h_0, g_0) = \lim_i d(h_{n_i}, g_{n_i}) = \lim_i d\text{-}\operatorname{dist}\big(h_{n_i}, (\operatorname{ext} B_{X^*})'\big) = d\text{-}\operatorname{dist}\big(h_0, (\operatorname{ext} B_{X^*})'\big) = 0.$ ■

Theorem 3.3 *For a separable Banach space X, the following statements are equivalent:*

(i) *X is isomorphically* (IV)*-polyhedral.*
(ii) *X is isomorphically* (K)*-polyhedral.*
(iii) *For some equivalent norm, X has a countable boundary.*

Proof The implications (i) ⇒ (ii) and (ii) ⇒ (iii) follow, respectively, from Theorem 1.2(a) and Theorem 2.1(a).

To prove the remaining implication (iii) ⇒ (i), suppose that $\{\pm f_n\}_1^\infty \subset S_{X^*}$ is a countable boundary for X. Fix a decreasing sequence $\{\varepsilon_n\}$ of positive numbers converging to 0, and define

$$|||x||| := \sup_n (1+\varepsilon_n)|f_n(x)|, \qquad C := w^*\text{-cl}\operatorname{co}\{\pm(1+\varepsilon_n)f_n\}_1^\infty.$$

Since $\|x\| \le |||x||| \le (1+\varepsilon_1)\|x\|$, the function $|||\cdot|||$ is an equivalent norm on X, whose dual unit ball is C. Using Milman's theorem and the properties of $\{\varepsilon_n\}$ we obtain

$$(\operatorname{ext} C)' \subset \Big(\{\pm(1+\varepsilon_n)f_n\}_1^\infty\Big)' = (\{\pm f_n\}_1^\infty)' \subset B_{X^*}.$$

Let $g \in (\operatorname{ext} C)'$ and $|||x||| = 1$. There exists $k \in \mathbb{N}$ such that $|f_k(x)| = \|x\|$. Then $g \in B_{X^*}$ and

$$g(x) \le \|x\| = |f_k(x)| < (1 + \varepsilon_k)|f_k(x)| \le |||x||| = 1.$$

Thus $(X, |||\cdot|||)$ is (IV)-polyhedral. ■

Theorem 3.4 *For a separable Banach space X, the following statements are equivalent:*

(i) *X is isomorphically* (III)*-polyhedral.*
(ii) *For some equivalent norm on X, the set* $\operatorname{ext} B_{X^*}$ *is countable.*

Proof (i) $\Rightarrow$ (ii) follows from Proposition 3.1. Let us prove the inverse implication.

Let $\operatorname{ext} B_{X^*} = \{\pm f_n\}_1^\infty$. As in the proof of Theorem 3.3, consider a sequence $\varepsilon_n \searrow 0$ and the corresponding equivalent norm $|||x||| := \sup_n (1 + \varepsilon_n)|f_n(x)|$ whose dual unit ball is the set $C := w^*\text{-cl}\operatorname{co}\{\pm(1 + \varepsilon_n) f_n\}_1^\infty$ satisfying $(\operatorname{ext} C)' \subset B_{X^*}$. Consider an arbitrary $g \in (\operatorname{ext} C)'$. By the Choquet theorem, g is a barycenter of a probability measure on $\operatorname{ext} B_{X^*}$. This means that there exist nonnegative numbers α_n, β_n ($n \in \mathbb{N}$) such that $\sum_{n=1}^\infty (\alpha_n + \beta_n) = 1$ and $g(x) = \sum_{n=1}^\infty (\alpha_n - \beta_n) f_n(x)$ for each $x \in X$. The definition of $|||\cdot|||$ implies

$$|g(x)| \le \sum_{n=1}^\infty (\alpha_n + \beta_n)|f_n(x)| \le \Big(\sum_{n=1}^\infty \frac{\alpha_n + \beta_n}{1 + \varepsilon_n}\Big) |||x|||.$$

Consequently $|||g||| \le \sum_{n=1}^\infty \frac{\alpha_n+\beta_n}{1+\varepsilon_n} < 1$. This proves that $(X, |||\cdot|||)$ is (III)-polyhedral. ■

The remarks on (IV)- and (V)-polyhedrality in Section 2 give immediately the following observation.

Observation 3.5 A Banach space X is (IV)-polyhedral if and only if X is (V)-polyhedral and satisfies (Δ).

Theorem 3.6 *Let X be a Banach space satisfying the property (Δ). If X is* (K)*-polyhedral then it is* (VI)*-polyhedral and its unit ball B_X has no extreme point.*

Proof Let x_0 be an arbitrary point of S_X. By the Krein-Milman theorem and the property (Δ), we have $D(x_0) = \operatorname{conv}\big(\operatorname{ext} D(x_0)\big)$ and the closed linear subspace

$$Y := \bigcap \{f^{-1}(0) : f \in D(x_0)\} = \bigcap \{f^{-1}(0) : f \in \operatorname{ext} D(x_0)\}$$

has finite codimension in X and does not contain x_0. The Hahn-Banach theorem implies that $(\mathbb{R}x_0 \oplus Y) \cap B_X$ (which is the unit ball of $\mathbb{R}x_0 \oplus Y$) has a unique supporting hyperplane in (the (K)-polyhedral space) $\mathbb{R}x_0 \oplus Y$, namely the hyperplane $x_0 + Y$. By Lemma 2 in [21], x_0 belongs to the relative interior of a solid face of $(\mathbb{R}x_0 \oplus Y) \cap B_X$ in

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$\mathbb{R}x_0 \oplus Y$. Note that this implies that x_0 is not an extreme point of B_X and, moreover, there exists a neighborhood V of 0 in Y such that $x_0 + V \subset S_X$.

