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FUNCTIONAL MEANS AND HARMONIC FUNCTIONAL MEANS

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For a continuous function $f(t)$ on $(0,\infty)$ which is strictly monotone and a probability measure μ on $[0,1]$ we introduce the functional mean $\mathfrak{M}_f(x,y;\mu)$ and the harmonic functional mean $\mathfrak{H}(x,y;\mu)$ of $x>0$ and $y>0$ with respect to μ by

$$\mathfrak{M}_f(x,y;\mu) = f^{-1}\left[\int_0^1 f(\lambda x + (1-\lambda)y)\, d\mu(\lambda)\right],$$
$$\mathfrak{H}(x,y;\mu) = \left[\mathfrak{M}_f\left(\frac{1}{x},\frac{1}{y};\mu\right)\right]^{-1},$$

which gives a unified approach to various famous means.

Moreover, functional means and harmonic means in n variables are also given and applied to get many interesting properties, such as

$$\mathfrak{H}_f(x_1,x_2,\ldots,x_n;\mu)\cdot\mathfrak{M}_f\big(x_1',x_2',\ldots,x_n';\mu\big) = \prod_{j=1}^n x_j$$

where $x_j' = \prod_{i\neq j} x_i$.

0. INTRODUCTION

The purpose of this paper is to give a unified approach to various familiar means.

Let $f(t)$ be a continuous function on $(0,\infty)$ which is strictly monotone and let μ be a probability measure on the interval on $[0,1]$. Then we define a functional mean $\mathfrak{M}_f(x,y;\mu)$ of positive numbers x and y ith respect to μ by

$$\mathfrak{M}_f(x,y;\mu) = f^{-1}\left[\int_0^1 f\big(\lambda x + (1-\lambda)y\big)d\mu(\lambda)\right].$$

Then it will be shown that various means (arithmetic mean, geometric mean, power mean, logarithmic mean, identric mean, et cetera) can be expressed as $\mathfrak{M}_f(x,y;\mu)$ for appropriate functions f.

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A harmonic functional mean $\mathfrak{H}_f(x, y, \mu)$ is introduced by

$$\mathfrak{H}_f(x,y;\mu) = \left[\mathfrak{M}_f\left(\frac{1}{x}, \frac{1}{y}; \mu\right)\right]^{-1},$$

so that

$$\mathfrak{H}_f(x,y;\mu) \cdot \mathfrak{M}_f(x,y;\mu) = \left\{\sqrt{xy}\right\}^2$$

if $f(t)$ satisfies some homogeneity condition .

The functional mean and the harmonic functional mean in n variables will be introduced and many interesting results will be derived. In particular,

$$\mathfrak{H}_f(x_1, x_2, \dots, x_n; \mu) \cdot \mathfrak{M}_f(x'_1, x'_2, \dots, x'_n; \mu) = \prod_{j=1}^{n} x_j$$

where $x'_j = \prod_{i \neq j} x_i$.

1. Functional Means

Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function which is strictly monotone. By the mean value theorem for each $x > 0$ and $y > 0$ we can find a unique z between x and y such that

$$\int_x^y f(t)dt = f(z)(x-y)$$

Here, $f(z)$ can be understood as an average value of $f(t)$ when t varies between x and y, so that z gives in a certain sense, a mean value of x and y which is expected to be related strongly to $f(t)$. Thus we define a functional mean as follows:

Defintion: Let $f(t)$ be a continuous function on $(0, \infty)$ which is strictly monotonic, and let μ be a probability measure supported by the interval $[0,1]$. For $x > 0$ and $y > 0$ we define a functional mean $\mathfrak{M}_f(x,y;\mu)$ with respect to the probability measure μ by

$$\mathfrak{M}_f(x,y;\mu) = f^{-1}\left[\int_0^1 f\big(\lambda x + (1-\lambda)y\big)d\mu(\lambda)\right].$$

By the mean value theorem it can be easily seen that the mean value $\mathfrak{M}_f(x,y;\mu)$ is uniquely determined. It is true that $\mathfrak{M}_f(x,x;\mu) = x$ for every $x > 0$ and $\mathfrak{M}_f(x,y;\mu)$ lies between x and y when $x \neq y$. On the other hand, $\mathfrak{M}_f(x,y;\mu)$ is usually symmetric in the sense that

$$\mathfrak{M}_f(x,y;\mu) \neq \mathfrak{M}_f(y,x;\mu)$$

unless μ is equally distributed on $[0,1]$.

