



Dilations of Markovian semigroups of measurable Schur multipliers

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Abstract. Using probabilistic tools, we prove that any weak* continuous semigroup $(T_t)_{t \geq 0}$ of self-adjoint unital completely positive measurable Schur multipliers acting on the space $B(L^2(X))$ of bounded operators on the Hilbert space $L^2(X)$, where X is a suitable measure space, can be dilated by a weak* continuous group of Markov *-automorphisms on a bigger von Neumann algebra. We also construct a Markov dilation of these semigroups. Our results imply the boundedness of the McIntosh's H^∞ functional calculus of the generators of these semigroups on the associated Schatten spaces and some interpolation results connected to BMO-spaces. We also give an answer to a question of Steen, Todorov, and Turowska on completely positive continuous Schur multipliers.

1 Introduction

The study of dilations of operators is of central importance in operator theory and has a long tradition in functional analysis. Indeed, dilations are powerful tools which allow to reduce general studies of operators to more tractable ones.

Suppose $1 < p < \infty$. In the spirit of Sz.-Nagy's dilation theorem for contractions on Hilbert spaces, Fendler [Fen97] proved a dilation result for any strongly continuous semigroup $(T_t)_{t \geq 0}$ of positive contractions on an L^p -space $L^p(\Omega)$. More precisely, this theorem says that there exist a measure space Ω' , two positive contractions $J: L^p(\Omega) \rightarrow L^p(\Omega')$ and $P: L^p(\Omega') \rightarrow L^p(\Omega)$, and a strongly continuous group of positive invertible isometries $(U_t)_{t \in \mathbb{R}}$ on $L^p(\Omega')$ such that

$$(1.1) \quad T_t = PU_tJ,$$

for any $t \geq 0$ (see also [Fen98]). Note that, in this situation, the map $J: L^p(\Omega) \rightarrow L^p(\Omega')$ is an isometric embedding, whereas the map $JP: L^p(\Omega') \rightarrow L^p(\Omega)$ is a contractive projection.

In the noncommutative setting, measure spaces and L^p -spaces are replaced by von Neumann algebras and noncommutative L^p -spaces and positive maps by completely positive maps. In their remarkable paper [JLM07], Junge and Le Merdy essentially¹

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¹The authors prove that there exists no “reasonable” analog of a variant of Fendler's result for a discrete semigroup $(T^k)_{k \geq 0}$ of completely positive contractions.



showed that there is no hope to have a “reasonable” analog of Fendler’s theorem for semigroups of completely positive contractions acting on noncommutative L^p -spaces. It is a *striking* difference with the world of classical (=commutative) L^p -spaces of measure spaces.

Let X be a σ -finite measure space. In this paper, our first main result (Theorem 5.2) gives a dilation on weak* continuous semigroups of self-adjoint unital completely positive measurable Schur multipliers acting on $B(L^2(X))$ in the spirit of (1.1) but at the level $p = \infty$. Our construction heavily relies on the use of Gaussian processes. Our dilation induces an isometric dilation similar to the one of Fendler’s theorem for the strongly continuous semigroup induced by the semigroup $(T_t)_{t \geq 0}$ on the Schatten space $S_X^p \stackrel{\text{def}}{=} S^p(L^2(X))$ for any $1 \leq p < \infty$. Finally, note that, in this paper, the weak* continuity of the semigroup means that the map $\mathbb{R}^+ \rightarrow \mathbb{C}$, $t \mapsto \langle T_t(x), y \rangle_{B(L^2(X)), S_X^1}$ is continuous for any $x \in B(L^2(X))$ and any $y \in S_X^1$.

Moreover, this result is analogous to the result of [Arh20], which provides a dilation of weak* continuous semigroups of self-adjoint unital completely positive Fourier multipliers acting on the group von Neumann algebra $VN(G)$ of a locally compact group G . Note also that in the case where X is a *finite set* equipped with the counting measure, it is possible to use the considerably more complicated approach of [Arh19] (relying on the use of ultrafilters) whose assumptions are satisfied by the results of [Arh13a] and [Ric08] for obtaining a dilation. But even in this case, our new and more general approach gives an explicit and a more useful dilation since the von Neumann algebra where the group $(U_t)_{t \in \mathbb{R}}$ acts is injective. This point is important for applications in vector-valued noncommutative L^p -spaces since we need injective von Neumann algebras in this context (see [Pis98]). We refer to the papers [AFM17, ALM14, Arh13b, HaM11] for related things. Finally, note that Schur multipliers are crucial operators in noncommutative analysis and are connected to a considerable number of topics as harmonic analysis, double operator integrals, perturbation theory, and Grothendieck’s theorem. More recently, the semigroups of Schur multipliers considered in this paper were connected to noncommutative geometry in the memoir [ArK22a].

One of the important consequences of Fendler’s theorem is the boundedness, for the generator of a strongly continuous semigroup $(T_t)_{t \geq 0}$ of positive contractions, of a bounded H^∞ functional calculus which is a fundamental tool in various areas: harmonic analysis of semigroups, multiplier theory, Kato’s square root problem, maximal regularity in parabolic equations, and control theory. For detailed information, we refer the reader to [Haa06, JMX06, KuW04], to the survey [LeM07], and to the recent book [HvNVW18] and the references therein. Our results also give a similar result on H^∞ functional calculus in the noncommutative context in the case of semigroups of measurable Schur multipliers.

Recall that a locally integrable function $f: \mathbb{R}^n \rightarrow \mathbb{C}$ is said to have bounded mean oscillation if

$$\|f\|_{\text{BMO}} \stackrel{\text{def}}{=} \sup_{Q \in \Omega} \frac{1}{|Q|} \int_Q |f(y) - f_Q|^2 dy < \infty,$$

where $f_Q \stackrel{\text{def}}{=} \frac{1}{|Q|} \int_Q f$ is the average of f over Q , $|Q|$ is the volume of Q , and Q is the set of all cubes in \mathbb{R}^n . The quotient space of bounded mean oscillation functions modulo the space of constant functions with the previous seminorm is a Banach space. The importance of the BMO-norm and the BMO-space lies in the fact that they arise as end-point spaces for estimating the bounds of linear maps on function spaces on \mathbb{R}^n . This includes many Calderón–Zygmund operators. By interpolation, BMO-spaces are an effective tool to obtain L^p -bounds of Fourier multipliers. In this sense, the BMO-space is a natural substitute of the space $L^\infty(\mathbb{R}^n)$.

Spaces of functions with bounded means oscillation can also be studied through semigroups. If $\mathcal{T} = (T_t)_{t \geq 0}$ is a Markov semigroup on an L^∞ -space $L^\infty(\Omega)$ of a (finite) measure space Ω , then we can define a BMO-seminorm by

$$\|f\|_{\text{BMO}_{\mathcal{T}}} \stackrel{\text{def}}{=} \sup_{t \geq 0} \|T_t(|f - T_t(f)|^2)\|_{\infty}^{\frac{1}{2}}.$$

More generally, Junge and Mei introduced in [JM12] several noncommutative semigroup BMO-spaces starting from a Markov semigroup on a semifinite von Neumann algebra (i.e., a noncommutative L^∞ -space). Relations between these spaces are studied, and interpolation results are obtained. A crucial ingredient of their approach is Markov dilations of semigroups that allow one to use martingale theory and derive results from this probabilistic setting. See also [FMS19, JiW17, JM10, JuZ15] for other applications of Markov dilations.

The second main result of this paper is Theorem 6.1, which gives the construction of a Markov dilation of each weak* continuous semigroup of self-adjoint unital completely positive measurable Schur multipliers acting on $B(L^2(X))$. Again, our construction relies on probabilistic tools. From this result, we obtain interpolation results with BMO-spaces as end-point for these semigroups (see Theorem 7.2 and Remark 7.3). Probably, our results will be useful for estimating norms of Schur multipliers on Schatten spaces. Finally, we equally refer to [Cas19, CJSZ20] for strongly related papers.

1.1 Structure of the paper

The paper is organized as follows. Section 2 gives background. In Section 3, we give useful observations on measurable positive-definite kernels and negative-definite kernels. In Section 4, we give complements on measurable Schur multipliers. We answer a question of Steen, Todorov, and Turowska in Remark 4.9. We also give a precise description of weak* continuous semigroups of unital self-adjoint completely positive measurable Schur multipliers on some suitable measure spaces. Section 5 gives a proof of our main result of dilation of these semigroups. Next, in Section 6, we also construct a Markov dilation for these semigroups. In Section 7, we describe some applications of our results to functional calculus and interpolation.

2 Preliminaries

In this paper, we suppose that the considered measure spaces are complete (i.e., every subset of every null set is measurable).

2.1 Isonormal processes

Let H be a real Hilbert space. An H -isonormal process on a probability space (Ω, μ) ([Nua06, Definition 1.1.1], [Neer07, Definition 6.5]) is a linear mapping $W: H \rightarrow L^0(\Omega)$ from H into the space $L^0(\Omega)$ of measurable functions on Ω with the following properties:

(2.1) For any $h \in H$, the random variable $W(h)$ is a centered real Gaussian.

(2.2) For any $h_1, h_2 \in H$, we have $\mathbb{E}(W(h_1)W(h_2)) = \langle h_1, h_2 \rangle_H$.

(2.3)

The linear span of the products $W(h_1)W(h_2) \cdots W(h_m)$, with $m \geq 0$ and h_1, \dots, h_m in H , is dense in the real Hilbert space $L^2_{\mathbb{R}}(\Omega)$.

Here, we make the convention that the empty product, corresponding to $m = 0$ in (2.3), is the constant function 1. Moreover, \mathbb{E} is used to denote the expected value.

If $(e_i)_{i \in I}$ is an orthonormal basis of H and if $(\gamma_i)_{i \in I}$ is a family of independent standard Gaussian random variables on a probability space Ω , then for any $h \in H$, the family $(\gamma_i \langle h, e_i \rangle_H)_{i \in I}$ is summable in $L^2(\Omega)$ and

(2.4)
$$W(h) \stackrel{\text{def}}{=} \sum_{i \in I} \gamma_i \langle h, e_i \rangle_H, \quad h \in H,$$

defines an H -isonormal process.