There exists a finite-dimensional subspace $Z \subset X$ such that

$$X = \mathbb{R}x_0 \oplus Y \oplus Z.$$

Since the unit ball of $\mathbb{R}x_0 \oplus Z$ is a polytope, for every point y in its boundary which is sufficiently near to x_0, the whole segment $[x, y]$ lies on the boundary of this polytope.

It is easy to see that the linear projection

$$P\colon X \to Y \oplus Z,\ P(tx_0 + y + z) = y + z \quad (t \in \mathbb{R}, y \in Y, z \in Z)$$

defines a homeomorphism of an open relative neighborhood of x_0 in S_X onto an open neighborhood N of 0 in $Y \oplus Z$. Let us denote by Q the inverse of this homeomorphism.

Let Y_0, Z_0 be open convex neighborhoods of 0 respectively in Y and Z such that

- $W := \operatorname{conv}(Y_0 \cup Z_0)$ is contained in N;
- $Y_0 \subset V$ (hence $Q(Y_0) = x_0 + Y_0$);
- for every $z \in Z_0$, the segment $[x_0, Q(z)]$ is contained in S_X.

To complete the proof, it suffices to show that $[x_0, Q(w)] \subset S_X$ whenever $w \in W$.

Let $w \in W$ be arbitrary. There exist $y \in Y_0$, $z \in Z_0$ and $\lambda \in [0, 1]$ such that $w = (1 - \lambda)y + \lambda z$. Consider the point $x = (1 - \lambda)Q(y) + \lambda Q(z)$. Certainly $x \in B_X$ (since B_X is convex) and $P(x) = (1-\lambda)y+\lambda z = w$. Since $[x_0, Q(z)] \subset S_X$, there exists a supporting hyperplane H to B_X that contains $[x_0, Q(z)]$. Then H is a supporting hyperplane at x_0, which implies (by the definition of Y) that H contains $x_0 + Y$ and hence also $[x_0, Q(y)]$. We conclude that H contains $\operatorname{co}\{x_0, Q(y), Q(z)\}$; in particular, $[x_0, x] \subset H$. This implies $[x_0, x] \subset S_X$ and $x = Q(w)$. The proof is complete. ■

4 Counterexamples

As we have seen at the end of Section 2, the space c_0 plays an important role in infinite-dimensional polyhedrality: it is (I)-polyhedral (and hence satisfies each of our eight definitions); and it is almost isometrically contained in every (K)-polyhedral space (and hence also in each space satisfying any of our eight definitions). Moreover, it is an easy exercise to show that every finite-dimensional polyhedral space is isometric to a subspace of c_0. So, it is not surprising that all our counterexamples, except Example 4.2, are renormings of c_0.

Example 4.1 There exists a (II)-polyhedral renorming X of c_0 such that X is not (I)-polyhedral.

Proof The direct sum $X = c_0 \oplus \mathbb{R}$ is isomorphic to c_0, and $X^* = \ell_1 \oplus \mathbb{R}$. Let X be equipped with the norm whose dual unit ball is the (w^*-closed bounded convex) set

$$U^* = \operatorname{conv}\left(B_{(\ell_1 \oplus_\infty \mathbb{R})} \cup \{(0, \pm 2)\}\right),$$

where $B_{(\ell_1 \oplus_\infty \mathbb{R})}$ denotes the unit ball of the ℓ_∞-sum $\ell_1 \oplus_\infty \mathbb{R}$. It is easy to see that the set $E = \operatorname{ext} U^*$ contains exactly the points

$$(0, \pm 2) \text{ and } (\pm e_n^*, \pm 1) \quad \text{(with all combinations of signs).}$$

Thus $(0,0) \notin E' = \{(0, \pm 1)\} \subset (1/2)U^*$. ■

Example 4.2 There exists a (III)-polyhedral separable Banach space X which is not isomorphic to any (II)-polyhedral space.

Proof Consider the space $C(K)$ where K is the (countable) interval of ordinals $[1, \omega^\omega]$ with the usual interval topology. Its dual space $C(K)^*$ is isomorphic (even isometric [20]) to ℓ_1 (see [7]) whose unit ball has countably many extreme points. By Theorem 3.4, $C(K)$ is isomorphic to a (III)-polyhedral space X. In view of Theorem 3.2(b), it remains to show that $C(K)$ is not isomorphic to any subspace of c_0. But this follows from Theorem 2 in [2] and from the easy fact that c_0 is isomorphic to $c \cong C\big([1, \omega]\big)$. ■

Example 4.3 There exists a (IV)-polyhedral renorming X of c_0 such that the set $\operatorname{ext} B_{X^*}$ is uncountable. In particular, X is not (III)-polyhedral.

Proof The idea, how to get uncountably many extreme points, is to find a norm on c_0 and a sequence $f = (f_1, f_2, f_3, \dots) \in \ell_1$ such that $f_k \neq 0$ $(k \geq 1)$ and all the points $(\pm f_1, \pm f_2, \pm f_3, \dots)$ are extreme points of the new dual ball; these points are clearly uncountably many. We are going to proceed in several steps.