When μ is the Lebesgue measure we simply write $\mathfrak{M}_f(x,y)$ instead of $\mathfrak{M}_f(x,y;\mu)$. In what follows, when we refer to $\mathfrak{M}_f(x,y;\mu)$ we always understand that f is a continuous function on $(0,\infty)$ which is strictly monotone, μ is a probability measure supported by $[0,1]$, and $x,y>0$.

EXAMPLE. (i) $\mathfrak{M}_t(x,y)=(x+y)/2$ is the arithmetic mean $A(x,y)$.

(ii) $\mathfrak{M}_{1/t}(x,y)=(x-y)/(\log x-\log y)$ is the logarithmic mean $L(x,y)$.

(iii) $\mathfrak{M}_{1/t^2}(x,y)=\sqrt{xy}$ is the geometric mean $G(x,y)$.

(iv) $\mathfrak{M}_{\log t}(x,y)=(1/e)\,(x^x/y^y)^{1/(x-y)}$ is the identric mean $I(x,y)$.

(v) $\mathfrak{M}_{1/\sqrt{t}}(x,y)=\left(\left(\sqrt{x}+\sqrt{y}\right)/2\right)^2$.

(vi) $\mathfrak{M}_{1/t^3}(x,y)=\sqrt[3]{2x^2y^2/(x+y)}$.

(vii) $\mathfrak{M}_{e^t}(x,y)=\log\left((e^x-e^y)/(x-y)\right)$.

(viii) Let μ be the measure concentrated on $\{0,1\}$ defined by

$$\mu(\{\lambda\})=\begin{cases}\dfrac{1}{p}, & \lambda=0\\[2mm] \dfrac{1}{q}, & \lambda=1\end{cases}$$

for $1/p+1/q=1$, $p>0$, $q>0$. Then for any f

$$\mathfrak{M}_f(x,y;\mu)=f^{-1}\left[\frac{f(x)}{p}+\frac{f(y)}{q}\right].$$

In particular, if $f(t)=t^r$ $(r\neq 0)$ then

$$\mathfrak{M}_{t^r}(x,y;\mu)=\left(\frac{x^r}{p}+\frac{y^r}{q}\right)^{1/r}$$

is the weighted r-th power mean.

The next few theorems parallel classical results in [4, Chapter 3]. The first theorem characterises functions which produce a common functional mean:

THEOREM 1.1. *In order that*

$$\mathfrak{M}_f(x,y;\mu)=\mathfrak{M}_g(x,y;\mu)$$

for all $x,y>0$ and all probability measures μ on $[0,1]$ it is necessary and sufficient that

$$f(x)=\alpha g(x)+\beta,\quad x\in(0,\infty)$$

for some constants α $(\alpha \neq 0)$ *and* β.

PROOF: The sufficiency is easy. We prove the necessity. By the assumption we may put

$$z = f^{-1}\left[\int_0^1 f\big(\lambda x + (1-\lambda)y\big)d\mu(\lambda)\right] = g^{-1}\left[\int_0^1 g\big(\lambda x + (1-\lambda)y\big)d\mu(\lambda)\right]$$

for all x, y and any probability measure μ. Take $x = a$ and $y = b$ $(a < b)$ arbitrarily on $(0, \infty)$ and a probability measure μ_t concentrated on $\{0, 1\}$ with

$$\mu_t(\{\lambda\}) = \begin{cases} \dfrac{t-a}{b-a}, & \lambda = 0, \\ \dfrac{b-t}{b-a}, & \lambda = 1. \end{cases}$$

