ON THE NUMBER OF SPHERES WHICH CAN HIDE A GIVEN SPHERE

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1. Several years ago H. Hornich suggested the following problem: find the minimal number of unit spheres which can hide a unit sphere in the sense that each ray emanating from the centre of that sphere meets at least one of the hiding spheres, with no two of the spheres overlapping. We shall call any set of spheres which hide a given unit sphere a *cloud*.

The first result concerning this and related questions can be found in a paper of Fejes Tóth (**4**; see also **5**, **7**, **8**, **6**, and **1**). With respect to the original problem, Fejes Tóth has given a lower estimate for the minimal number N of the spheres of a cloud. His proof of the inequality $N \geqslant 19$ was based on an earlier estimate of his, referring to the minimal number of spherical caps of given radius which can cover the unit sphere. An upper bound for N has been provided by a result of Danzer (**2**). He constructed a cloud consisting of 42 spheres. Thus we have

$$19 \leqslant N \leqslant 42.$$

The gap between these two bounds is rather broad and our aim is to make it somewhat narrower. We shall prove that $N \geqslant 24$.

2. In the following we shall use the terms *middle sphere* and *shadow of a sphere*. By the first we mean the sphere to be hidden, and by the second the cap of the middle sphere determined by the rays intersecting the hiding sphere in question. Obviously no sphere has a shadow of radius larger than 30°. Since the shadows together cover the sphere, and on the other hand the area of any shadow is $\leqslant 2\pi(1 - \cos 30°)$, we have

$$N \geqslant 4\pi/\{2\pi(1 - \cos 30°)\} = 4/(2 - \sqrt{3}) = 4(2 + \sqrt{3}) = 14.92\ldots,$$

and therefore $N \geqslant 15$.

3. A sharper inequality can be given by taking into consideration the fact that if n caps of radius r form a covering they must more or less overlap. This has been done by Fejes Tóth (**3**). We give here an outline of his method.

The surface of the sphere can be decomposed into n convex spherical polygons in such a way that each polygon is contained in one cap. It may be supposed that the decomposition contains only trihedral vertices. Among the spherical

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k-gons contained in a cap the regular k-gon inscribed in the cap has the greatest area. Now the function

$$A(k, r) = 2\pi - 2k \arctan[\cos r \cdot \tan(\pi/k)]$$

describing the area of this regular k-gon is a concave monotone function of the variable k. Therefore, using Jensen's inequality, we obtain that the average area of a polygon cannot exceed the value $A(\bar{k}, r)$, where $\bar{k}$ denotes the average number of sides in the polygonal decomposition. Since all the vertices are trihedral Euler's formula yields $\bar{k} \leqslant 6 - 12/n$. Thus n has to satisfy the inequality

$$4\pi \leqslant nA(\bar{k}, r) \leqslant nA(6(n-2)/n, r),$$

and hence

$$n \geqslant \frac{2 \arctan[(1/\surd 3) \cos r]}{\{\arctan[(1/\surd 3) \cos r] - \pi/6\}}. \tag{1}$$

For $r = 30°$ this gives $n \geqslant 18.2 \ldots$. Therefore

$$N \geqslant 19. \tag{2}$$

4. Let us now consider the cloud constructed by Danzer. The 42 spheres can be grouped according to the radii of their shadows. Five groups consist of eight spheres each with shadow radii 30°, 21°19′, 17°12′, 14°4′, and 11°32′. The radii of the remaining two shadows equal 25°46′. The fact that in this cloud more than half of the spheres have a shadow of rather small radius (less than 18°) suggests that an essentially better lower estimate can be given only if we take into consideration that some of the spheres contribute essentially smaller shadows to the covering of the middle sphere.

5. Our estimate will be based on the following lemma.

LEMMA 1. *If three shadow caps have a point in common, then the radius of the smallest is* $\leqslant r_0 = \arccos\surd[(3 + \surd 6)/6] < 17°37'56''$. *Equality holds only if two of the spheres touch the middle sphere and each other, while the third is tangent to the first two and to the ray through their common point.*

Let us denote the centre of the middle sphere by O and the centres of the others by C_1, C_2, and C_3, where $OC_1 \leqslant OC_2 \leqslant OC_3$.