Step 1: Definition of X. Fix an arbitrary $\varrho \in (0,1)$ and any sequence $\{\lambda_i\}_1^\infty$ of positive numbers such that $\sum_{i=1}^\infty \lambda_i = 1$. Let us define

$$a = \frac{1}{\lambda_1} \quad \text{and} \quad a_n = \frac{a \sum_1^n \lambda_i}{1 - \varrho \sum_{n+1}^\infty \lambda_i}.$$

It is not difficult to see that the following three properties are satisfied:

(i) $\frac{a}{a_n} \sum_1^n \lambda_i + \varrho \sum_{n+1}^\infty \lambda_i = 1$ $(n \in \mathbb{N})$;
(ii) $a_n \to a$ (as $n \to \infty$);
(iii) $1 < a_1 < a_2 < a_3 < \cdots$.

(To see the last property, note that the inequality $a_n < a_{n+1}$ is equivalent to $\sum_1^n \lambda_i (1 - \varrho \sum_{n+2}^\infty \lambda_i) < (\sum_1^n \lambda_i + \lambda_{n+1})(1 - \varrho \sum_{n+2}^\infty \lambda_i - \varrho \lambda_{n+1})$, which, in its turn, can be written as $0 < 1 - \varrho(\sum_1^n \lambda_i + \lambda_{n+1} + \sum_{n+2}^\infty \lambda_i)$; and this is true by the choice of ϱ and $\{\lambda_i\}$.)

For $m \in \mathbb{N}$ define the following subset of ℓ_1:

$$A_m = \Big\{ a_m \Big(\sum_1^m \lambda_i \Big)^{-1} \sum_1^m \varepsilon_k \lambda_k e_k^* : \varepsilon_k = \pm 1 \Big\}.$$

Then the set

$$U^* = w^*\text{-cl co}\Big(\{\pm e_n^*\}_1^\infty \cup \bigcup_{m=1}^{\infty} A_m\Big)$$

is the dual unit ball of an equivalent norm $|||\cdot|||$ on c_0. It is easy to see that

$$\begin{aligned}
|||x||| &= \max\Big\{\sup_{n\in\mathbb{N}} |x(n)|, \sup_{m\in\mathbb{N}} \max_{\varepsilon_k=\pm 1} a_m\Big(\sum_1^m \lambda_i\Big)^{-1} \sum_1^m \varepsilon_k\lambda_k x(k)\Big\} \\
&= \max\Big\{\|x\|_\infty, \sup_{m\in\mathbb{N}} a_m\Big(\sum_1^m \lambda_i\Big)^{-1} \sum_1^m \lambda_k|x(k)|\Big\}.
\end{aligned}$$

Put $X = (c_0, |||\cdot|||)$.

Step 2: X is (IV)*-Polyhedral.* Denote $E = \text{ext}\, U^*$ and consider $f \in E'$ and $x \in c_0$ be such that $|||x||| = 1$. We want to show that $f(x) < 1$. Suppose the contrary, *i.e.*, $f(x) = 1$. By Milman's theorem, f must be a w^*-limit point of $\{e_n^*\}_1^\infty \cup \bigcup_{m=1}^\infty A_m$, and hence also of $\bigcup_{m=1}^\infty A_m$ (since $e_n^* \xrightarrow{w^*} 0$).

We *claim* that f is necessarily of the form $f = a\sum_{k=1}^\infty \varepsilon_k\lambda_k e_k^*$ for some choice of signs $(\varepsilon_k) \in \{\pm 1\}^{\mathbb{N}}$. Indeed, there exists a sequence of positive integers $m_j \to \infty$ and functionals $g_j \in A_{m_j}$ such that $f = w^*\text{-}\lim g_j$. If $f = \sum_1^\infty f_k e_k^*$, $g_j = a_{m_j}(\sum_1^{m_j} \lambda_i)^{-1} \sum_1^{m_j} \varepsilon_k^{(j)}\lambda_k e_k^*$, and e_k denotes the k-th vector of the standard basis of c_0, we have

$$f_k = f(e_k) = \lim_j g_j(e_k) = \lim_j a_{m_j}\Big(\sum_1^{m_j} \lambda_i\Big)^{-1} \varepsilon_k^{(j)}\lambda_k = a\lambda_k \lim_j \varepsilon_k^{(j)}.$$

Thus, for each k, $(\varepsilon_k^{(j)})_{j=1}^\infty$ is convergent to some $\varepsilon_k \in \{\pm 1\}$, and $f_k = a\varepsilon_k\lambda_k$ as claimed.

Choose m so large that $a\cdot\max\{|x(k)|\}_{k=m+1}^\infty < \varrho/2$. Then the definition of $|||\cdot|||$ implies

$$\begin{aligned}
1 = f(x) &= a\sum_{k=1}^m \varepsilon_k\lambda_k x(k) + a\sum_{k=m+1}^\infty \varepsilon_k\lambda_k x(k) \\
&\le \frac{a}{a_m}\Big[a_m\Big(\sum_1^m \lambda_i\Big)^{-1}\sum_{k=1}^m \lambda_k|x(k)|\Big]\sum_1^m \lambda_i + \big(a\cdot\max_{k>m}|x(k)|\big)\sum_{k=m+1}^\infty \lambda_k \\
&< \frac{a}{a_m}\cdot\sum_1^m \lambda_i + \frac{\varrho}{2}\cdot\sum_{m+1}^\infty \lambda_i < 1
\end{aligned}$$

(the last inequality follows from the property (i)). This contradiction shows that X is (IV)-polyhedral.