Recall that the span of elements $e^{iW(h)}$ is weak* dense in $L^\infty(\Omega)$ by [Jan97, Remark 2.15]. Using [HvNVW18, Proposition E.2.2] with t instead of ξ and by observing by (2.2) that the variance $\mathbb{E}(W(h)^2)$ of the Gaussian variable $W(h)$ is equal to $\|h\|_H^2$, we see that

(2.5)
$$\mathbb{E}(e^{itW(h)}) = e^{-\frac{t^2}{2} \|h\|_H^2}, \quad t \in \mathbb{R}, h \in H.$$

If $u: H \rightarrow H$ is a contraction, we denote by $\Gamma_\infty(u): L^\infty(\Omega) \rightarrow L^\infty(\Omega)$ the (symmetric) second quantization of u acting on the complex Banach space $L^\infty(\Omega)$. Recall that the map $\Gamma_\infty(u): L^\infty(\Omega) \rightarrow L^\infty(\Omega)$ preserves the integral.² If u is a surjective isometry, we have

(2.6)
$$\Gamma_\infty(u)(e^{iW(h)}) = e^{iW(u(h))}, \quad h \in H,$$

and $\Gamma_\infty(u): L^\infty(\Omega) \rightarrow L^\infty(\Omega)$ is a *-automorphism of the von Neumann algebra $L^\infty(\Omega)$.

Furthermore, the second quantization functor Γ satisfies the following elementary result [Arh20, Lemma 2.1]. In the part I, we suppose that the construction³ is given by the concrete representation (2.4).

Lemma 2.1 (1) *If $L^\infty(\Omega)$ is equipped with the weak* topology, then the map $H \rightarrow L^\infty(\Omega)$, $h \mapsto e^{iW(h)}$ is continuous.*

²That means that for any $f \in L^\infty(\Omega)$, we have $\int_\Omega \Gamma_\infty(u) f \, d\mu = \int_\Omega f \, d\mu$.

³The existence of a proof of Lemma 2.1 without (2.4) is unclear.

- (2) If $(U_t)_{t \in \mathbb{R}}$ is a strongly continuous group of orthogonal operators acting on the Hilbert space H , then $(\Gamma_\infty(U_t))_{t \in \mathbb{R}}$ is a weak* continuous group of operators acting on the Banach space $L^\infty(\Omega)$.

Let H be a real Hilbert space. Following [NVW15, Definition 2.2] and [Neer07, Definition 6.11], we say that an $L^2_{\mathbb{R}}(\mathbb{R}^+, H)$ -isonormal process W is an H -cylindrical Brownian motion. In this case, for any $t \geq 0$ and any $h \in H$, we let

$$(2.7) \quad W_t(h) \stackrel{\text{def}}{=} W(1_{[0,t]} \otimes h).$$

We introduce the filtration $(\mathcal{F}_t)_{t \geq 0}$ defined by

$$(2.8) \quad \mathcal{F}_t \stackrel{\text{def}}{=} \sigma(W_r(h) : r \in [0, t], h \in H),$$

that is, the σ -algebra generated by the random variables $W_r(h)$ for $r \in [0, t]$ and $h \in H$.

By [Neer07, p. 77], for any fixed $h \in H$, the family $(W_t(h))_{t \geq 0}$ is a Brownian motion. This means that [Neer07, Definition 6.2]

$$(2.9) \quad W_0(h) = 0 \text{ almost surely,}$$

$$(2.10) \quad W_t(h) - W_s(h) \text{ is Gaussian with variance } (t - s)\|h\|_H^2 \text{ for any } 0 \leq s \leq t,$$

$$(2.11) \quad W_t(h) - W_s(h) \text{ is independent of } \{W_r(h) : r \in [0, s]\} \text{ for any } 0 \leq s \leq t.$$

Indeed, by [Neer07, p. 163],

$$(2.12) \quad \text{the increment } W_t(h) - W_s(h) \text{ is independent of the } \sigma\text{-algebra } \mathcal{F}_s.$$

Moreover, by [Neer07, p. 163], the family $(W_t(h))_{t \geq 0}$ is a martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$. In particular, the random variable $W_t(h)$ is \mathcal{F}_t -measurable. If $0 \leq s \leq t$, note that

$$\|1_{]s,t]} \otimes h\|_{L^2_{\mathbb{R}}(\mathbb{R}^+, H)}^2 = \|1_{]s,t]}\|_{L^2_{\mathbb{R}}(\mathbb{R}^+)}^2 \|h\|_H^2 = (t - s)\|h\|_H^2.$$

Using (2.5) together with the previous computation, we obtain

$$(2.13) \quad \mathbb{E}(e^{iW(1_{]s,t]} \otimes h)}) = e^{-\frac{t-s}{2}\|h\|_H^2}, \quad 0 \leq s \leq t, \quad h \in H.$$

2.2 Probabilities

Let Ω be a probability space, and let X be a Banach space. If $f \in L^1(\Omega, X)$ is independent of the sub- σ -algebra \mathcal{F} , then, by [HvNVW16, Proposition 2.6.35], its conditional expectation $\mathbb{E}(f|\mathcal{F})$ with respect to \mathcal{F} is given by the constant function:

$$(2.14) \quad \mathbb{E}(f|\mathcal{F}) = \mathbb{E}(f).$$

2.3 Hilbert–Schmidt operators

Let X be a σ -finite measure space. We will use the space $S_X^\infty \stackrel{\text{def}}{=} S^\infty(L^2(X))$ of compact operators, its dual S_X^1 , and the space $B(L^2(X))$ of bounded operators on the Hilbert space $L^2(X)$. If $f \in L^2(X \times X)$, we denote the associated Hilbert–Schmidt operator by

$$(2.15) \quad \begin{aligned} K_f: L^2(X) &\longrightarrow L^2(X) \\ \xi &\longmapsto \int_X f(\cdot, y)\xi(y) dy \end{aligned}$$

We will also use the notation $K((x, y) \mapsto f(x, y))$ for K_f . With the notation $\check{f}(x, y) \stackrel{\text{def}}{=} f(y, x)$, we have

$$(2.16) \quad (K_f)^* = K_{\check{f}}.$$

The operator K_f is self-adjoint if and only if $f(x, y) = \overline{f(y, x)}$ almost everywhere.

For any $f, g \in L^2(X \times X)$, we have $K_f K_g = K_h$ where $h \in L^2(X \times X)$ is defined by

$$(2.17) \quad h(x, y) = \int_X f(x, z)g(z, y) dz$$

with $\|h\|_2 \leq \|f\|_2 \|g\|_2$.

2.4 Kernels

Let X be a set. A function $\varphi: X \times X \rightarrow \mathbb{C}$ is called *Hermitian* if $\varphi(y, x) = \overline{\varphi(x, y)}$ for any $x, y \in X$. A function $\varphi: X \times X \rightarrow \mathbb{C}$ is a *positive-definite kernel* if for any integer $n \geq 1$, any $c_1, \dots, c_n \in \mathbb{C}$, and any $x_1, \dots, x_n \in X$, we have

$$(2.18) \quad \sum_{i,j=1}^n c_i \bar{c}_j \varphi(x_i, x_j) \geq 0.$$

In this definition, it is enough to consider mutually different elements x_1, \dots, x_n of X . By [BCR84, Section 1.5, p. 68], any positive-definite kernel is Hermitian.

Example 2.2 If X is a finite set, then the function $\varphi: X \times X \rightarrow \mathbb{C}$ is a positive-definite kernel if and only if the matrix $[\varphi(x_i, x_j)]$ is positive-definite.

Example 2.3 Let H be a Hilbert space, and let $\alpha: X \rightarrow H$ be a map. By [Boz87, p. 9], the function $\varphi: X \times X \rightarrow \mathbb{C}$ defined by

$$(2.19) \quad \varphi(x, y) = \langle \alpha_x, \alpha_y \rangle_H, \quad x, y \in X,$$

is a positive-definite kernel.

Conversely, by [BCR84, p. 82], every positive-definite kernel φ is of this form for some suitable Hilbert space H and some map $\alpha: X \rightarrow H$. Moreover, if φ is real-valued, we can take a real Hilbert space.

We say that $\varphi: X \times X \rightarrow \mathbb{C}$ is a *negative-definite kernel* if φ is Hermitian and if for any integer $n \geq 1$, any $c_1, \dots, c_n \in \mathbb{C}$ with $c_1 + \dots + c_n = 0$, and any $x_1, \dots, x_n \in X$, we have

$$\sum_{i,j=1}^n c_i \bar{c}_j \varphi(x_i, x_j) \leq 0.$$

Example 2.4 Let H be a real Hilbert space, and let $\alpha: X \rightarrow H$ be a map. The map $\psi: X \times X \rightarrow \mathbb{R}$ defined by $\psi(x, y) = \|\alpha_x - \alpha_y\|_H^2$ is a negative-definite kernel.

By [BCR84, Proposition 3.2, p. 82], every *real-valued* negative-definite kernel ψ which vanishes on the diagonal $\{(x, x) : x \in X\}$ is of this form for a real Hilbert space H .

The following is the famous Schoenberg’s theorem [BCR84, Theorem 2.2, p. 74], which relates negative-definite kernels and positive-definite kernels.

Theorem 2.5 *Let X be a set, and let $\psi: X \times X \rightarrow \mathbb{C}$ be a map. Then ψ is a negative-definite kernel if and only if for any $t > 0$ the map $e^{-t\psi}: X \times X \rightarrow \mathbb{C}$ is a positive-definite kernel.*

3 Complements on measurable kernels

The following lemma is folklore. For the sake of completeness, we give an argument following Cartier [Car81, Lemme, p. 287].

Lemma 3.1 *Let H be a Hilbert space. Suppose that X is a set equipped with a σ -algebra \mathcal{A} . Let $\alpha: X \rightarrow H$ be a function. For any $x, y \in X$, we define $\varphi: X \times X \rightarrow \mathbb{C}$ by (2.19). If φ is measurable with respect to $\mathcal{A} \otimes \mathcal{A}$, then the map α is weak measurable, i.e., for any $h \in H$, the map $X \rightarrow \mathbb{C}, x \mapsto \langle \alpha_x, h \rangle_H$ is measurable.*

Proof We denote by H_1 the set of vectors h of H such that the function $X \rightarrow \mathbb{C}, x \mapsto \langle \alpha_x, h \rangle_H$ is measurable. It is obvious that H_1 is a closed subspace of H . It is clear that any vector of H which is orthogonal to the range of α belongs to H_1 , i.e., $\alpha(X)^\perp \subset H_1$. We infer that $\overline{\text{span } \alpha(X)}^\perp \subset H_1$. Since the partial functions of φ are measurable, for any $y \in X$, the map $X \rightarrow \mathbb{C}, x \mapsto \langle \alpha_x, \alpha_y \rangle_H$ is measurable. So we also have the inclusion $\alpha(X) \subset H_1$, hence $\overline{\text{span } \alpha(X)} \subset H_1$. We conclude that $H \subset H_1$. ■

Lemma 3.2 *Suppose that X is a set equipped with a σ -algebra \mathcal{A} . Let $\psi: X \times X \rightarrow \mathbb{R}$ be a measurable real-valued negative-definite kernel which vanishes on the diagonal $\{(x, x) : x \in X\}$. Then there exist a real Hilbert space H and a weak measurable map $\alpha: X \rightarrow H$ such that*

$$\psi(x, y) = \|\alpha_x - \alpha_y\|_H^2, \quad x, y \in X.$$

Proof Fix $x_0 \in X$. Recall that by [BCR84, Lemma 2.1, p. 74], the function $\varphi: X \times X \rightarrow \mathbb{R}$ defined by

$$(3.1) \quad \varphi(x, y) = \frac{1}{2} [\psi(x, x_0) + \psi(x_0, y) - \psi(x, y)], \quad x, y \in X,$$

is a positive-definite kernel. Since ψ is measurable, this kernel is measurable. So there exist a real Hilbert space H and a function $\alpha: X \rightarrow H$ satisfying (2.19), which is weak measurable by Lemma 3.1. Now, for any $x, y \in X$, we have

$$\begin{aligned} \|\alpha_x - \alpha_y\|_H^2 &= \|\alpha_x\|_H^2 + \|\alpha_y\|_H^2 - 2\langle \alpha_x, \alpha_y \rangle_H \stackrel{(2.19)}{=} \varphi(x, x) + \varphi(y, y) - 2\varphi(x, y) \\ &\stackrel{(3.1)}{=} \frac{1}{2} (\psi(x, x_0) + \psi(x_0, x) - \psi(x, x)) + \frac{1}{2} (\psi(y, x_0) + \psi(x_0, y) - \psi(y, y)) \\ &\quad - (\psi(x, x_0) + \psi(x_0, y) - \psi(x, y)) = \psi(x, y). \end{aligned} \quad \blacksquare$$

We introduce the following definition in the spirit of [Mer21, p. 25].