We first show that if there is a common point of the three shadows in the interior of the smallest cap, then the position of this third sphere can be changed so as to increase its shadow. In this case there exists a ray s starting from O and, after meeting the first two spheres, going through an inner point of the third. Consider a plane p containing the points O and C_3, which is orthogonal to the plane through O, C_1, and C_2. If we move C_3 in p, then, obviously, the distance of C_3 from any fixed point P in space depends strictly monotonically on the distance of C_3 from the orthogonal projection P' of P on the plane p. Thus any motion of C_3 along the boundary of the circle with centre the projection C'_1 of C_1

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on p which brings C_3 nearer to O prevents overlapping of the hiding spheres (in Figure 1 C'_2 denotes the projection of C_2). Therefore in the best arrangement of the three spheres there exists a ray which meets all of them and which is tangent only to the one with smallest shadow.

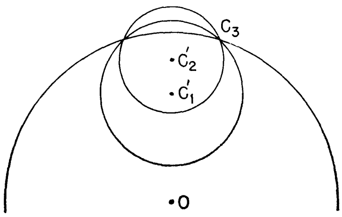

FIGURE 1

Now we use the following result **(4)**. If a straight line meets three non-overlapping unit spheres S_1, S_2, and S_3 in this order, then the "level difference" of the centres of S_1 and S_3, measured in the direction of the line, is at least $\sqrt{2}$. Equality holds only if the spheres are mutually tangent and the line is tangent to all three spheres and passes through the common point of two of them. Since the "level difference" between the middle sphere and our first hiding sphere, measured along the ray which touches the third sphere, is $\geqslant \sqrt{3}$, the "level difference" of O and C_3 is $\sqrt{2} + \sqrt{3}$. Therefore

$$OC \geqslant \sqrt{[(\sqrt{3} + \sqrt{2})^2 + 1]} = \sqrt{(6 + 2\sqrt{6})}$$

and consequently the radius of the smallest shadow cap is

$$\leqslant \arccos\sqrt{[(3 + \sqrt{6})/6]} = r_0.$$

We remark that a direct application of Fejes Tóth's result would give only the weaker lower bound $\sqrt{2} + \sqrt{3}$ for the distance OC_3 and correspondingly the upper bound $18°31'56''$ for the radius of the smallest shadow.

6. We next prove the following lemma.

LEMMA 2. *To any cloud there belong at least* 8 *small caps, i.e. caps having a radius* $\leqslant r_0 = \arccos\sqrt{[(3 + \sqrt{6})/6]}$.

First we give a polygonal decomposition of the surface of the middle sphere in such a way that each shadow cap will contain a unique polygon.

Consider a cloud and suppose that there is no superfluous sphere in it. Then each cap has interior points not covered by any other cap. The convex polyhedron determined by the planes containing the boundary circles of the individual caps has as many faces as the number of caps and is contained in the

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sphere. Therefore the central projection of the edges of this polyhedron provides a polygonal decomposition of the sphere into convex spherical polygons each lying in the corresponding cap. Since any point of the sphere lies in a polygon having a circumcircle of radius $\leqslant 30°$, it follows that every point is within 30° of some vertex. On the other hand all the vertices of our polygons are covered by at least three caps and therefore, by Lemma 1, every vertex has to be covered by a small cap. Hence the caps of radius $30° + r_0$, concentric with the small caps, form a covering of the sphere. The estimate (1) for the case $r = 30° + r_0$ now yields that this covering consists of

$$n \geqslant \frac{2\arctan[(1/\sqrt{3})\cos(30° + r_0)]}{\arctan[(1/\sqrt{3})\cos(30° + r_0)] - \pi/6} > 7$$

caps, which implies that the number of the small caps is at least 8.