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Step 3: It is standard to prove that the second dual norm of the norm $|||\cdot|||$ is given by

$$|||\Phi||| = \max\Big\{ \|\Phi\|_\infty, \sup_{m\in\mathbb{N}} a_m \Big(\sum_1^m \lambda_i\Big)^{-1} \sum_1^m \lambda_k |\Phi_k| \Big\} \quad \big(\Phi = (\Phi_k) \in \ell_\infty\big).$$

Step 4: The Set $E = \operatorname{ext} U^*$ is Uncountable. We shall show that, for each choice of signs $(\varepsilon_k) \in \{\pm 1\}^{\mathbb{N}}$, the functional

$$f = a \sum_{k=1}^\infty \varepsilon_k \lambda_k e_k^*$$

is an exposed (end hence extreme) point of U^*.

Because of symmetry we can (and do) suppose that $\varepsilon_k = 1$ for all k. Note that $|||f||| \le 1$ since $f = \lim_m h_m$ with $h_m \in A_m$ $(m \ge 1)$. Consider the functional

$$F = (1, 1, 1, \dots) \in \ell_\infty.$$

We have $|||F||| = \max\{1, \sup_m a_m\} = a$ and $F(f) = a\sum_1^\infty \lambda_k = a$. Consequently, F attains its $|||\cdot|||$-norm at f.

It remains to show that F exposes U^* at f. Let $g = (g_k)_1^\infty \in U^*$ be such that $F(g) = a$. For each $s \in \mathbb{N}$ we have $U^* = \operatorname{co}[P_s \cup Q_s]$, where

$$P_s = w^*\text{-cl co}\Big(\{\pm e_n^*\}_{n=1}^\infty \cup \bigcup_{m=1}^s A_m\Big)$$

$$Q_s = w^*\text{-cl co}\Big(\bigcup_{m=s+1}^\infty A_m\Big).$$

Let $p_s \in P_s$, $q_s \in Q_s$ and $\mu_s \in [0, 1]$ be such that $g = \mu_s p_s + (1 - \mu_s) q_s$. Then we have

$$\begin{aligned} a = F(g) &= \mu_s F(p_s) + (1 - \mu_s) F(q_s) \\ &\le \mu_s \max\Big\{ \|F\|_\infty, \max F\Big(\bigcup_1^s A_m\Big) \Big\} + (1 - \mu_s)|||F||| \\ &= \mu_s \max\{\|F\|_\infty, \max_{m\le s} a_m\} + (1 - \mu_s)|||F||| \\ &= \mu_s a_s + (1 - \mu_s) a. \end{aligned}$$

This implies $\mu_s = 0$ since $a_s < a$. Consequently, $g \in Q_s$ for each $s \in \mathbb{N}$.

We *claim* that $g_k \le a\lambda_k$ for all $k \in \mathbb{N}$. If not, there exists $k \in \mathbb{N}$ such that

$$g_k - a\lambda_k =: 2\delta > 0.$$

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By the property (ii) of $\{a_n\}$, there exists $s \in \mathbb{N}$ such that

$$\Big|a - a_m\Big(\sum_1^m \lambda_i\Big)^{-1}\Big| < \delta \quad \text{for each } m > s.$$

Then, for each $m > s$, we have

$$\begin{aligned} g_k - a_m\Big(\sum_1^m \lambda_i\Big)^{-1}\lambda_k &\geq g_k - a\lambda_k - \Big|a - a_m\Big(\sum_1^m \lambda_i\Big)^{-1}\Big|\lambda_k \\ &> 2\delta - \delta\lambda_k > \delta, \end{aligned}$$

In other words, $a_m(\sum_1^m \lambda_i)^{-1}\lambda_k < g_k - \delta$. But this implies that the k-th coordinate of each element of $\bigcup_{m=s+1}^{\infty} A_m$ is smaller that $g_k - \delta$, which is in contradiction with $g \in Q_s = w^*\text{-cl}\,\mathrm{co}(\bigcup_{m=s+1}^{\infty} A_m)$. Our claim is proved.

Since $\sum_1^\infty g_k = F(g) = a = \sum_1^\infty a\lambda_k$ and $g_k \leq a\lambda_k$ ($k \in \mathbb{N}$), we have necessarily $g_k = a\lambda_k$ for every k. In other words, $g = a\sum_1^\infty \lambda_k e_k^* = f$. This completes the proof that that f is exposed by F.

The last assertion of the statement follows from Proposition 3.1. ∎

Example 4.4 There exists a (VI)-polyhedral renorming X of c_0 such that X satisfies (Δ) but X is not (V)-polyhedral.

Proof Fix an arbitrary sequence $\{\varepsilon_n\}_1^\infty \subset (0,1)$ such that $\varepsilon_n \to 0$. For $n \geq 2$, consider the functionals

$$h_n = (1-\varepsilon_n)e_1^* + 2\varepsilon_n e_n^* \in \ell_1 = (c_0)^*$$

and the set

$$U^* = w^*\text{-cl}\,\mathrm{co}(\{\pm e_n^*\}_1^\infty \cup \{\pm h_n\}_2^\infty).$$

Since U^* is symmetric and $B_{\ell_1} \subset U^* \subset 2B_{\ell_1}$, it follows that U^* is the dual unit ball of an equivalent norm $|||\cdot|||$ on c_0. We shall show that $X = \big(c_0, |||\cdot|||\big)$ has the required properties.