Definition 3.1 Let X be a measure space. We say that a measurable map $\varphi: X \times X \rightarrow \mathbb{C}$ is an integrally positive-definite kernel if for any $\xi \in L^1(X)$ the function $(x, y) \mapsto \varphi(x, y)\overline{\xi(x)}\xi(y)$ is integrable on $X \times X$ and

$$(3.2) \quad \iint_{X \times X} \varphi(x, y)\overline{\xi(x)}\xi(y) \, dx \, dy \geq 0.$$

Let $\varphi: X \times X \rightarrow \mathbb{C}$ be a measurable positive-definite kernel on a σ -finite measure space X . Cartier showed in [Car81, p. 302] that (3.2) is satisfied for any measurable function $\xi: X \rightarrow \mathbb{C}$ if the previous integral is absolutely convergent. Moreover, Cartier proved the following result [Car81, Corollaire, p. 301].

Proposition 3.3 Suppose that X is a σ -finite measure space. Let $\varphi: X \times X \rightarrow \mathbb{C}$ be a measurable and bounded function. Then φ is equal almost everywhere to a bounded and measurable positive-definite kernel on $X \times X$ if and only if φ is an integrally positive-definite kernel.

Remark 3.4 The author naively thought that he proved the “if” part of an $L^2(X)$ -variant of this result in a previous version of this paper relying on the version of Mercer’s theorem for finite measure spaces stated in [Kon86, Theorem 3.a.1, p. 145]. Unfortunately, the proof of [Kon86, Theorem 3.a.1, p. 145] has a large gap.⁴ In particular, the estimate $\sup_n \|f_n\|_\infty < \infty$ is not proved and seems doubtful, and consequently the arguments of convergence of [Kon86] are inexact.

Remark 3.5 It would be interesting to study variants of Definition 3.1 by replacing the space $L^1(X)$ by another space of functions in the spirit of [HuW75] and the references therein.

The following is [Car81, Théorème, p. 283]⁵ and would be used in the next section.

Proposition 3.6 Let X be a locally compact space equipped with a Radon measure with support X , and let $\varphi: X \times X \rightarrow \mathbb{C}$ be a continuous function. Then φ is a positive-definite kernel if and only if for any continuous function $\xi: X \rightarrow \mathbb{C}$ with compact support, we have (3.2).

4 A description of Markov semigroups of measurable Schur multipliers

Let X be a σ -finite measure space. We say that a function $\varphi \in L^\infty(X \times X)$ induces a measurable Schur multiplier on $B(L^2(X))$ if the map $S_X^2 \mapsto B(L^2(X)), K_f \mapsto K_{\varphi f}$ induces a bounded operator from S_X^∞ into $B(L^2(X))$. In this case, the operator $S_X^\infty \mapsto B(L^2(X)), K_f \mapsto K_{\varphi f}$ admits by [BLM04, Lemma A.2.2, p. 360] a unique weak* extension $M_\varphi: B(L^2(X)) \rightarrow B(L^2(X))$ called the measurable Schur multiplier associated with φ . It is known that in this case

$$(4.1) \quad \|\varphi\|_{L^\infty(X \times X)} \leq \|M_\varphi\|_{B(L^2(X)) \rightarrow B(L^2(X))}.$$

We refer to the surveys [Tod15, ToT10] for more information. See also [Spr04].

⁴It was confirmed by email by the author of the book.

⁵The “only if” part is true without the assumption “with support X ”

Example 4.1 If the set $X = \{1, \dots, n\}$ is equipped with the counting measure, we can identify the space $B(L^2(X))$ with the matrix algebra M_n . Then each operator K_f identifies to the matrix $[f(i, j)]$. A Schur multiplier is given by a map $M_\varphi: M_n \rightarrow M_n$, $[f(i, j)] \mapsto [\varphi(i, j)f(i, j)]$.

We say that a measurable Schur multiplier $M_\varphi: B(L^2(X)) \rightarrow B(L^2(X))$ is self-adjoint if its restriction $S_X^2 \mapsto S_X^2$, $K_f \mapsto K_{\varphi f}$ on the Hilbert space S_X^2 is self-adjoint. See [JMX06, Section 5] for more information on a related notion for operators acting on von Neumann algebras.

Proposition 4.2 Suppose that X is a σ -finite measure space. A measurable Schur multiplier $M_\varphi: B(L^2(X)) \rightarrow B(L^2(X))$ is self-adjoint if and only if the function φ is real-valued almost everywhere.

Proof The self-adjointness property is equivalent to

$$\operatorname{Tr}(M_\varphi(K_f)K_g^*) = \operatorname{Tr}(K_f(M_\varphi(K_g))^*), \quad \text{i.e.,} \quad \operatorname{Tr}(K_{\varphi f}K_g^*) = \operatorname{Tr}(K_f(K_{\varphi g})^*)$$

for any $f, g \in L^2(X \times X)$. Since $L^2(X \times X) \rightarrow S_X^2$, $f \mapsto K_f$ is an isometry from the Hilbert space $L^2(X \times X)$ onto the Hilbert space S_X^2 of Hilbert–Schmidt operators on $L^2(X)$, that means that

$$\iint_{X \times X} \varphi(x, y) f(x, y) \overline{g(x, y)} \, dx \, dy = \iint_{X \times X} f(x, y) \overline{\varphi(x, y) g(x, y)} \, dx \, dy.$$

Note that $f\bar{g}$ belongs to $L^1(X \times X)$ and that each function of $L^1(X \times X)$ has this form. By duality, we deduce that $\varphi = \bar{\varphi}$ almost everywhere. ■

We will show that a measurable Schur multiplier $M_\varphi: B(L^2(X)) \rightarrow B(L^2(X))$ is unital if and only if its restriction on S_X^1 is trace-preserving. We need some information on Arens products to prove some preliminary result. Let A be a Banach algebra. Recall that we can equip its bidual A^{**} with two natural products, the left and right Arens products defined in [BLM04, pp. 78–79] on A^{**} . We say A is Arens regular if these products coincide on A^{**} . Note that the algebra S_X^∞ of compact operators is Arens regular by [Pal01, Theorem 9.1.39, p. 838].

Proposition 4.3 Let X be a σ -finite measure space. The map $S_X^2 \rightarrow S_X^2$, $K_f \mapsto K_{\check{f}}$ extends to an involutive normal $*$ -antiautomorphism $R: B(L^2(X)) \rightarrow B(L^2(X))$.

Proof For any $f \in L^2(X \times X)$, we let $R(K_f) \stackrel{\text{def}}{=} K_{\check{f}}$. On the one hand, for any $f, g \in L^2(X \times X)$, the Hilbert–Schmidt operator $K_f K_g: L^2(X) \rightarrow L^2(X)$ is associated with the function h defined by (2.17), and for almost all $(x, y) \in X \times X$, we have

$$\check{h}(x, y) = \int_X f(y, z) g(z, x) \, dz.$$

On the other hand, by (2.17), the Hilbert–Schmidt operator $K_{\check{g}} K_{\check{f}}: L^2(X) \rightarrow L^2(X)$ is associated with the function

$$(x, y) \mapsto \int_X \check{g}(x, z) \check{f}(z, y) \, dz = \int_X f(y, z) g(z, x) \, dz.$$

We obtain that $R(K_f K_g) = R(K_g)R(K_f)$. Moreover, we have

$$(R(K_f))^* = (K_{\check{f}})^* \stackrel{(2.16)}{=} K_{\check{f}} = R(K_{\check{z}}) \stackrel{(2.16)}{=} R((K_f)^*).$$

We conclude that the map $S_X^2 \rightarrow S_X^2, K_f \mapsto K_{\check{f}}$ is an involutive $*$ -antiautomorphism. From the formula $(K_f)^* = K_{\check{z}}$, it is not difficult to see that the Banach adjoint of K_f is $K_{\check{f}}$. Hence, $\|K_f\|_{B(L^2(X))} = \|K_{\check{f}}\|_{B(L^2(X))}$. We deduce an involutive $*$ -antiautomorphism $R: S_X^\infty \rightarrow S_X^\infty$. By [BLM04, Lemma A.2.2, p. 360], we deduce a unique weak* continuous extension $R: (S_X^\infty)^{**} \rightarrow B(L^2(X))$. Moreover, by [BLM04, Section 2.5.5, p. 79], this map is a $*$ -antiautomorphism where the double dual $(S_X^\infty)^{**}$ is equipped with the Arens product. Finally, note that by [Pal01, Theorem 9.1.39, p. 838], the algebra $(S_X^\infty)^{**}$ identifies with the algebra $B(L^2(X))$. ■

We introduce the following duality bracket:

$$(4.2) \quad \langle z, y \rangle_{B(L^2(X)), S_X^1} \stackrel{\text{def}}{=} \text{Tr}(R(z)y), \quad z \in B(L^2(X)), y \in S_X^1.$$

Proposition 4.4 *Let X be a σ -finite measure space. The preadjoint $(M_\varphi)_*: S_X^1 \rightarrow S_X^1$ of a measurable Schur multiplier $M_\varphi: B(L^2(X)) \rightarrow B(L^2(X))$ coincides with its restriction on S_X^1 , i.e., for any $z \in B(L^2(X))$ and $y \in S_X^1$, we have*

$$(4.3) \quad \langle M_\varphi(z), y \rangle_{B(L^2(X)), S_X^1} = \langle z, M_\varphi(y) \rangle_{B(L^2(X)), S_X^1}.$$

Proof Recall that the map $L^2(X \times X) \rightarrow S_X^2, f \mapsto K_f$ is an isometry from the Hilbert space $L^2(X \times X)$ onto the Hilbert space S_X^2 of Hilbert–Schmidt operators on $L^2(X)$. For any $K_f, K_g \in S_X^1$, we deduce that

$$\begin{aligned} \langle M_\varphi(K_f), K_g \rangle_{B(L^2(X)), S_X^1} &= \langle K_{\varphi f}, K_g \rangle_{B(L^2(X)), S_X^1} \stackrel{(4.2)}{=} \text{Tr}(R(K_{\varphi f})K_g) \\ &= \text{Tr}(K_{\check{\varphi f}} K_g) = \iint_{X \times X} \varphi(y, x) f(y, x) g(y, x) \, dx \, dy. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \langle K_f, K_{\varphi g} \rangle_{B(L^2(X)), S_X^1} &\stackrel{(4.2)}{=} \text{Tr}(R(K_f)K_{\varphi g}) = \text{Tr}(K_{\check{f}} K_{\varphi g}) \\ &= \iint_{X \times X} f(y, x) \varphi(y, x) g(y, x) \, dx \, dy. \end{aligned}$$

We conclude by density. ■

Now, we prove the following equivalence by a standard argument.