7. Lemma 2 enables us to improve the lower bound (2) by applying the simple area-estimate used in §2. The number n of the caps has to satisfy the inequality

$$2\pi[(n - 8)(1 - \cos 30°) + 8(1 - \cos r_0)] \geqslant 4\pi.$$

Since $1 - \cos 30° < 0.134$ and $1 - \cos r_0 < 0.047$, it follows that

$$N \geqslant 21.$$

8. Further improvement can be attained by taking it into consideration that the caps partly overlap. For the sake of simplicity, we first enlarge the caps in the following way: we replace each *small cap* (cap having a radius $\leqslant r_0$) by a concentric one of radius r_0 and the remaining caps by concentric ones of radius 30°. The new system of caps forms a covering such that all the vertices of the corresponding polygonal decomposition still lie in the small caps. Let us denote the number of small caps and large caps by s and l, respectively, and the number of vertices and edges of the polygonal decomposition by v and e, respectively. We may suppose that the decomposition contains only trihedral vertices; $3v = 2e$. Then we obtain from Euler's formula

$$(l + s) + v = e + 2$$

the equation

(3) $$2e = 6(l + s - 2).$$

On the other hand

(4) $$2e = lk_l + sk_s,$$

where k_l and k_s denote the average number of sides of the polygons corresponding to the large caps and small caps, respectively.

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Following the method described in §3, the concavity of the function $A(k, r)$ implies the inequality

$$z = lA(k_l, 30°) + sA(k_s, r_0) \geqslant 4\pi.$$

We shall prove that $N \geqslant 24$ by showing that for $l + s = 23$ we always have $z < 4\pi$.

In the course of the proof we distinguish two cases according as the average number k_s of sides in the *small polygons* satisfies $k_s \geqslant 6$ or $k_s < 6$. By Lemma 2, $s \geqslant 8$ in both cases.

(a) $k_s \geqslant 6$. It follows easily from (3) and (4) that in this case $k_l \leqslant 5.2$. Since $A(k, r)$ is a monotonic function of k and since each cap contains the corresponding polygon, this implies that

$$z < lA(5.2, 30°) + s2\pi(1 - \cos r_0) < 0.680l + 0.2953s$$
$$\leqslant 15 \times 0.680 + 8 \times 0.2953 = 12.5624 < 4\pi.$$

(b) $k_s < 6$. In this case $A(k_s, r_0) < A(6, r_0)$. Therefore

$$z < lA(k_l, 30°) + sA(6, r_0) = z'.$$

Thus it is enough to prove that $z' \leqslant 4\pi$. The number s of the small caps cannot be greater than 12, because for $s \geqslant 13$ even the sum of the area of the whole caps is

$$13 \times 0.2953 + 10 \times 0.842 < 12.3 < 4\pi.$$

To settle the remaining cases $s = 8, 9, 10, 11$, and 12 we need an upper bound for k_l depending on the value of s. Since all the vertices lie in the small caps, $sk_s \geqslant v$ and hence from (4)

$$k_l = (3v - sk_s)/l \leqslant 2v/l = 84/l.$$

Therefore

$$z' \leqslant lA(84/l, 30°) + sA(6, r_0) = z'', \qquad A(6, r_0) < 0.247.$$

Table I shows that z'' never exceeds 4π. This completes the proof of our assertion that $N \geqslant 24$.

TABLE I

s	l	$84/l$	$A(84/l, 30°)$	z''
8	15	5.6	$\leqslant A(5.6, 30°) < 0.7017$	$< 12.502 < 4\pi$
9	14	6.0	$\leqslant A(6.0, 30°) < 0.7195$	$< 12.296 < 4\pi$
10	13	6.5	$\leqslant A(6.5, 30°) < 0.7373$	$< 12.055 < 4\pi$
11	12	7.0	$\leqslant A(7.0, 30°) < 0.7515$	$< 11.735 < 4\pi$
12	11	7.7	$\leqslant A(7.7, 30°) < 0.7670$	$< 11.401 < 4\pi$

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