It is easy to see that

$$|||x||| = \max\Big\{\, \|x\|_\infty, \sup_{n\geq 2} |h_n(x)| \Big\}\,.$$

Milman's theorem asserts that the set $\mathrm{ext}\, U^* = \mathrm{ext}\, B_{X^*}$ is contained in the w^*-closure of the set $\{\pm e_n^*\}_1^\infty \cup \{\pm h_n\}_2^\infty$. Note that $h_n \xrightarrow{w^*} 0$ and $\mathrm{ext}\, U^*$ cannot contain the origin. Thus we have

$$\mathrm{ext}\, U^* \subset \{\pm e_n^*\}_1^\infty \cup \{\pm h_n\}_2^\infty.$$

Observe that, for every fixed $n \geq 2$, we have

$$\begin{gathered} |e_i^*(e_1+e_n)| \leq 1 \quad \forall i \geq 1, \\ |h_i(e_1+e_n)| = 1 - \varepsilon_i < 1 \quad \forall i \geq 1,\ i \neq n, \\ |h_n(e_1+e_n)| = h_n(e_1+e_n) = 1 + \varepsilon_n. \end{gathered}$$

This easily implies that $e_1 + e_n$ exposes (even strongly) U^* at h_n. Thus $\{\pm h_n\}_2^\infty \subset \operatorname{ext} U^*$. Moreover, $h_n \notin D(e_1)$ since $|h_n(e_1)| = 1-\varepsilon_n < 1 = |||e_1|||$ $(n \geq 2)$. It follows that

$$(\operatorname{ext} U^* \setminus D(e_1))' \supset (\{\pm h_n\}_2^\infty)' = \{\pm e_1\}.$$

Moreover, $e_1^* \in D(e_1)$, since $e_1^* \in U^*$ and $e_1^*(e_1) = 1$. Consequently, X is not (V)-polyhedral.

It remains to show that X is (VI)-polyhedral and satisfies (Δ). Fix $x \in S_X$ and $\delta \in (0, \frac{1}{2})$. There exists $N \geq 2$ such that $|x(n)| < \delta$ whenever $n > N$. If $y \in S_X$ and $\|x - y\|_\infty < \delta$, we have for all $n > N$

$$|y(n)| \leq |x(n)| + |y(n) - x(n)| < 2\delta < 1$$

and

$$|h_n(y)| = |(1-\varepsilon_n)y(1) + 2\varepsilon_n x(n)| < (1-\varepsilon_n)|||y||| + 2\varepsilon_n\delta = 1 - (1-2\delta)\varepsilon_n < 1.$$

It follows that, for $y \in S_X$ with $\|y - x\|_\infty < \delta$,

$$D(y) \cap \operatorname{ext} U^* \subset D(y) \cap (\{\pm e_n^*\}_1^\infty \cup \{\pm h_n\}_2^\infty) \subset \{\pm e_n^*\}_1^N \cup \{\pm h_n\}_2^N =: A.$$

This implies that $D(x) \cap \operatorname{ext} U^*$ is finite, therefore X satisfies (Δ).

Moreover, since the set $B := A \setminus D(x)$ is finite, there exists $\lambda \in (0,1)$ such that $\max\{f(x) : f \in B\} < \lambda$. Hence, for some $\delta' \in (0, \delta)$,

$$\max\{f(x) : f \in B\} < \lambda \quad \text{whenever } y \in S_X, \|y - x\|_\infty < \delta'.$$

For such y we have $D(y) \cap \operatorname{ext} U^* \subset A \setminus B \subset D(x)$. Consequently $D(y) \subset D(x)$ whenever $y \in S_X$, $\|y - x\|_\infty < \delta'$. By Section 2 (remarks on (VI)-polyhedrality), X is (VI)-polyhedral. ∎

Example 4.5 There exists a (VII)-polyhedral renorming X of c_0 such that X is not (VI)-polyhedral.

Proof Fix an arbitrary sequence $\{\omega_n\}_2^\infty \subset (\frac{5}{6}, 1)$ such that $\omega_n \to 1$. Then the formula

$$|||x||| = \max\Big\{\max_{n\geq 2}|x(n)|, \sup_{n\geq 2}\Big[|x(1)| + \frac{1}{3}|x(n)|\Big], \sup_{n\geq 2}\Big[\omega_n|x(1)| + \frac{1}{2}|x(n)|\Big]\Big\}$$

defines an equivalent norm on c_0. Let us put $X = (c_0, |||\cdot|||)$. In several steps, we are going to show that X has the required properties.

Step 1: It is easy to see that $B_{X^*} = w^*\text{-cl}\operatorname{co} A$ where

$$A = \{\pm e_n^*\}_2^\infty \cup \{\pm e_1^* \pm (1/3)e_n^*\}_2^\infty \cup \{\pm\omega_n e_1^* \pm (1/2)e_n^*\}_2^\infty$$

(here and later on we consider all possible combinations of signs).

Let (τ_n, σ_n) be the solution of the system

$$\begin{cases} \tau_n + \frac{1}{3}\sigma_n = 1 \\ \omega_n \tau_n + \frac{1}{2}\sigma_n = 1. \end{cases}$$

The properties of ω_n imply, via elementary calculation, that $\tau_n, \sigma_n \in (0,1)$, $\tau_n \to 1$, $\sigma_n \to 0$.

Step 2: $\operatorname{ext} B_{X^*} = A$. By Milman's theorem, $\operatorname{ext} B_{X^*}$ is contained in the set

$$w^*\text{-cl}A = A \cup \{\pm e_1^*\} \cup \{0\}.$$

Consequently, $\operatorname{ext} B_{X^*} \subset A$ since e_1^* is the midpoint of the points $e_1^* \pm (1/3)e_2^*$. On the other hand, for each $f \in A$ there exists $x_f \in X$ such that

$$\sup\{g(x_f) : g \in A, g \neq f\} < f(x_f)$$

(indeed, for $f = e_n^*$ take $x_f = e_n$, for $f = e_1^* + (1/3)e_n^*$ take $x_f = (1+\tau_n)e_1 + \sigma_n e_n$, for $f = \omega_n e_1^* + (1/2)e_n^*$ take $x_f = e_1 + e_n$, analogously for different signs). This easily implies that B_{X^*} is w^*-exposed at $f \in A$ by x_f; in particular $f \in \operatorname{ext} B_{X^*}$.