for each parameter t with $a < t < b$. Then it follows that

$$f(z) = \frac{b-t}{b-a}f(a) + \frac{t-a}{b-a}f(b) \tag{1.1}$$

and

$$g(z) = \frac{b-t}{b-a}g(a) + \frac{t-a}{b-a}g(b) \tag{1.2}$$

for $a < t < b$. Of course, this is still true for $t = a$ and $t = b$ and as t varies from a to b, z assumes all values in $[a, b]$. From (1.2) we have

$$t = \frac{(b-a)g(z) + ag(b) - bg(a)}{g(b) - g(a)}.$$

If we substitute this for t in (1.1) we obtain

$$\begin{aligned} f(z) &= \frac{g(b) - g(z)}{g(b) - g(a)}f(a) + \frac{g(z) - g(a)}{g(b) - g(a)}f(b) \\ &= \frac{f(b) - f(a)}{g(b) - g(a)}g(z) + \frac{g(b)f(a) - g(a)f(b)}{g(b) - g(a)}, \end{aligned}$$

which implies that

$$f(z) = \alpha g(z) + \beta \text{ on } [a, b]$$

where α and β are constants, possibly depending on the choice of a and b. But in fact these constants do not depend on the choice of a and b. To see this, let a_1 and b_1 be such that $a < a_1 < b < b_1$. Then as in the argument above there are constants α_1 and β_1 such that

$$f(z) = \alpha_1 g(z) + \beta_1 \text{ on } [a_1, b_1].$$

But they must coincide on the interval $[a_1, b]$, say,

$$(\alpha - \alpha_1)g(z) = \beta_1 - \beta \text{ on } [a_1, b],$$

This implies that $g(z)$ must be constant on $[a_1, b]$, which is impossible, since g is strictly monotone. This completes the proof. □

Write $f \sim g$ if the functions, f and g produce the same functional mean. In view of the above theorem

$$f \sim g \text{ if and only if } f(x) = \alpha g(x) + \beta, \quad x \in (0, \infty)$$

for some $\alpha \neq 0$ and β. Moreover, we may assume that the function f which is concerned with $\mathfrak{M}_f(\cdot, \cdot; \mu)$, is always strictly increasing.

Most of the standard examples of means have a property of homogeneity

$$\mathfrak{M}_f(kx, ky; \mu) = k\mathfrak{M}_f(x, y; \mu), \quad k > 0$$

for all x, y and μ. So it is quite natural to ask what kind of functions give a homogeneous functional mean.

THEOREM 1.2. *In order that*

$$\mathfrak{M}_f(kx, ky; \mu) = k\mathfrak{M}_f(x, y; \mu)$$

for every $x, y, k > 0$ and every probability measure μ on $[0, 1]$, it is necessary and sufficient that either $f(t) \sim t^r$ for some $r \neq 0$ or $f(t) \sim \log t$.

PROOF: We prove only the necessity here. By Theorem 1.1 we may assume that $f(1) = 0$. If we put $g(x) = f(kx)$ then the relation

$$\mathfrak{M}_f(kx, ky; \mu) = k\mathfrak{M}_f(x, y; \mu)$$

implies that

$$\begin{aligned}\mathfrak{M}_f(x, y; \mu) &= k^{-1} f^{-1}\left[\int_0^1 f\big[\lambda kx + (1-\lambda)ky\big]\, d\mu(\lambda)\right] \\ &= g^{-1}\left[\int_0^1 g\big(\lambda x + (1-\lambda)y\big)\, d\mu(\lambda)\right] \\ &= \mathfrak{M}_g(x, y; \mu).\end{aligned}$$