Proposition 4.5 *Suppose that X is a σ -finite measure space. A measurable Schur multiplier $M_\varphi: B(L^2(X)) \rightarrow B(L^2(X))$ is unital if and only if it is trace-preserving when restricted to the space S_X^1 .*

Proof It is elementary to check that a $*$ -antiautomorphism is necessarily unital. If the measurable Schur multiplier $M_\varphi: B(L^2(X)) \rightarrow B(L^2(X))$ is unital, then for any $y \in S_X^1$, we have $\langle M_\varphi(1), y \rangle_{B(L^2(X)), S_X^1} \stackrel{(4.3)}{=} \langle 1, M_\varphi(y) \rangle_{B(L^2(X)), S_X^1}$, that is, $\text{Tr } y = \text{Tr } M_\varphi(y)$. We conclude that M_φ is trace-preserving.

If the measurable Schur multiplier $M_\varphi: B(L^2(X)) \rightarrow B(L^2(X))$ is trace-preserving, then for any $x \in S_X^1$, we have

$$\begin{aligned} \text{Tr}(R(M_\varphi(1)x)) &\stackrel{(4.2)}{=} \langle M_\varphi(1), x \rangle_{B(L^2(X)), S_X^1} \stackrel{(4.3)}{=} \langle 1, M_\varphi(x) \rangle_{B(L^2(X)), S_X^1} \\ &\stackrel{(4.2)}{=} \text{Tr}(R(1)M_\varphi(x)) = \text{Tr}(R(1)x). \end{aligned}$$

We conclude by duality that $M_\varphi(1) = 1$. ■

It is proved in [Duq22, Theorem 3.5] that a measurable Schur multiplier $M_\varphi: B(L^2(X)) \rightarrow B(L^2(X))$ is unital if and only if $\varphi(x, x) = 1$ almost everywhere on X . We deduce the following result.

Corollary 4.6 *Suppose that X is a σ -finite measure space. Let $M_\varphi: B(L^2(X)) \rightarrow B(L^2(X))$ be a measurable Schur multiplier. The following conditions are equivalent.*

- (1) M_φ is trace-preserving when restricted to the space S_X^1 .
- (2) M_φ is unital.
- (3) $\varphi(x, x) = 1$ almost everywhere on X .

In the following statement, we say that φ induces a (completely) positive measurable Schur multiplier if the linear map $M_\varphi: B(L^2(X)) \rightarrow B(L^2(X))$ is (completely) positive.

Proposition 4.7 *Let X be a σ -finite measure space. Let $\varphi \in L^\infty(X \times X)$. Consider the following properties.*

- (1) φ induces a completely positive measurable Schur multiplier.
- (2) φ induces a positive measurable Schur multiplier.
- (3) The function φ is an integrally positive-definite kernel.

We have $1 \iff 2 \implies 3$.

Proof $1. \implies 2$. It is obvious.

$2. \implies 1$. That is essentially a standard application of [STT14, Lemma 4.3]. See also [Tod14, Exercise 4.15].

$2. \implies 3$. Let $\xi \in L^1(X)$. We can write $\xi = \xi_1 \xi_2$ for some functions $\xi_1, \xi_2 \in L^2(X)$. Note that the operator $K_{\overline{\xi_1} \otimes \xi_1}: L^2(X) \rightarrow L^2(X)$ is positive. Indeed, for any $\eta \in L^2(X)$, we have

$$\begin{aligned} \langle K_{\overline{\xi_1} \otimes \xi_1}(\eta), \eta \rangle_{L^2(X)} &\stackrel{(2.15)}{=} \left(\int_X \xi_1 \eta \right) \langle \overline{\xi_1}, \eta \rangle_{L^2(X)} \\ &= \overline{\langle \overline{\xi_1}, \eta \rangle_{L^2(X)}} \langle \overline{\xi_1}, \eta \rangle_{L^2(X)} = |\langle \overline{\xi_1}, \eta \rangle_{L^2(X)}|^2 \geq 0. \end{aligned}$$

We deduce that $\langle M_\varphi(K_{\overline{\xi_1} \otimes \xi_1})\xi_2, \xi_2 \rangle_{L^2(X)} \geq 0$. Finally, it suffices to observe that

$$\begin{aligned} \langle M_\varphi(K_{\overline{\xi_1} \otimes \xi_1})\xi_2, \xi_2 \rangle_{L^2(X)} &= \langle K_{\varphi(\overline{\xi_1} \otimes \xi_1)}\xi_2, \xi_2 \rangle_{L^2(X)} = \int_X (K_{\varphi(\overline{\xi_1} \otimes \xi_1)}\xi_2)(x) \overline{\xi_2(x)} \, dx \\ &\stackrel{(2.15)}{=} \int_X \left(\int_X \varphi(x, y) \overline{\xi_1(x)} \xi_1(y) \xi_2(y) \, dy \right) \overline{\xi_2(x)} \, dx \\ &= \iint_{X \times X} \varphi(x, y) \overline{\xi_1(x)} \xi_1(y) \xi_2(y) \overline{\xi_2(x)} \, dx \, dy = \iint_{X \times X} \varphi(x, y) \overline{\xi(x)} \xi(y) \, dx \, dy. \end{aligned}$$

So (3.2) is satisfied. ■

Remark 4.8 Let X be a locally compact space equipped with a Radon measure with full support. Let $\varphi \in L^\infty(X \times X)$ be a function which induces a measurable Schur multiplier $M_\varphi: B(L^2(X)) \rightarrow B(L^2(X))$. The statement “3” implies “2” is true. Suppose that φ is an integrally positive-definite kernel. By Proposition 3.3, we can suppose that φ is equal to a bounded and measurable positive-definite kernel on $X \times X$. Let $K_f: L^2(X) \rightarrow L^2(X)$ be a positive Hilbert–Schmidt operator with a *continuous* symbol $f: X \times X \rightarrow \mathbb{C}$. Then the inequality

$$\iint_{X \times X} f(x, y) \overline{\xi(x)} \xi(y) \, dx \, dy \geq 0$$

is clearly satisfied for any $\xi \in L^2(X)$, in particular for any function $\xi \in C_c(X)$. By Proposition 3.6, we deduce that f is the class of a positive-definite kernel. Using the stability under pointwise product of the set of definite-positive kernels [BCR84, Theorem 1.12, p. 69], we see that the function φf of $L^2(X \times X)$ is also the class of a positive-definite kernel. By the observation following Definition 3.1, we deduce that for any $\xi \in L^2(X)$, we have

$$\iint_{X \times X} \varphi(x, y) f(x, y) \overline{\xi(x)} \xi(y) \, dx \, dy \geq 0.$$

Consequently $M_\varphi(K_f) = K_{\varphi f}$ is a positive operator on the Hilbert space $L^2(X)$. We conclude with Kaplansky’s theorem using the cone of positive Hilbert–Schmidt operators with continuous kernels.

Remark 4.9 Let X be a locally compact space equipped with a Radon measure with full support. Here, we explain how to give an answer to [STT14, Question 4.13(ii)]. Consider a positive continuous Schur multiplier $M_\varphi: B(L^2(X)) \rightarrow B(L^2(X))$, i.e., a measurable Schur multiplier with a *continuous* symbol $\varphi: X \times X \rightarrow \mathbb{C}$. By Proposition 4.7, the map φ is an integrally positive-definite kernel. In particular, for any continuous function $\xi: X \rightarrow \mathbb{C}$ with compact support, we have (3.2). By Proposition 3.6, we infer that the continuous function φ is a positive-definite kernel. Consequently, by [BCR84, p. 82], there exist a Hilbert space H and some map $\alpha: X \rightarrow H$ satisfying (2.19). Since the function φ is continuous, we conclude by the (classical) result [Car81, Lemme, p. 279]⁶ that the map $\alpha: X \rightarrow H$ is necessarily continuous (here the Hilbert space H is equipped with its usual topology). Since $\varphi \in L^\infty(X \times X)$, the function $x \mapsto \varphi(x, x) = \langle \alpha_x, \alpha_x \rangle_H$ is bounded. We conclude that α is bounded.

With sharp contrast, we think that the answer to the similar question [STT14, Question 4.13(i)] for (not necessarily positive) continuous Schur multipliers is negative.

The following is a generalization of [Arh13a, Proposition 5.4]. Recall that in this paper, the weak* continuity of a semigroup $(T_t)_{t \geq 0}$ means that the map $\mathbb{R}^+ \rightarrow \mathbb{C}$, $t \mapsto \langle T_t(x), y \rangle_{B(L^2(X)), S_X^1}$ is continuous for any $x \in B(L^2(X))$ and any $y \in S_X^1$.

Proposition 4.10 (1) *Let X be a second countable Radon measure space. If $(T_t)_{t \geq 0}$ is a weak* continuous semigroup of self-adjoint unital completely positive measurable Schur multipliers on $B(L^2(X))$ with continuous symbols, then there exist a real Hilbert space H and a continuous function $\alpha: X \rightarrow H$ such that the symbol of T_t is*

⁶If (2.19) is satisfied on a topological space X , then φ is continuous if and only if α is continuous.

$$(4.4) \quad \phi_t(x, y) = e^{-t\|\alpha_x - \alpha_y\|_H^2}, \quad \text{for almost all } x, y \in X.$$

(2) Let X be a (complete) σ -finite measure space such that if $(T_t)_{t \geq 0}$ is a weak* continuous semigroup of self-adjoint unital completely positive measurable Schur multipliers on $B(L^2(X))$, then there exist a real Hilbert space H and a weak measurable function $\alpha: X \rightarrow H$ such that the symbol of T_t is given by (4.4).