Step 3: X is Not (VI)*-Polyhedral.* Fix $n \geq 2$ and observe that

$$|||te_1 + se_n||| = \max\left\{ |s|, |t| + \frac{1}{3}|s|, \omega_n|t| + \frac{1}{2}|s| \right\}.$$

It follows easily that the $|||\cdot|||$-unit ball of span $\{e_1, e_n\}$ is the polygon whose vertices are the following ten points: $\pm e_1$, $\pm\tau_n e_1 \pm \sigma_n e_n$, $\pm\frac{1}{2\omega_n}e_1 \pm e_n$. Moreover, an easily made diagram shows that the point e_1 is a common endpoint of two sides of this polygon, both of them of length

$$|||e_1 - (\tau_n e_1 \pm \sigma_n e_n)||| = |||(1-\tau_n)e_1 \pm \sigma_n e_n||| \to 0 \quad (\text{as } n \to \infty).$$

But this implies that X is not (VI)-polyhedral.

Step 4: X is (VII)*-Polyhedral.* Fix $x, v \in S_X$ such that $\max\langle D(x), v\rangle \leq 0$. We want to prove that $\max\langle D(y), v\rangle \leq 0$ for all $y \in S_X$ sufficiently near to x. By symmetry, we can (and do) suppose that $x(n) \geq 0$ for each $n \geq 1$. We shall distinguish two cases.

1-st Case: $x(1) < 1$. Choose λ such that $x(1) < \lambda < 1$ and note that there exists $N \geq 2$ such that

$$\left.\begin{aligned} x(n) &< \lambda \\ x(1) + \frac{1}{3}x(n) &< \lambda \\ \omega_n x(1) + \frac{1}{2}x(n) &< \lambda \end{aligned}\right\} \quad \text{whenever } n > N.$$

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Consequently, denoting

$$A_0 = \{\pm e_n^*\}_2^N \cup \{\pm e_1^* \pm (1/3)e_n^*\}_2^N \cup \{\pm\omega_n e_1^* \pm (1/2)e_n^*\}_2^N,$$

we obtain $1 = |||x||| = \max\langle A_0, x\rangle$ and $\sup\langle A \setminus A_0, x\rangle \leq \lambda < 1$. This implies that $D(x) \cap \operatorname{ext} B_{X^*} \subset A_0$ and, putting $B = A \setminus D(x)$, that

$$\sup\langle B, x\rangle < 1 = \sup\langle D(x), x\rangle.$$

By continuity, for every $y \in S_X$ sufficiently close to x, we have

$$\sup\langle B, y\rangle < \sup\langle D(x), y\rangle.$$

Remembering that $\operatorname{ext} B_{X^*} = A = B \cup \big(D(x) \cap \operatorname{ext} B_{X^*}\big)$, we conclude from the last inequality that $D(y) \cap \operatorname{ext} B_{X^*} \subset D(x)$. Consequently, $D(y) \subset D(x)$ for each $y \in S_X$ sufficiently near to x.

2-nd Case: $x(1) = 1$. In this case, the definition of $|||\cdot|||$ (and the fact that $|||x||| = 1$) implies that $x = e_1$. Moreover,

$$D(e_1) \cap \operatorname{ext} B_{X^*} = \left\{ e_1^* \pm \frac{1}{3} e_n^* \right\}_2^\infty$$

since, for each $n \geq 2$,

$$\begin{gathered} e_n^*(e_1) = 0, \\ \big(e_1^* \pm (1/3)e_n^*\big)(e_1) = 1, \\ \big(-e_1^* \pm (1/3)e_n^*\big)(e_1) = -1, \\ \big|\big(\omega_n e_1^* \pm (1/2)e_n^*\big)(e_1)\big| = \omega_n < 1. \end{gathered}$$

The assumption

$$0 \geq \max\langle D(e_1), v\rangle = \sup\langle [D(e_1) \cap \operatorname{ext} B_{X^*}], v\rangle = \sup_{n\geq 2}\Big[v(1) + \frac{1}{3}|v(n)| \Big]$$

implies $\frac{1}{3}\sup_{n\geq 2}|v(n)| \leq -v(1)$. Consequently $v(1) < 0$ (note that $v(1) = 0$ would imply $v = 0$). Suppose that arbitrarily near to x there are points $y \in S_X$ with $\max\langle D(y), v\rangle > 0$. This means that there exist a sequence $\{x_k\}_1^\infty \subset S_X$ and functionals $h_k \in \operatorname{ext} B_{X^*} = A$ such that:

(1) $x_k \to e_1$,
(2) $\|x_k - e_1\|_\infty < 1$,
(3) $h_k(x_k) = 1$,
(4) $h_k(v) > 0$.