Thus in view of Theorem 1.1 we may write

$$g(x) = f(kx) = \alpha(k) f(x) + \beta(k)$$

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for some $\alpha(k) \neq 0$ and $\beta(k)$. We obtain from this that

$$g(1) = f(k) = \beta(k).$$

Substituting y for k we find that for all $x, y > 0$

$$f(xy) = \alpha(y)f(x) + f(y)$$

or, equivalently,

$$f(xy) = \alpha(x)f(y) + f(x).$$

These give

$$\frac{\alpha(x)-1}{f(x)} = \frac{\alpha(y)-1}{f(y)}$$

when $f(x) \neq 0$ and $f(y) \neq 0$. But since f is strictly monotone and continuous the final conclusion in the last part of this proof must be true on $(0,\infty)$. Each of these functions must reduce to a constant K, so that $\alpha(y) = 1 + Kf(y)$. Then we obtain

$$f(xy) = Kf(x)f(y) + f(x) + f(y).$$

Here if $K = 0$ then this functional equation reduces to the famous equation

$$f(xy) = f(x) + f(y).$$

It is well known that the only continuous solution of this functional equation for $x > 0$ is $f(x) = C\log x$ where C is an arbitrary constant.

Secondly, if $K \neq 0$ we put $Kf(x) + 1 = F(x)$. Then the equation becomes

$$F(xy) = F(x)F(y)$$

whose general solution is $F(x) = x^r$, where r is a constant. In both cases the constants C and r must be nonzero in order that f should be strictly monotonic. This completes the proof. ∎

We shall now discuss the comparability of two functional means with respect to the same probability measure. Many results about comparability have been developed (see [**1, 2, 4, 6, 7, 8, 9, 10**]). Many of those can be restated by the following theorem:

Theorem 1.3. *Let f and g be continuous and strictly increasing on $(0,\infty)$. Then a necessary and sufficient condition in order that*

$$\mathfrak{M}_f(x,y;\mu) \leqslant \mathfrak{M}_g(x,y;\mu)$$

for all x, y and μ, is that $g \circ f^{-1}$ is convex.

PROOF: In view of Jensen's inequality it follows that

$$(g \circ f^{-1}) \left[\int_0^1 f(\lambda x + (1-\lambda)y) d\mu(\lambda) \right] \leqslant \int_0^1 g(\lambda x + (1-\lambda)y) d\mu(\lambda).$$

Since g^{-1} is also increasing we obtain

$$f^{-1} \left[\int_0^1 f(\lambda x + (1-\lambda)y) d\mu(\lambda) \right] \leqslant g^{-1} \left[\int_0^1 g(\lambda x(1-\lambda)y) d\mu(\lambda) \right],$$

which is the required result.

Now to prove the converse we assume that $\mathfrak{M}_f(x, y; \mu) \leqslant \mathfrak{M}_g(x, y; \mu)$ holds for all x, y and μ. For $0 < t < 1$, let μ_t be the probability measure concentrated on $\{0, 1\}$ given by

$$\mu_t\{\lambda\} = \begin{cases} t, & \lambda = 0, \\ 1-t, & \lambda = 1. \end{cases}$$

If z_1 and z_2 belong to the range of f such that $f(x_1) = z_1$ and $f(x_2) = z_2$ where $x_1, x_2 > 0$ then the hypothesis gives that

$$f^{-1}[tf(x_1) + (1-t)f(x_2)] \leqslant g^{-1}[tg(x_1) + (1-t)g(x_2)].$$

Then it follows that for all t in $(0, 1)$

$$(g \circ f^{-1})[tz_1 + (1-t)z_2] \leqslant t(g \circ f^{-1})(z_1) + (1-t)(g \circ f^{-1})z_2,$$

which implies the convexity of $g \circ f^{-1}$. $\square$

EXAMPLE. In view of the above theorem we can easily obtain the well known inequality

$$G(x, y) \leqslant L(x, y) \leqslant I(x, y) \leqslant A(x, y)$$

by expressing these respectively as functional means.

We now prove the monotonicity and continuity of the functional mean.

THEOREM 1.4. *For any function f on $(0, \infty)$ which is continuous and strictly monotone the functional mean $\mathfrak{M}_f(x, y; \mu)$ is continuous on $(0, \infty) \times (0, \infty)$ and increasing in the sense that*

$$\text{if } x_1 \leqslant x_2 \text{ and } y_1 \leqslant y_2 \text{ then } \mathfrak{M}_f(x_1, y_1; \mu) \leqslant \mathfrak{M}_f(x_2, y_2; \mu)$$

for any probability measure μ *on* $[0,1]$.