Proof 1. For any $t \geq 0$, let $\phi_t: X \times X \rightarrow \mathbb{C}$ be the continuous symbol of the Schur multiplier T_t . Since the map T_t is self-adjoint, the function ϕ_t is real-valued by Proposition 4.2. By Corollary 4.6, we have $\phi_t(x, x) = 1$ for any $x \in X$ and any $t \geq 0$. By Proposition 4.7, each ϕ_t is an integrally positive-definite kernel. We deduce by Proposition 3.6 that each ϕ_t is a positive-definite kernel. Finally, we have $\phi_0 = 1$.

For any $t, t' \geq 0$ and any $f \in L^2(X \times X)$, the relation $T_t T_{t'}(K_f) = T_{t+t'}(K_f)$ gives the equality $K_{\phi_t \phi_{t'} f} = K_{\phi_{t+t'} f}$. We infer that $\phi_t \phi_{t'} f = \phi_{t+t'} f$ in the space $L^2(X \times X)$. It is apparent that $(\phi_t)_{t \geq 0}$ defines a semigroup $(\text{Mult}_t)_{t \geq 0}$ of multiplication operators acting on $L^2(X \times X)$ defined by

$$(4.5) \quad \text{Mult}_t(f) \stackrel{\text{def}}{=} \phi_t f, \quad t \geq 0, f \in L^2(X \times X).$$

Note that (4.1) implies that $(\text{Mult}_t)_{t \geq 0}$ is a contraction semigroup. We will show the weak continuity of this semigroup. For any functions $f, g \in L^2(X \times X)$ such that $K_g \in S_X^1$, we have using the weak* continuity of the semigroup $(T_t)_{t \geq 0}$

$$\begin{aligned} \langle \text{Mult}_t(f), g \rangle_{L^2(X \times X)} &\stackrel{(4.5)}{=} \langle \phi_t f, g \rangle_{L^2(X \times X)} = \langle K_{\phi_t f}, K_g \rangle_{S_X^2} \\ &= \langle T_t(K_f), K_g \rangle_{B(L^2(X)), S_X^1} \xrightarrow{t \rightarrow 0} \langle K_f, K_g \rangle_{B(L^2(X)), S_X^1} = \langle f, g \rangle_{L^2(X \times X)} \end{aligned}$$

(here, we make a slight abuse of notation for the definition of the brackets). Note that the set of functions $g \in L^2(X \times X)$ such that $K_g \in S_X^1$ is dense in $L^2(X \times X)$ since S_X^1 is dense in the space S_X^2 . Now, with [Meg98, Exercise 2.71(a), p. 234], it is clear (using the contractivity of the semigroup) that this semigroup is weak continuous. By [EnN00, Theorem 5.8, p. 40], this semigroup is even strongly continuous. By [EnN00, Proposition 4.12, p. 32], there exists a measurable function $\psi: X \times X \rightarrow \mathbb{R}$ such that for almost all $x, y \in X$,

$$\phi_t(x, y) = e^{-t\psi(x, y)}, \quad t \geq 0.$$

We deduce that for almost all $x, y \in X$, we have if $t > 0$,

$$\psi(x, y) = -\frac{1}{t} \log \phi_t(x, y).$$

Since ϕ_t is continuous, we can suppose that ψ is continuous. Moreover, for any $x \in X$, we have

$$\psi(x, x) = -\frac{1}{t} \log \phi_t(x, x) = 0.$$

By Schoenberg’s theorem (Theorem 2.5), the function ψ defines a real-valued negative definite kernel which vanishes on the diagonal of $X \times X$. Finally, we use the characterization of continuous real-valued negative-definite kernels of [BHV08, Theorem C.2.3].

2. For any $t \geq 0$, let $\phi_t: X \times X \rightarrow \mathbb{C}$ be the symbol of T_t . Since the map T_t is self-adjoint, the function ϕ_t is real-valued almost everywhere by Proposition 4.2. Consequently, we can suppose that each function ϕ_t is real-valued. By Proposition 4.7, each ϕ_t is an integrally positive-definite kernel. Consequently, we can suppose by Proposition 3.3 that the function ϕ_t is a real-valued positive-negative kernel and bounded (by 1).

By Corollary 4.6, we have $\phi_t(x, x) = 1$ for almost all $x \in X$ and any $t \geq 0$. For any $t > 0$, let A_t be the subset of X such that $\phi_t(x, x) \neq 1$. The subset $\{(x, x) : x \in A_t\}$ is a subset of $A_t \times A_t$, hence negligible. By modifying ϕ_t on A_t , we can suppose that $\phi_t(x, x) = 1$ for any $x \in X$. Indeed, it is clear by observing the definition (2.18) that the property *positive-definite* is not lost by this change.

The end of the proof is similar to the proof of the first point replacing [BHV08, Theorem C.2.3] by Lemma 3.2. ■

Remark 4.11 In the opposite direction, consider a *separable* real Hilbert space H and a weak measurable function $\alpha: X \rightarrow H$. Let $(e_n)_{n \geq 1}$ be an orthonormal basis of H . We can write

$$(4.6) \quad \langle \alpha_x, \alpha_y \rangle_H = \sum_{n \geq 1} \langle \alpha_x, e_n \rangle_H \langle e_n, \alpha_y \rangle_H, \quad x, y \in X.$$

So the map $X \times X \rightarrow \mathbb{C}, (x, y) \mapsto \langle \alpha_x, \alpha_y \rangle_H$ is measurable with respect to the σ -algebra $\mathcal{A} \otimes \mathcal{A}$. Since we have

$$\|\alpha_x - \alpha_y\|_H^2 = \|\alpha_x\|_H^2 + \|\alpha_y\|_H^2 - 2\langle \alpha_x, \alpha_y \rangle_H,$$

we see that the symbol ϕ_t defined in (4.4) is measurable.

Note that ϕ_t converges to the function 1 when $t \rightarrow 0$ for the weak* topology of $L^\infty(X \times X)$. Indeed, for any $f \in L^1(X \times X)$, we have using dominated convergence theorem

$$\begin{aligned} \langle \phi_t, f \rangle_{L^\infty(X \times X), L^1(X \times X)} &= \int_{X \times X} e^{-t\|\alpha_x - \alpha_y\|_H^2} f(x, y) \, dx \, dy \xrightarrow{t \rightarrow 0} \int_{X \times X} f(x, y) \, dx \, dy \\ &= \langle 1, f \rangle_{L^\infty(X \times X), L^1(X \times X)}. \end{aligned}$$

With this convergence, it is left to the reader to show with [ArK22b, Lemma 4.25] that the symbols (4.4) define a weak* continuous semigroup $(T_t)_{t \geq 0}$.

If H is not separable, we can obtain the measurability of the symbols (4.4) if we make the stronger assertion that α is strongly measurable [HvNVW16, Definition 1.1.14, p. 8], which is equivalent by the Pettis measurability theorem [HvNVW16, Theorem 1.1.20, p. 10] to say that α is weak measurable and essentially separably valued, i.e., there exists a negligible subset N of X such that $\alpha(X - N)$ is separable. In this case, we can find a separable Hilbert space H containing $\alpha(X - N)$. Consequently, with an orthonormal basis $(e_n)_{n \geq 1}$ of H (4.6) is true on $(X - N) \times (X - N)$. We conclude as previously (if X is complete).

5 Dilations of semigroups of measurable Schur multipliers

Let X be a σ -finite measure space. If $g \in L^\infty(X)$, we will use the multiplication operator $\text{Mult}_g: L^2(X) \rightarrow L^2(X), \xi \mapsto g\xi$. Recall that $(\text{Mult}_g)^* = \text{Mult}_{\bar{g}}$. We will use the

following elementary result, which describes the composition of a Hilbert–Schmidt operator $K_f: L^2(X) \rightarrow L^2(X)$ with multiplication operators. Recall that the space of Hilbert–Schmidt operators is an ideal of the algebra $B(L^2(X))$ of bounded operators on the Hilbert space $L^2(X)$.

Lemma 5.1 *For any $\xi \in L^2(X)$, any $g, h \in L^\infty(X)$, and any $f \in L^2(X \times X)$, we have for almost all $x \in X$*

$$(5.1) \quad ((\text{Mult}_g K_f \text{Mult}_h)(\xi))(x) = \int_X g(x)h(y)f(x, y)\xi(x) \, dy.$$

So the Hilbert–Schmidt operator $\text{Mult}_g K_f \text{Mult}_h: L^2(X) \rightarrow L^2(X)$ is associated with the function $(x, y) \mapsto g(x)h(y)f(x, y)\xi(x)$.

Proof For almost all $x \in X$, we have

$$\begin{aligned} ((\text{Mult}_g K_f \text{Mult}_h)(\xi))(x) &= ((\text{Mult}_g K_f)(h\xi))(x) \\ &\stackrel{(2.15)}{=} \left(\text{Mult}_g \left(\int_X f(\cdot, y)h(y)\xi(y) \, dy \right) \right)(x) \\ &= \left(g(\cdot) \int_X f(\cdot, y)h(y)\xi(y) \, dy \right)(x) = \int_X g(x)h(y)f(x, y)\xi(x) \, dx. \quad \blacksquare \end{aligned}$$

The following is the first main result of this paper.

Theorem 5.2 *Let X be a σ -finite measure space. Let H be a real Hilbert space, and let $\alpha: X \rightarrow H$ be a strongly measurable map. Then the symbols (4.4) for $t \geq 0$ define a weak* continuous semigroup $(T_t)_{t \geq 0}$ of self-adjoint unital completely positive Schur multipliers on $B(L^2(X))$. Moreover, there exist an injective von Neumann algebra M equipped with a normal semifinite faithful trace, a weak* continuous group $(U_t)_{t \in \mathbb{R}}$ of trace-preserving $*$ -automorphisms of M , and a unital trace-preserving injective normal $*$ -homomorphism $J: B(L^2(X)) \rightarrow M$ such that*

$$(5.2) \quad T_t = \mathbb{E}U_t J,$$

for any $t \geq 0$, where $\mathbb{E}: M \rightarrow B(L^2(X))$ is the canonical trace-preserving normal faithful conditional expectation associated with J .