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Then (4) implies that h_k cannot be one of the vectors $e_1^* \pm (1/3)e_n^*$ (since these vectors belong to $D(e_1)$ and $\max\langle D(x), v\rangle \le 0$). Using (2), it is easy to see that

$$f(x_k) < 1 \quad \text{whenever } f \in \{\pm e_n^*\}_2^\infty \cup \{-e_1^* \pm (1/3)e_n^*\}_2^\infty \cup \{-\omega_n e_1^* \pm (1/2)e_n^*\}_2^\infty.$$

Thus, by (3), we must have

$$h_k = \omega_{n(k)} e_1^* + \frac{\vartheta_k}{2} e_{n(k)}^* \quad \text{for some } n(k) \ge 2,\ \vartheta_k \in \{\pm 1\}.$$

By (4), $0 < h_k(v) = \omega_{n(k)} v(1) + \vartheta_k (1/2) v\bigl(n(k)\bigr)$, but the last quantity is negative whenever $n(k)$ is sufficiently large since $v(1) < 0$ and $v \in c_0$. Thus there are only finitely many possible values for $n(k)$. Passing to a suitable subsequence of $\{x_k\}_1^\infty$, we can (and do) suppose that

$$h_k = \omega_{n_0} e_1^* + \frac{\vartheta}{2} e_{n_0}^* \quad \text{for each } k \text{ and some fixed } n_0 \ge 2,\ \vartheta \in \{\pm 1\}.$$

But in this case

$$1 = \lim_k h_k(x_k) = \lim_k \Bigl[\omega_{n_0} x_k(1) + \frac{\vartheta}{2} x_k(n_0)\Bigr] = \omega_{n_0} < 1.$$

This contradiction completes the proof. ∎

Example 4.6 There exists a (K)-polyhedral renorming X of c_0 such that X is not (VII)-polyhedral.

Proof Fix a sequence $\{\omega_n\}_3^\infty \subset (0,1)$ such that $\omega_n \to 1$. Let us define a new norm on c_0 by the formula

$$|||x||| = \max\{G(x), H(x)\}$$

where

$$G(x) = \frac{1}{2}\max_{n\ge 3} |x(n)| + \max\{|x(1)|, |x(2)|\},$$

$$H(x) = \sup_{n\ge 3}[|x(n)| + \omega_n |x(1)| + (1-\omega_n)|x(2)|].$$

It is easy to see that $(1/2)\|x\|_\infty \le |||x||| \le 2\|x\|_\infty$. Let us denote $X = \bigl(c_0, |||\cdot|||\bigr)$.

Step 1: X is (K)*-Polyhedral.* It is sufficient to prove that $B_X \cap L$ is a polytope for every finite-dimensional subspace $L \subset X$ that contains the vectors e_1, e_2. Let $\{x_i\}_1^K$ be a basis of $L \cap \overline{\operatorname{span}}\{e_n\}_3^\infty$ (thus $\{x_1, x_2, \ldots, x_K, e_1, e_2\}$ is a basis for L). For $a = (a_1, \ldots, a_K) \in \mathbb{R}^K$ and $(\beta, \gamma) \in \mathbb{R}^2$, we put

$$\begin{aligned} p(a,\beta,\gamma) &= \Bigl|\Bigl|\Bigl| \sum_{i=1}^K a_i x_i + \beta e_1 + \gamma e_2 \Bigr|\Bigr|\Bigr| \\ &= \max\Bigl\{ \frac{1}{2} g(a) + \max\{|\beta|, |\gamma|\}, \sup_{n\ge 3}\bigl[h_n(a) + \omega_n|\beta| + (1-\omega_n)|\gamma|\bigr] \Bigr\} \end{aligned}$$

where

$$g(a) = \Big\| \sum_{i=1}^{K} a_i x_i \Big\|_\infty, \quad h_n(a) = \Big| \sum_{i=1}^{K} a_i x_i(n) \Big|.$$

Clearly, p is an equivalent norm on $\mathbb{R}^{K+2}$, and $(L, |||\cdot|||)$ is isometric to $(\mathbb{R}^{K+2}, p)$.

Claim *There exists $N \geq 3$ such that $h_n(a) \leq \frac{1}{2}g(a)$ for each $a \in \mathbb{R}^K$ and each $n > N$.*

To show this, note that, since g is an equivalent norm on $\mathbb{R}^K$, there exists $r > 0$ such that $\|a\|_\infty \leq rg(a)$ for every $a \in \mathbb{R}^K$. Choose an integer $N \geq 3$ such that $r\sum_{i=1}^{K} |x_i(n)| \leq \frac{1}{2}$ whenever $n > N$. Then, for each $a \in \mathbb{R}^K$ and $n > N$, we have

$$h_n(a) \leq \|a\|_\infty \sum_{i=1}^{K} |x_i(n)| \leq rg(a) \sum_{i=1}^{K} |x_i(n)| \leq \frac{1}{2}g(a),$$

which proves our *claim*.

This implies that the definition of p can be equivalently written in the following way:

$$p(a,\beta,\gamma) = \max\Big\{ \frac{1}{2}g(a) + \max\{|\beta|, |\gamma|\}, \sup_{3\leq n\leq N} \big[h_n(a) + \omega_n|\beta| + (1-\omega_n)|\gamma| \big] \Big\}.$$

This means that the unit ball of $(\mathbb{R}^{K+2}, p)$ is the set $B \cap \bigcap_{n=3}^{N} C_n$ where

$$B = \Big\{ (a,\beta,\gamma) : \frac{1}{2}g(a) + \max\{|\beta|, |\gamma|\} \Big\} \leq 1,$$
$$C_n = \big\{ (a,\beta,\gamma) : h_n(a) + \omega_n|\beta| + (1-\omega_n)|\gamma| \leq 1 \big\}.$$

It is easy to see that the (finite-dimensional normed) space $Y = \big(\mathbb{R}^K, \frac{1}{2}g(\cdot)\big)$ is polyhedral (indeed, B_Y is the intersection of 2^K halfspaces). Since B is isometric to the unit ball of the ℓ_1-sum

$$Y \oplus_1 (\mathbb{R}^2, \|\cdot\|_\infty),$$

B is a polytope (indeed, it is the convex hull of a K-dimensional polytope and a 2-dimensional one). Moreover, each C_n is the intersection of eight halfspaces in $\mathbb{R}^{K+2}$. It follows that the unit ball of $(L, |||\cdot|||)$ is a polytope.