PROOF: In view of Theorem 1.1 we may assume that f is strictly increasing, by replacing f by $-f$ if necessary. Let $(x_0, y_0) \in (0,\infty)\times(0,\infty)$ and let $\{(x_n, y_n)\}$ be a sequence in $(0,\infty)\times(0,\infty)$ converging to (x_0, y_0). Then since both f and $-f$ are continuous the convergence theorem for the integral implies

$$\begin{aligned}\lim_{n\to\infty} \mathfrak{M}_f(x_n, y_n;\mu) &= \lim_{n\to\infty} f^{-1}\left[\int_0^1 f\big(\lambda x_n + (1-\lambda)y_n\big)d\mu(\lambda)\right] \\ &= f^{-1}\left[\lim_{n\to\infty}\int_0^1 f\big(\lambda x_n + (1-\lambda)y_n\big)d\mu(\lambda)\right] \\ &= f^{-1}\left[\int_0^1 f\big(\lambda x_0 + (1-\lambda)y_0\big)d\mu(\lambda)\right],\end{aligned}$$

which gives the continuity of $\mathfrak{M}_f(x,y;\mu)$.

Now let $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$ on $(0,\infty)$. Then we have

$$\lambda x_1 + (1-\lambda)y_1 \leqslant \lambda x_2 + (1-\lambda)y_2, \quad 0\leqslant\lambda\leqslant 1.$$

Since f and f^{-1} are both increasing it follows easily that

$$\mathfrak{M}_f(x_1, y_1;\mu) \leqslant \mathfrak{M}_f(x_2, y_2;\mu).$$

∎

2. FUNCTIONAL HARMONIC MEAN

The harmonic mean $H(x,y)$ of two positive numbers x and y is given by

$$H(x,y) = \frac{2xy}{x+y} = \left[A\left(\frac{1}{x},\frac{1}{y}\right)\right]^{-1}.$$

It is of interest to introduce the functional harmonic mean with respect to a probability measure.

Let f be a continuous function on $(0,\infty)$ which is strictly monotonic and let μ be a probability measure supported by $[0,1]$, as before.

For positive numbers x and y we define the functional harmonic mean $\mathfrak{H}_f(x,y;\mu)$ by

$$\mathfrak{H}_f(x,y;\mu) = \left[\mathfrak{M}_f\left(\frac{1}{x},\frac{1}{y};\mu\right)\right]^{-1}.$$

In particular, if μ is Lebesgue measure we write simply $\mathfrak{H}_f(x,y)$ instead of $\mathfrak{H}_f(x,y;\mu)$.

We consider some examples here.

Example.

(i) $\mathfrak{H}_t(x,y) = \left[A\left(1/x, 1/y\right)\right]^{-1} = 2xy/(x+y) = H(x,y)$. If μ is the probability measure concentrated on $\{0,1\}$ with

$$\mu(\{\lambda\}) = \begin{cases} \frac{1}{3}, & \lambda = 0 \\ \frac{2}{3}, & \lambda = 1 \end{cases},$$

then $\mathfrak{M}_f(x,y;\mu) = \int_0^1 [\lambda x + (1-\lambda)y]\,d\mu(\lambda) = (2x+y)/3$, so that

$$\mathfrak{H}_t(x,y;\mu) = \frac{3xy}{x+2y}$$

(ii) Since $\mathfrak{M}_{1/t^2}(x,y) = \sqrt{xy} = G(x,y)$ it follows that

$$\mathfrak{H}_{1/t^2} = \left[\mathfrak{M}_{1/t^2}\left(\frac{1}{x}, \frac{1}{y}\right)\right]^{-1} = \left(\sqrt{\frac{1}{xy}}\right)^{-1} = G(x,y).$$

Thus we obtain the interesting conclusion

$$\mathfrak{M}_{1/t^2}(x,y) = \mathfrak{H}_{1/t^2}(x,y) = G(x,y)$$

for all $x, y > 0$. Moreover, it is true that

$$\mathfrak{H}_{1/t^2}(x,y;\mu) \cdot \mathfrak{M}_{1/t^2}(x,y;\mu) = xy$$

for every probability measure μ (seen later in Theorem 2.1).