Proof Let $W: L^2_{\mathbb{R}}(\mathbb{R}, H) \rightarrow L^0(\Omega)$ be an $L^2_{\mathbb{R}}(\mathbb{R}, H)$ -isonormal process on a probability space (Ω, μ) as in (2.4). We define the von Neumann algebra $M \stackrel{\text{def}}{=} L^\infty(\Omega) \overline{\otimes} B(L^2(X))$. Note that M is an injective von Neumann algebra. We equip this von Neumann algebra with the normal semifinite faithful trace $\tau_M \stackrel{\text{def}}{=} \int_\Omega \cdot \otimes \text{Tr}$. By [Sak71, Theorem 1.22.13], we have a $*$ -isomorphism $M = L^\infty(\Omega, B(L^2(X)))$. We define the canonical unital normal injective $*$ -homomorphism

$$(5.3) \quad \begin{array}{ccc} J: B(L^2(X)) & \longrightarrow & L^\infty(\Omega) \overline{\otimes} B(L^2(X)) \\ z & \longmapsto & 1 \otimes z \end{array} .$$

It is clear that the map J preserves the traces. We denote by $\mathbb{E}: L^\infty(\Omega) \overline{\otimes} B(L^2(X)) \rightarrow B(L^2(X))$ the canonical trace-preserving normal faithful conditional expectation of

M onto $B(L^2(X))$. Note that $\mathbb{E} = \int_{\Omega} \cdot \otimes \text{Id}_{B(L^2(X))}$. For almost all $\omega \in \Omega$ and any $t \geq 0$, we introduce with (2.7)

$$(5.4) \quad k_{t,\omega}(x) \stackrel{\text{def}}{=} e^{\sqrt{2i}(W_t(\alpha_x))(\omega)}, \quad x \in X.$$

Let (f_i) be an orthonormal basis of the separable Hilbert space $L^2_{\mathbb{R}}(\mathbb{R}^+)$. By the Pettis measurability theorem [HvNVW16, Theorem 1.1.20, p. 10], there exists a negligible subset N of X such that $\alpha(X - N)$ is subset of a separable closed subspace H_0 of H . Let (e_j) be an orthonormal basis of H_0 . For any $t > 0$ fixed, we can write almost everywhere (with [HvNVW18, Corollary 6.4.4, pp. 35–36])

$$\begin{aligned} (W_t(\alpha_x))(\omega) &\stackrel{(2.4)(2.7)}{=} \sum_{i,j \in I} \gamma_{i,j}(\omega) \langle 1_{[0,t]} \otimes \alpha_x, f_i \otimes e_j \rangle_{L^2_{\mathbb{R}}(\mathbb{R}^+, H)} \\ &= \sum_{i,j \in I} \gamma_{i,j}(\omega) \langle 1_{[0,t]}, f_i \rangle \langle \alpha_x, e_j \rangle_H. \end{aligned}$$

We conclude that the function $X \times \Omega \rightarrow \mathbb{C}, (x, \omega) \mapsto k_{t,\omega}(x)$ is measurable.

If $t < 0$, we define $k_{t,\omega}(x)$ by the same formula but with $W_t(\alpha_x) \stackrel{\text{def}}{=} W(-1_{[t,0]} \otimes \alpha_x)$. For any $t \in \mathbb{R}$, we can define an element V_t of $L^\infty(\Omega, B(L^2(X)))$ by

$$(5.5) \quad V_t(\omega) \stackrel{\text{def}}{=} \text{Mult}_{k_{t,\omega}}.$$

Indeed, the function V_t is weak* measurable since if we consider a normal linear form $f: B(L^2(X)) \rightarrow \mathbb{C}, z \mapsto \sum_{n \geq 1} \langle z(\xi_n), \eta_n \rangle_{L^2(X)}$ with $\sum_{n \geq 1} (\|\xi_n\|_{L^2(X)} + \|\eta_n\|_{L^2(X)}) < \infty$ [Li92, Proposition 1.2.2, p. 9], we see that

$$f(\text{Mult}_{k_{t,\omega}}) = \sum_{n \geq 1} \langle \text{Mult}_{k_{t,\omega}} \xi_n, \eta_n \rangle_{L^2(X)} = \sum_{n \geq 1} \int_X k_{t,\omega} \xi_n \eta_n$$

is measurable since each integral defines a measurable function $\omega \rightarrow \int_X k_{t,\omega} \xi_n \eta_n$ by Fubini's theorem. Note that V_t is a unitary element of $L^\infty(\Omega, B(L^2(X)))$. Now, for any $t \in \mathbb{R}$, we define the linear map

$$(5.6) \quad \begin{array}{ccc} \mathcal{U}_t: L^\infty(\Omega, B(L^2(X))) & \longrightarrow & L^\infty(\Omega, B(L^2(X))) \\ g & \longmapsto & V_t g V_t^* \end{array}.$$

It is obvious that each map \mathcal{U}_t is a trace-preserving *-automorphism of the von Neumann algebra M . Moreover, for any elements ξ, η of the Hilbert space $L^2(\Omega, L^2(X)) = L^2(\Omega \times X)$, we obtain using⁷ the first part of Lemma 2.1 since $\bar{\xi}\eta \in L^1(\Omega \times X)$

$$\begin{aligned} \langle V_t(\xi), \eta \rangle_{L^2(\Omega, L^2(X))} &= \int_{\Omega \times X} \overline{V_t(\xi)(\omega, x)} \eta(\omega, x) \, d\mu(\omega) \, dx \\ &\stackrel{(5.5)}{=} \int_{\Omega \times X} \overline{k_{t,\omega}(x) \xi(\omega, x)} \eta(\omega, x) \, d\mu(\omega) \, dx \\ &\stackrel{(5.4)}{=} \int_{\Omega \times X} e^{-\sqrt{2i}(W_t(\alpha_x))(\omega)} \overline{\xi(\omega, x)} \eta(\omega, x) \, d\mu(\omega) \, dx \\ &\xrightarrow{t \rightarrow 0} \int_{\Omega \times X} e^{-\sqrt{2i}(W_{t_0}(\alpha_x))(\omega)} \overline{\xi(\omega, x)} \eta(\omega, x) \, d\mu(\omega) \, dx = \langle V_{t_0}(\xi), \eta \rangle_{L^2(\Omega, L^2(X))}. \end{aligned}$$

⁷We have $1_{[0,t]} \xrightarrow{t \rightarrow t_0} 1_{[0,t_0]}$ and $1_{[t,0]} \xrightarrow{t \rightarrow t_0} 1_{[t_0,0]}$ in $L^2_{\mathbb{R}}(\mathbb{R})$.

So $(V_t)_{t \in \mathbb{R}}$ is a weak continuous family of unitaries hence a strongly continuous family since by [Str81, p. 41] the weak operator topology and the strong operator topology coincide on the unitary group. By composition on bounded subsets, the map $\mathbb{R} \rightarrow L^\infty(\Omega, B(L^2(X)))$, $t \mapsto \mathcal{U}_t(g)$ is strong operator continuous for any $g \in L^\infty(\Omega, B(L^2(X)))$ and clearly even strong* operator continuous, hence weak* continuous. We conclude that $(\mathcal{U}_t)_{t \in \mathbb{R}}$ is a point-weak* continuous family of *-automorphisms. For any $f \in L^2(X \times X)$ and any $t \in \mathbb{R}$, we have

$$\begin{aligned}
 (5.7) \quad \mathcal{U}_t(1 \otimes K_f) &\stackrel{(5.6)}{=} \omega \mapsto (V_t(1 \otimes K_f)V_t^*)(\omega) = \omega \mapsto V_t(\omega)K_fV_t(\omega)^* \\
 &\stackrel{(5.5)}{=} \omega \mapsto \text{Mult}_{k_{t,\omega}}K_f(\text{Mult}_{k_{t,\omega}})^* \stackrel{(5.1)}{=} \omega \mapsto K((x, y) \mapsto k_{t,\omega}(x)\overline{k_{t,\omega}(y)}f(x, y)) \\
 &\stackrel{(5.4)}{=} \omega \mapsto K((x, y) \mapsto e^{\sqrt{2}i(W_t(\alpha_x - \alpha_y))(\omega)}f(x, y)).
 \end{aligned}$$

In particular, for almost all $\omega \in \Omega$, any $\xi \in L^2(X)$, and almost all $y \in X$, we have

$$(5.8) \quad \left([(\mathcal{U}_t(1 \otimes K_f))(\omega)](\xi) \right)(y) \stackrel{(2.15)}{=} \int_X e^{\sqrt{2}i(W_t(\alpha_x - \alpha_y))(\omega)} f(x, y)\xi(x) dx.$$

For any $t \in \mathbb{R}$, we introduce the right shift $S_t: L^2_{\mathbb{R}}(\mathbb{R}) \rightarrow L^2_{\mathbb{R}}(\mathbb{R})$. We consider the operator $S_t \stackrel{\text{def}}{=} \Gamma_\infty(S_t) \otimes \text{Id}_{B(L^2(X))}: L^\infty(\Omega, B(L^2(X))) \rightarrow L^\infty(\Omega, B(L^2(X)))$. Essentially, by the second part of Lemma 2.1, $(S_t)_{t \in \mathbb{R}}$ is a weak* continuous semigroup of *-automorphisms. We have

$$(5.9) \quad S_t(e^{iW(1_{[s,s'] \otimes h})} \otimes z) \stackrel{(2.6)}{=} e^{iW(1_{[s+t,s'+t] \otimes h})} \otimes z, \quad t \in \mathbb{R}, h \in H, z \in B(L^2(X)), s < s'.$$

For any $t \in \mathbb{R}$, we define the *-automorphism $U_t \stackrel{\text{def}}{=} \mathcal{U}_t S_t$. We will prove that $(U_t)_{t \in \mathbb{R}}$ is a group of operators. On the one hand, for any $t, t' \in \mathbb{R}$, we have

$$\begin{aligned}
 U_{t'}U_t(e^{iW(1_{[s,s'] \otimes h})} \otimes 1) &= \mathcal{U}_{t'}S_{t'}\mathcal{U}_tS_t(e^{iW(1_{[s,s'] \otimes h})} \otimes 1) \\
 &\stackrel{(5.9)}{=} \mathcal{U}_{t'}S_{t'}\mathcal{U}_t(e^{iW(1_{[s+t,s'+t] \otimes h})} \otimes 1) \\
 &= \mathcal{U}_{t'}S_{t'}(e^{iW(1_{[s+t,s'+t] \otimes h})} \otimes 1) \stackrel{(5.9)}{=} \mathcal{U}_{t'}(e^{iW(1_{[s+t+t',s'+t+t'] \otimes h})} \otimes 1) \\
 &= e^{iW(1_{[s+t+t',s'+t+t'] \otimes h})} \otimes 1 = U_{t+t'}(e^{iW(1_{[s,s'] \otimes h})} \otimes 1).
 \end{aligned}$$