Step 2: X is Not (VII)*-Polyhedral.* It is elementary to see that, for each $n \geq 3$, the solution (ξ_n, τ_n) of the system

$$\begin{cases} \frac{1}{2}\xi_n + \tau_n = 1 \\ \xi_n + \omega_n\tau_n = 1 \end{cases}$$

satisfies $\tau_n \in (0,1)$, $\xi_n > 0$ and, moreover, $\tau_n \to 1$, $\xi_n \to 0$. Let us define $u_n = \tau_n e_1 + \xi_n e_n$ ($n \geq 3$). Note that $u_n \to e_1$ and

$$|||u_n||| = \max\big\{ (1/2)\xi_n + \tau_n, \xi_n + \omega_n\tau_n, \sup_{\substack{j\geq 3\\ j\neq n}} \omega_j\tau_j \big\} = 1 = |||e_1|||.$$

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The functionals $f_n = e_n^* + \omega_n e_1^* + (1-\omega_n)e_2^*$ $(n \geq 3)$ satisfy $f_n(u_n) = \xi_n + \omega_n \tau_n = 1$ and

$$|f(x)| \leq |x(n)| + \omega_n |x(1)| + (1-\omega_n)|x(2)| \leq |||x||| \quad \text{for every } x \in c_0.$$

Thus $f_n \in D(u_n)$ $(n \geq 3)$. Moreover,

$$\max\langle D(u_n), e_2\rangle \geq f_n(e_2) = 1 - \omega_n > 0.$$

The proof that X is not (VII)-polyhedral will be complete if we show that $\max\langle D(e_1), e_2\rangle \leq 0$.

Let φ be an arbitrary element of $D(e_1)$. Consider the two points $z_\pm = e_1 \pm e_2$. We have

$$e_1 = \frac{1}{2}(z_+ + z_-) \quad \text{and} \quad |||z_+||| = |||z_-||| = |||e_1||| = 1.$$

Then necessarily $\varphi(z_+) = \varphi(z_-) = \varphi(e_1) = 1$. But this implies $\varphi(e_2) = \varphi(z_+ - z_-) = 0$. We have proved that $\max\langle D(e_1), e_2\rangle = 0$. ■

5 Appendix Related to Remark 1.5

Definition 5.1 ([5], p. 634) Let us define the following two properties of a Banach space X:

(lm) For each $x \in S_X$, there exists a finite set $I_x \subset \operatorname{ext} B_{X^*}$ such that $\sup\{f(x) : f \in \operatorname{ext} B_{X^*} \setminus I_x\} < 1$.

(PH) For each $x \in S_X$, there exist $\delta > 0$ and a nonempty finite set $\|x - y\| < \delta$ and $f(y) \leq 1\ \forall f \in J_x$.

The aim of the present section is to prove briefly the following proposition, as promised in Remark 1.5(b).

Proposition 5.2 *Let X be a Banach space. Then*

(a) *X satisfies* (lm) *if and only if X is* (V)*-polyhedral and satisfies* (Δ)*;*

(b) *X satisfies* (PH) *if and only if X is* (VI)*-polyhedral and satisfies* (Δ).

By the above proposition, our results imply the following facts.

- By Theorem 1.2, (lm) implies (PH) (this was known: [5, p. 634]).
- By Theorem 1.4, (PH) does not imply (lm) (this solves an open problem; see Remark 1.5).
- By Observation 3.5, (lm) is equivalent to (IV)-polyhedrality.

To prove Proposition 5.2(a), it suffices to note that (lm) can be equivalently reformulated considering only $I_x = \operatorname{ext} D(x)$ (see [5]). Considering the fact that $\operatorname{ext} D(x) = \operatorname{ext} B_{X^*} \cap D(x)$, the assertion follows immediately from definitions.

To prove Proposition 5.2(b), we shall need the following lemma.

Lemma 5.3 *Let $x \in S_X$. Then the following are equivalent:*

(i) $\|y\| \leq 1$ *whenever* $y \in X$ *is sufficiently close to* x *and such that* $f(y) \leq 1$ $\forall f \in \operatorname{ext} D(x)$.

(ii) $D(y) \cap D(x) \neq \varnothing$ *whenever* $y \in S_X$ *is sufficiently close to* x.

Proof By the Krein-Milman theorem, (i) is equivalent to

(i′) For every $y \in X$ sufficiently close to x, we have the equivalence $[\max\langle y, D(x)\rangle \leq 1 \Leftrightarrow \|y\| \leq 1]$.

Using homogeneity of the functions $\max\langle \cdot, D(x)\rangle$ and $\|\cdot\|$, it is an elementary exercise to show that (i′) is equivalent to:

- For every $y \in X$ sufficiently close to x, we have $\max\langle y, D(x)\rangle = \|y\|$.

  By positive homogeneity again, this is equivalent to:

- For every $y \in S_X$ sufficiently close to x, we have $\max\langle y, D(x)\rangle = 1$, which is obviously equivalent to (ii). ■

It was proved in [5, p. 635], that (PH) can be equivalently reformulated considering only $J_x = \operatorname{ext} D(x)$. Using Lemma 5.3, we conclude that X satisfies (PH) if and only if X satisfies (Δ) and the property (ii) from Lemma 5.3 holds for every $x \in S_X$. Then Proposition 5.2(b) follows from Theorem 3 in [6].

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