(iii) Since $\mathfrak{M}_{1/t}(x,y) = (x-y)/(\log x - \log y)$ $(= L(x,y))$

$$\begin{aligned}\mathfrak{H}_{1/t}(x,y) &= \left[L\left(\frac{1}{x}, \frac{1}{y}\right)\right]^{-1} = \frac{\log x - \log y}{x-y} \cdot xy \\ &= \left[L(x,y)\right]^{-1} xy.\end{aligned}$$

Hence, we obtain also

$$\mathfrak{H}_{1/t}(x,y) \cdot \mathfrak{M}_{1/t}(x,y) = xy = \left[G(x,y)\right]^2.$$

We state a general result concerning the above arguments.

THEOREM 2.1. *If $f(t)$ is a continuous function on $(0,\infty)$ which is strictly monotone and is equivalent to a homogeneous function in the sense that*

$$f(kt) = \alpha(k)f(t) + \beta(k), \quad t > 0,\ k > 0 \tag{2.1}$$

for some real functions $\alpha(k) \neq 0$ and $\beta(k)$, then

$$\mathfrak{H}_f(x,y;\mu)\cdot\mathfrak{M}_f(x,y;\mu) = [G(x,y)]^2 \tag{2.2}$$

for all $x, y > 0$ and for every probability measure μ.

PROOF: The functional relation (2.1) reduces to either $f(t) \sim t^r$ $(r \neq 0)$ or $f(t) \sim \log t$. (In fact, this can be seen by the same method as in the proof of Theorem 1.1.) In view of the equivalence we may assume that either $f(t) = t^r$ or $f(t) = \log t$.

We first assume $f(t) = t^r$ $(r \neq 0)$. Then

$$\begin{aligned}\mathfrak{M}_f\left(\frac{1}{x},\frac{1}{y};\mu\right) &= \left[\int_0^1 \left(\frac{\lambda}{x} + \frac{1-\lambda}{y}\right)^r d\mu(\lambda)\right]^{1/r} \\ &= \left[\int_0^1 \frac{[\lambda y + (1-\lambda)x]^r}{(xy)^r} d\mu(\lambda)\right]^{1/r} \\ &= \frac{1}{xy}\mathfrak{M}_f(x,y;\mu)\end{aligned}$$

which implies

$$\mathfrak{H}_f(x,y;\mu)\cdot\mathfrak{M}_f(x,y;\mu) = xy = [G(x,y)]^2.$$

On the other hand if $f(t) = \log t$ then

$$\begin{aligned}\mathfrak{M}_f\left(\frac{1}{x},\frac{1}{y};\mu\right) &= \exp\left[\int_0^1 \log\left[\frac{\lambda}{x} + \frac{1-\lambda}{y}\right] d\mu(\lambda)\right] \\ &= \exp\left[\int_0^1 \log[\lambda y + (1-\lambda)x]\, d\mu(\lambda) - \log xy\right] \\ &= \mathfrak{M}_f(x,y;\mu)/xy.\end{aligned}$$

This completes the proof. ∎

3. FUNCTIONAL MEAN IN n VARIABLES

We have discussed so far the functional mean only in two variables. Now we establish here the functional mean in several variables and derive its basic properties.