On the other hand, for any $t, t' \geq 0$ and any $\xi \in L^2(X)$, we have

$$\begin{aligned}
 U_{t'}U_t(1 \otimes K_f) &= \mathcal{U}_{t'}S_{t'}\mathcal{U}_tS_t(1 \otimes K_f) \\
 &\stackrel{(5.9)}{=} \mathcal{U}_{t'}S_{t'}\mathcal{U}_t(1 \otimes K_f) \\
 &\stackrel{(5.7)}{=} \mathcal{U}_{t'}S_{t'}(\omega \mapsto K((x, y) \mapsto e^{\sqrt{2}i(W_t(\alpha_x - \alpha_y))(\omega)}f(x, y))) \\
 &= \mathcal{U}_{t'}(\omega \mapsto K((x, y) \mapsto e^{\sqrt{2}i(W(1_{[t',t'+t] \otimes (\alpha_x - \alpha_y))}(\omega)}f(x, y))) \\
 &\stackrel{(5.6)(5.5)}{=} \omega \mapsto \text{Mult}_{k_{t',\omega}}K((x, y) \mapsto e^{\sqrt{2}i(W(1_{[t',t'+t] \otimes (\alpha_x - \alpha_y))}(\omega)}f(x, y)) \\
 &\quad (\text{Mult}_{k_{t',\omega}})^* \\
 &\stackrel{(5.1)}{=} \omega \mapsto K((x, y) \mapsto k_{t',\omega}(x)\overline{k_{t',\omega}(y)}e^{\sqrt{2}i(W(1_{[t',t'+t] \otimes (\alpha_x - \alpha_y))}(\omega)}f(x, y))
 \end{aligned}$$

$$\begin{aligned}
 &\stackrel{(5.4)}{=} \omega \mapsto K((x, y) \mapsto e^{\sqrt{2i}(W_{t'}(\alpha_x - \alpha_y))(\omega)} e^{\sqrt{2i}(W(1_{[t', t+t'] \otimes (\alpha_x - \alpha_y)}) (\omega))} f(x, y)) \\
 &= \omega \mapsto K((x, y) \mapsto e^{\sqrt{2i}(W_{t+t'}(\alpha_x - \alpha_y))(\omega)} f(x, y)) \\
 &\stackrel{(5.8)}{=} \mathcal{U}_{t+t'}(1 \otimes K_f) = U_{t+t'}(1 \otimes K_f).
 \end{aligned}$$

Similarly for any $t \leq 0$ and any $t' \geq 0$ such that $t' \leq -t$ and any $\xi \in L^2(X)$, we have

$$\begin{aligned}
 U_{t'} U_t(1 \otimes K_f) &= \mathcal{U}_{t'} S_{t'} \mathcal{U}_t S_t(1 \otimes K_f) \stackrel{(5.9)}{=} \mathcal{U}_{t'} S_{t'} \mathcal{U}_t(1 \otimes K_f) \\
 &\stackrel{(5.7)}{=} \mathcal{U}_{t'} S_{t'}(\omega \mapsto K((x, y) \mapsto e^{\sqrt{2i}(W(-1_{[t, 0] \otimes (\alpha_x - \alpha_y)}) (\omega))} f(x, y))) \\
 &= \mathcal{U}_{t'}(\omega \mapsto K((x, y) \mapsto e^{\sqrt{2i}(W(-1_{[t'+t, t'] \otimes (\alpha_x - \alpha_y)}) (\omega))} f(x, y))) \\
 &\stackrel{(5.6)(5.5)}{=} \omega \mapsto \text{Mult}_{k_{t', \omega}} K((x, y) \mapsto e^{\sqrt{2i}(W(-1_{[t'+t, t'] \otimes (\alpha_x - \alpha_y)}) (\omega))} f(x, y)) \\
 &\quad (\text{Mult}_{k_{t', \omega}})^* \\
 &\stackrel{(5.1)}{=} \omega \mapsto K((x, y) \mapsto k_{t', \omega}(x) \overline{k_{t', \omega}(y)} e^{\sqrt{2i}(W(-1_{[t'+t, t'] \otimes (\alpha_x - \alpha_y)}) (\omega))} f(x, y)) \\
 &\stackrel{(5.4)}{=} \omega \mapsto K((x, y) \mapsto e^{\sqrt{2i}(W(1_{[0, t'] \otimes (\alpha_x - \alpha_y)}) (\omega))} e^{\sqrt{2i}(W(-1_{[t'+t, t'] \otimes (\alpha_x - \alpha_y)}) (\omega))} \\
 &\quad f(x, y)) \\
 &= \omega \mapsto K((x, y) \mapsto e^{\sqrt{2i}(W(-1_{[t'+t, 0] \otimes (\alpha_x - \alpha_y)}) (\omega))} f(x, y)) \quad \text{since } t + t' \leq 0 \leq t' \\
 &\stackrel{(5.8)}{=} \mathcal{U}_{t+t'}(1 \otimes K_f) = U_{t+t'}(1 \otimes K_f).
 \end{aligned}$$

The remaining cases are left to the reader. With these computations, it is easy to check that $(U_t)_{t \in \mathbb{R}}$ is a weak* continuous group of *-automorphisms (consider the preadjoint maps of the U_t 's and use [EnN00, Lemme, B.15] to prove the weak* continuity of the group).

For any $f \in L^2(X \times X)$, any $\xi \in L^2(X)$, all $t \geq 0$, and almost all y of X , we finally have⁸

$$\begin{aligned}
 &([\mathbb{E}U_t J(K_f)](\xi))(y) \stackrel{(5.3)}{=} ([\mathbb{E}U_t(1 \otimes K_f)](\xi))(y) = ([\mathbb{E}\mathcal{U}_t S_t(1 \otimes K_f)](\xi))(y) \\
 &= ([\mathbb{E}\mathcal{U}_t(1 \otimes K_f)](\xi))(y) = \left(\left[\int_{\Omega} (\mathcal{U}_t(1 \otimes K_f))(\omega) d\mu(\omega) \right] (\xi) \right)(y) \\
 &= \int_{\Omega} ([(\mathcal{U}_t(1 \otimes K_f))(\omega)](\xi))(y) d\mu(\omega) \\
 &\stackrel{(5.8)}{=} \int_{\Omega} \left(\int_X e^{\sqrt{2i}(W_t(\alpha_x - \alpha_y))(\omega)} f(x, y) \xi(x) dx \right) d\mu(\omega) \\
 &= \int_X \left(\int_{\Omega} e^{\sqrt{2i}(W_t(\alpha_x - \alpha_y))(\omega)} d\mu(\omega) \right) f(x, y) \xi(x) dx \\
 &\stackrel{(2.13)}{=} \int_X e^{-t\|\alpha_x - \alpha_y\|_H^2} f(x, y) \xi(x) dx \\
 &\stackrel{(4.4)}{=} \int_X \phi_t(x, y) f(x, y) \xi(x) dx \stackrel{(2.15)}{=} (K_{\phi_t f}(\xi))(y).
 \end{aligned}$$

⁸Note that the function $f(\cdot, y)$ belongs to $L^2(X)$ for almost all $y \in X$.

Hence, we have $(\mathbb{E}U_t J(K_f))(\xi) = (K_{\phi_{t,f}}(\xi))$ and finally $\mathbb{E}U_t J(K_f) = K_{\phi_{t,f}}$. By weak* density, we deduce the existence of T_t and (5.2). For any $t, t' \geq 0$ and all $x, y \in X$, we have

$$e^{-t\|\alpha_x - \alpha_y\|_H^2} e^{-t'\|\alpha_x - \alpha_y\|_H^2} = e^{-(t+t')\|\alpha_x - \alpha_y\|_H^2}.$$

We deduce that $(T_t)_{t \geq 0}$ is a semigroup. Furthermore, the function (4.4) is real-valued. By Proposition 4.2, we conclude that each T_t is self-adjoint. The factorization (5.2) shows that each T_t is completely positive and unital. The weak* continuity is a consequence of the one of the group $(U_t)_{t \in \mathbb{R}}$. ■

Remark 5.3 It would be interesting to give a direct proof of the first part, i.e., without the factorization (5.2).

6 Markov dilations of semigroups of measurable Schur multipliers

Our second main result is the following theorem, which gives a standard Markov dilation.

Theorem 6.1 *Let X be a σ -finite measure space. Let H be a real Hilbert space, and let $\alpha: X \rightarrow H$ be a strongly measurable map. Consider the weak* continuous semigroup $(T_t)_{t \geq 0}$ of self-adjoint unital completely positive Schur multipliers on $B(L^2(X))$ defined in Theorem 5.2 by the symbols (4.4). There exist an injective von Neumann algebra N equipped with a normal semifinite faithful trace, an increasing filtration $(N_s)_{s \geq 0}$ of N with associated trace-preserving conditional normal faithful expectations $\mathbb{E}_s: N \rightarrow N_s$, and trace-preserving unital normal injective $*$ -homomorphisms $\pi_s: B(L^2(X)) \rightarrow N_s$ such that*

$$(6.1) \quad \mathbb{E}_s \pi_t = \pi_s T_{t-s}, \quad 0 \leq s \leq t.$$

Proof Let $W: L^2_{\mathbb{R}}(\mathbb{R}^+, H) \rightarrow L^0(\Omega)$ be an H -cylindrical Brownian motion on a probability space (Ω, μ) (see Section 2). As in the proof of Theorem 5.2, for almost all $\omega \in \Omega$ and any $t \geq 0$, we introduce the element (5.4) and for any $t \geq 0$ the element V_t of $L^\infty(\Omega, B(L^2(X)))$ defined by (5.5). Recall that V_t is a unitary element of the von Neumann algebra $N \stackrel{\text{def}}{=} L^\infty(\Omega, B(L^2(X)))$, which is clearly injective. Again with the $*$ -homomorphism J defined in (5.3) and the $*$ -automorphism \mathcal{U}_t defined in (5.6), we define for any $t \geq 0$ the unital normal injective $*$ -homomorphism

$$(6.2) \quad \pi_t \stackrel{\text{def}}{=} \mathcal{U}_t J: \begin{matrix} B(L^2(X)) & \longrightarrow & L^\infty(\Omega, B(L^2(X))) \\ z & \longmapsto & V_t(1 \otimes z)V_t^* \end{matrix}.$$

It is obvious that π_t is trace-preserving. For any $t \geq 0$, we also define the canonical normal conditional expectations $\mathbb{E}_{\mathcal{F}_t}: L^\infty(\Omega) \rightarrow L^\infty(\Omega, \mathcal{F}_t)$ and $\mathbb{E}_t \stackrel{\text{def}}{=} \mathbb{E}_{\mathcal{F}_t} \otimes \text{Id}_{B(L^2(X))}: L^\infty(\Omega, B(L^2(X))) \rightarrow L^\infty(\Omega, \mathcal{F}_t, B(L^2(X)))$ where the σ -algebra \mathcal{F}_t is defined in (2.8). We introduce the von Neumann algebra $N_t \stackrel{\text{def}}{=} L^\infty(\Omega, \mathcal{F}_t, B(L^2(X)))$. Note that the range of π_t is included in N_t .

For almost all $\omega \in \Omega$, the Hilbert–Schmidt operator

$$(\pi_t(K_f))(\omega) \stackrel{(6.2)(5.3)}{=} (\mathcal{U}_t(1 \otimes K_f))(\omega)$$

is associated by the computation (5.7) with the function

$$(6.3) \quad (x, y) \mapsto e^{\sqrt{2}i(W_t(\alpha_x - \alpha_y))(\omega)} f(x, y).$$

Similarly, for any $0 \leq s \leq t$ and almost all $\omega \in \Omega$, the operator

$$(\pi_s(T_{t-s}(K_f)))(\omega) = (\pi_s(K_{\phi_{t-s}, f}))(\omega)$$

is Hilbert–Schmidt and associated with the function

$$(6.4) \quad (x, y) \mapsto e^{\sqrt{2}i(W_s(\alpha_x - \alpha_y))(\omega)} e^{-(t-s)\|\alpha_x - \alpha_y\|_H^2} f(x, y).$$

Now, recall that $(W_t(h))_{t \geq 0}$ is a Brownian motion for any fixed $h \in H$. Hence, for any $0 \leq s \leq t$ and any $x, y \in X$, the random variable

$$(6.5) \quad \begin{aligned} W(1_{]s, t]} \otimes (\alpha_x - \alpha_y)) &= W(1_{[0, t]} \otimes (\alpha_x - \alpha_y)) - W(1_{[0, s]} \otimes (\alpha_x - \alpha_y)) \\ &\stackrel{(2.7)}{=} W_t(\alpha_x - \alpha_y) - W_s(\alpha_x - \alpha_y) \end{aligned}$$

is independent by (2.12) from the σ -algebra $\mathcal{F}_s \stackrel{(2.8)}{=} \sigma(W_r(h) : r \in [0, s], h \in H)$.

Consequently, the random variable $e^{\sqrt{2}iW(1_{]s, t]} \otimes (\alpha_x - \alpha_y))}$ is also independent from the σ -algebra \mathcal{F}_s . Using [HvNVW16, Proposition 2.6.31] in the fourth equality since each random variable $W_s(\alpha_x)$ is \mathcal{F}_s -measurable, we finally obtain, for any $0 \leq s \leq t$ and any $f \in L^2(X \times X)$,

$$\begin{aligned} \mathbb{E}_s \pi_t(K_f) &\stackrel{(6.3)}{=} \mathbb{E}_s \left(\omega \mapsto [K((x, y) \mapsto e^{\sqrt{2}iW_t(\alpha_x - \alpha_y)(\omega)} f(x, y))] \right) \\ &\stackrel{(6.5)}{=} \mathbb{E}_s \left(\omega \mapsto [K((x, y) \mapsto e^{\sqrt{2}iW_s(\alpha_x - \alpha_y)(\omega)} e^{\sqrt{2}iW(1_{]s, t]} \otimes (\alpha_x - \alpha_y))(\omega)} f(x, y))] \right) \\ &\stackrel{(5.1)}{=} \mathbb{E}_s \left(\omega \mapsto \text{Mult}_{k_s, \omega} [K((x, y) \mapsto e^{\sqrt{2}iW(1_{]s, t]} \otimes (\alpha_x - \alpha_y))(\omega)} f(x, y))] \text{Mult}_{k_s, \omega}^{-1} \right) \\ &= [\omega \mapsto \text{Mult}_{k_s, \omega}] \mathbb{E}_s \left(\omega \mapsto [K((x, y) \mapsto e^{\sqrt{2}iW(1_{]s, t]} \otimes (\alpha_x - \alpha_y))(\omega)} f(x, y))] \right) \\ &\quad [\omega \mapsto \text{Mult}_{k_s, \omega}^{-1}] \\ &\stackrel{(2.14)}{=} [\omega \mapsto \text{Mult}_{k_s, \omega}] \mathbb{E} \left(\omega \mapsto [K((x, y) \mapsto e^{\sqrt{2}iW(1_{]s, t]} \otimes (\alpha_x - \alpha_y))(\omega)} f(x, y))] \right) \\ &\quad [\omega \mapsto \text{Mult}_{k_s, \omega}^{-1}]. \end{aligned}$$

Now, for any $\xi \in L^2(X)$ and almost all $y \in Y$, we see that

$$\begin{aligned} \mathbb{E}(\omega \mapsto [K((x, y) \mapsto e^{\sqrt{2}iW(1_{]s, t]} \otimes (\alpha_x - \alpha_y))(\omega)} f(x, y))](\xi)(y) &= \left(\int_{\Omega} K((x, y) \mapsto e^{\sqrt{2}iW(1_{]s, t]} \otimes (\alpha_x - \alpha_y))(\omega)} f(x, y)) d\mu(\omega) \right)(\xi)(y) \\ &= \int_X \left(\int_{\Omega} e^{\sqrt{2}iW(1_{]s, t]} \otimes (\alpha_x - \alpha_y))(\omega)} d\mu(\omega) \right) f(x, y) \xi(x) dx \\ &\stackrel{(2.13)}{=} \int_X e^{-(t-s)\|\alpha_x - \alpha_y\|_H^2} f(x, y) \xi(x) dx. \end{aligned}$$

Consequently, the previous expectation is

$$\mathbb{E}(\omega \mapsto [K((x, y) \mapsto e^{\sqrt{2}iW(1_{]s, t]} \otimes (\alpha_x - \alpha_y))(\omega)} f(x, y)))]$$

$$= K((x, y) \mapsto e^{-(t-s)\|\alpha_x - \alpha_y\|_H^2} f(x, y)).$$

Finally, we obtain

$$\begin{aligned} \mathbb{E}_s \pi_t(K_f) &= [\omega \mapsto \text{Mult}_{k_s, \omega}] K((x, y) \mapsto e^{-(t-s)\|\alpha_x - \alpha_y\|_H^2} f(x, y)) [\omega \mapsto \text{Mult}_{k_s, \omega}^{-1}] \\ &\stackrel{(5.1)}{=} \omega \mapsto K((x, y) \mapsto e^{\sqrt{2}iW_s(\alpha_x - \alpha_y)(\omega)} e^{-(t-s)\|\alpha_x - \alpha_y\|_H^2} f(x, y)) \\ &\stackrel{(6.4)}{=} \pi_s(T_{t-s}(K_f)). \end{aligned}$$

By weak* density, the proof is complete. ■

Similarly, we can prove the following reversed Markov dilation.

Theorem 6.2 *Let X be a σ -finite measure space. Let H be a real Hilbert space, and let $\alpha: X \rightarrow H$ be a strongly measurable map. Consider the weak* continuous semigroup $(T_t)_{t \geq 0}$ of self-adjoint unital completely positive Schur multipliers on $B(L^2(X))$ defined in Theorem 5.2 by the symbols (4.4). There exist an injective von Neumann algebra N equipped with a normal semifinite faithful trace, a decreasing filtration $(N_s)_{s \geq 0}$ of N with associated trace-preserving normal faithful conditional expectations $\mathbb{E}_s: N \rightarrow N_s$, and trace-preserving unital normal injective $*$ -homomorphisms $\pi_s: B(L^2(X)) \rightarrow N_s$ such that*

$$(6.6) \quad \mathbb{E}_s \pi_t = \pi_s T_{s-t}, \quad 0 \leq t \leq s.$$

7 Applications to functional calculus and interpolation

We refer to [Haa06, HvNVW18, JMX06, KuW04, LeM07] for background on sectoriality and H^∞ functional calculus. In the spirit of the result of [Arh20, Theorem 4.1], we can prove the following theorem with a similar argument.

Theorem 7.1 *Let X be a σ -finite measure space. Let H be a real Hilbert space, and let $\alpha: X \rightarrow H$ be a strongly measurable map. Consider the weak* continuous semigroup $(T_t)_{t \geq 0}$ of self-adjoint unital completely positive Schur multipliers on $B(L^2(X))$ defined in Theorem 5.2 by the symbols (4.4). Suppose $1 < p < \infty$. We let $-A_p$ be the generator of the induced strongly continuous semigroup $(T_{t,p})_{t \geq 0}$ on the Banach space S_X^p . Then, for any $\theta > \pi|\frac{1}{p} - \frac{1}{2}|$, the operator A_p has a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus.*

Let $\mathcal{T} = (T_t)_{t \geq 0}$ be a Markov semigroup on a von Neumann algebra \mathcal{M} equipped with a normal semifinite faithful trace. We set $\mathcal{M}_0 \stackrel{\text{def}}{=} \{x \in \mathcal{M} : \lim_{t \rightarrow \infty} T_t(x) = 0\}$ where the limit is a weak* limit. For $1 \leq p < \infty$, we let $L_0^p(\mathcal{M}) \stackrel{\text{def}}{=} \{x \in L^p(\mathcal{M}) : \lim_{t \rightarrow \infty} T_t(x) = 0\}$ where the limit is for the L^p -norm.

Recall that the column-BMO-seminorm is defined by

$$\|x\|_{\text{BMO}_{\mathcal{T}}^c} \stackrel{\text{def}}{=} \sup_{t \geq 0} \|T_t(|x - T_t(x)|^2)\|_\infty^{\frac{1}{2}}, \quad x \in L_0^2(\mathcal{M}).$$

Then we can define the row-BMO-seminorm $\|x\|_{\text{BMO}_{\mathcal{T}}} \stackrel{\text{def}}{=} \|x^*\|_{\text{BMO}_{\mathcal{T}}^c}$ and finally the BMO-seminorm

$$\|x\|_{\text{BMO}_{\mathcal{T}}} \stackrel{\text{def}}{=} \max \{ \|x\|_{\text{BMO}_{\mathcal{T}}^c}, \|x^*\|_{\text{BMO}_{\mathcal{T}}^c} \}.$$

In this context, we can introduce a Banach space $\text{BMO}_{\mathcal{T}}$ and we refer to [Mei08, Section 3] and [Cas19, JM12] for a precise definition. Combining Theorem 6.2 and [JM12, Theorem 5.12], we obtain the following interpolation formula by observing that each operator T_t of the semigroup is trace-preserving by the factorization of Theorem 5.2.

Theorem 7.2 *Let X be a σ -finite measure space. Let H be a real Hilbert space, and let $\alpha: X \rightarrow H$ be a strongly measurable map. Consider the weak* continuous semigroup $(T_t)_{t \geq 0}$ of self-adjoint unital completely positive Schur multipliers on $B(L^2(X))$ defined in Theorem 5.2 by the symbols (4.4). Suppose $1 \leq p < \infty$. Then*

$$\left(\text{BMO}_{\mathcal{T}}, S_{X,0}^1 \right)_{\frac{1}{p}} \approx S_{X,0}^p$$

with equivalence of norms up to a constant $\approx p$.

Remark 7.3 We can replace the space $\text{BMO}_{\mathcal{T}}$ by other BMO-spaces (see [JM12, Theorem 5.12]).

Finally, we could give applications in ergodic theory in the spirit of the paper [JiW17].

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