A motivation comes from [**3, 5**] as follows: the logarithmic mean $L(x,y)$ of x and y is given by

$$L(x,y) = \left[\int_0^1 \frac{d\lambda}{\lambda y + (1-\lambda)x}\right]^{-1}$$

and the logarithmic mean of $x_1, x_2, \dots, x_n$ is given by

$$L(x_1, x_2, \dots, x_n) = \int_{A_{n-1}} (x \cdot \nu)^{-1}(n-1)!\, d\nu$$

where $d\nu$ denotes the differential of volume in A_{n-1}, where A_{n-1} is the simplex

$$A_{n-1} = \left\{(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) \mid 0 \leqslant \lambda_j \leqslant 1,\ j = 1, 2, \dots, n,\ \sum_{j=1}^{n-1} \lambda_j \leqslant 1\right\},$$

$x \cdot \nu = \sum_{j=1}^{n} x_j \lambda_j$ and $\lambda_n = 1 - \lambda_1 - \cdots - \lambda_{n-1}$. Since we have already shown that $\mathfrak{M}_{1/t}(x,y) = L(x,y)$ for $x, y > 0$ it is quite natural to define a functional mean as follows:

Definition: Let f be a continuous function on $(0,\infty)$ which is strictly monotone and let μ be a probability measure supported by A_{n-1}. Then the functional mean $\mathfrak{M}_f(x;\mu)$ with respect to the probability measure μ is defined for $x = (x_1, x_2, \dots, x_n)$, $x_j > 0$, $j = 1, 2, \dots, n$ by

$$\mathfrak{M}_f(x;\mu) = f^{-1}\left[\int_{A_{n-1}} f(x \cdot \nu)\, d\mu(\nu)\right].$$

Of course, the mean value theorem guarantees the unique existence of the value $\mathfrak{M}_f(x;\mu)$. When μ is Lebesgue measure it can be written using the iterated integral as

$$\mathfrak{M}_f(x;\mu) = f^{-1}\left[\int_0^1 \int_0^{1-\lambda_1} \cdots \int_0^{1-\lambda_1-\cdots-\lambda_{n-2}} \Big[f\big(x_1\lambda_1 + \cdots + x_{n-1}\lambda_{n-1} + (1-\lambda_1-\cdots-\lambda_{n-1})x_n\big)\Big](n-1)!\, d\lambda_{n-1}\cdots d\lambda_1\right].$$

As we have done before, when μ is Lebesgue measure we write $\mathfrak{M}_f(x)$ instead of $\mathfrak{M}_f(x;\mu)$.

Now we consider some examples.

EXAMPLE. (i) $\mathfrak{M}_t(x) = \left(\sum_{j=1}^{n} x_j\right)\Big/n$ is the arithmetic mean.

(ii) $\mathfrak{M}_{1/t^n}(x) = \sqrt[n]{\prod_{j=1}^{n} x_j}$ is the geometric mean (see [**5**]).

(iii) Let μ be the measure concentrated on the vertices $\nu_1, \nu_2, \dots, \nu_n$ of the simplex A_{n-1}, defined by

$$\mu(\{\nu_j\}) = \frac{1}{p_j} > 0, \quad j = 1, 2, \dots, n$$

with $\sum_{j=1}^{n}(1/p_j) = 1$. Then for any f and $x = (x_1, x_2, \dots, x_n)$

$$\begin{aligned}\mathfrak{M}_f(x;\mu) &= \mathfrak{M}_f(x_1, x_2, \dots, x_n;\mu) \\ &= f^{-1}\left[\sum_{j=1}^{n} \frac{f(x_j)}{p_j}\right].\end{aligned}$$

This is shown for example in [**4**]. For instance if $f(t) = t^r$ $(r \neq 0)$ then

$$\mathfrak{M}_{t^r}(x;\mu) = \left(\sum_{j=1}^{n} \frac{x_j^r}{p_j}\right)^{1/r}.$$

The functional harmonic mean $\mathfrak{H}(x;\mu)$ in n variables is defined by

$$\mathfrak{H}(x;\mu) = \left[\mathfrak{M}_f\left(\frac{1}{x}, \mu\right)\right]^{-1}$$

where $1/x$ denotes $(1/x_1, 1/x_2, \dots, 1/x_n)$ for $x = (x_1, x_2, \dots, x_n)$, $x_j > 0$ for $j = 1, 2, \dots, n$.

Then we can restate all the theorems which hold for two variables. We mention them without proofs. We denote by $\mathbb{R}_+^n$ the set $\{(x_1, x_2, \dots, x_n) \mid x_j > 0,\ j = 1, 2, \dots, n\}$.

THEOREM 1.1′. *In order that*

$$\mathfrak{M}_f(x;\mu) = \mathfrak{M}_g(x;\mu)$$

for all $x, y \in \mathbb{R}_+^n$ and all probability measures μ on A_{n-1} it is necessary and sufficient that

$$f(x) = \alpha g(x) + \beta, \quad x \in \mathbb{R}_+^n$$

for some constants $\alpha \neq 0$ *and* β.

THEOREM 1.2′. *In order that*

$$\mathfrak{M}_f(kx;\mu) = k\mathfrak{M}_f(x;\mu), \quad k > 0$$

for all $x \in \mathbb{R}^n_+$ *and all* μ *it is necessary and sufficient that*

$$\text{either } f(t) \sim t^r \text{ for some } r \neq 0 \text{ or } f(t) \sim \log t.$$

THEOREM 1.3′. *Let* f *and* g *be strictly increasing continuous functions on* $(0,\infty)$. *Then a necessary and sufficient condition that*

$$\mathfrak{M}_f(x;\mu) \leqslant \mathfrak{M}_g(x;\mu)$$

for all $x, y \in \mathbb{R}^n$ *with* $x_j \leqslant y_j$, $j = 1, 2, \ldots, n$ *and all* μ, *is that* $g \circ f^{-1}$ *is convex.*

For any $x, y \in \mathbb{R}^n_+$ we now write $x \prec y$ if

$$x_j \leqslant y_j \text{ for } j = 1, 2, \ldots, n.$$

THEOREM 1.4′. *The functional mean* $\mathfrak{M}_f(x;\mu)$ *is continuous on* $\mathbb{R}^n_+$ *and is increasing in the sense that*

$$x \prec y \text{ implies } \mathfrak{M}_f(x;\mu) \leqslant \mathfrak{M}_f(y;\mu)$$

for all μ.

THEOREM 2.1′. *If* $f(t)$ *is equivalent to a homogeneous function in the sense that*

$$f(kt) = \alpha(k)f(t) + \beta(k), \quad t > 0,\ k > 0$$

for some $\alpha(k) \neq 0$ *and* $\beta(k)$ *then*

$$\mathfrak{H}_f(x;\mu) \cdot \mathfrak{M}_f(x';\mu) = \prod_{j=1}^{n} x_j$$

for all μ, *where* $x' = (x'_1, x'_2, \ldots, x'_n)$ *with* $x'_j = \prod_{i \neq j} x_i$, $j = 1, 2, \ldots, n$.

This result is a very interesting one.

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References

[1] J.M. Borwein and P.B. Borwein, 'Inequalities for compound mean iterations with logarithmic asymptotes', *J. Math. Anal. Appl.* **177** (1993), 572–582.

[2] F. Burk, 'Geometric, logarithmic, and arithmetic mean inequalities', *Amer. Math. Monthly* **94** (1987), 523–528.

[3] B. C. Carlson, 'The logarithmic mean', *Amer. Math. Monthly* **79** (1972), 615–618.

[4] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities* (Cambridge Univ. Press, Cambridge, 1934).

[5] A. O. Pittenger, 'The logarithmic mean in n variables', *Amer. Math. Monthly* **92** (1985), 99–104.

[6] J. Sandor, 'On the identric and logarithmic means', *Aequationes Math.* **40** (1990), 261–270.

[7] J. Sandor, 'A note on some inequalities for means', *Arch. Math.* **56** (1991), 471–473.

[8] J. Sandor, 'On certain inequalities for means', *J. Math. Anal. Appl.* **189** (1995), 602–606.

[9] J. Sandor, 'On certain inequalities for means II', *J. Math. Anal. Appl.* **189** (1996), 629–635.

[10] M. K. Vamanamurthy and M. Vuorinen, 'Inequalities for means', *J. Math. Anal. Appl.* **183** (1994), 155–166.

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