Sets of Semi-Commutative Matrices

By J. WILLIAMSON, Johns Hopkins University.

(*Received 7th July,* 1932. *Received in revised form 8th November,* 1932. *Read 4th November,* 1932.)

Introduction. In a paper entitled "Sets of anticommuting matrices" Eddington[1] proved that *if* $E_1, E_2, \ldots, E_q$ *form a set of* q *four-rowed square matrices satisfying the relations,*

$$(1) \qquad E_i E_j = -E_j E_i, \quad E_i^2 = -E, \quad i, j = 1, 2, \ldots, q, \quad i \neq j,$$

where E *is the unit matrix, then the maximum value of* q *is five.* Later Newman[2] showed that this result is a particular case of the general theorem that *if* $E_1, E_2, \ldots, E_q$ *form a set of* q *t-rowed square matrices satisfying* (1), *where* $t = 2^p \tau$ *and* τ *is odd, then the maximum value of* q *is* $2p + 1$.

In this paper we consider a generalization of Newman's theorem and prove the following result.

THEOREM I. *If* ω *is a primitive nth root of unity, and if* $E_1, E_2, \ldots, E_q$ *form a set of* q *t-rowed square matrices satisfying the relations*

$$(2) \qquad E_i E_j = \omega E_j E_i, \quad E_i^n = E, \quad i, j = 1, 2, \ldots, q, \quad i < j,$$

where E *is the unit matrix and* $t = n^p \tau$, $\tau \neq 0$ *mod* n, *then the maximum value of* q *is* $2p + 1$. *Moreover, for every value of* t, *sets of* $2p + 1$ *matrices satisfying* (2) *exist.*

We shall call a set of q matrices satisfying (2) an E-set; or in the case where q is maximal, a *maximal* E-set. While Eddington and Newman proved a theorem on the number of *real* matrices in a maximal E-set for the case $n = 2$, we shall see that no such theorem is true in the general case. However, if n is *even*, there does exist a general theorem on the number of matrices of a special type in a maximal E-set.

[As a consequence of this it may be shown that, when $t = n^p$, every matrix of order t can be expressed as a polynomial, with complex number coefficients, in the matrices of any maximal E-set. It is

---

[1] *Journal London Math. Soc.*, 7 (1932), 58-68.

[2] *Ibid*, 7 (1932), 94-99.

also shown that any two maximal E-sets are similar, provided that $p \neq 0$. That is, if $F_1, F_2, \ldots, F_q$ form a set of matrices which satisfy the equivalent of (2), then there exists a non-singular matrix A such that $AE_i A^{-1} = F_i$, $i = 1, 2, \ldots, q$. It is also possible to relate the E-sets of matrices of order t with periodic collineations in space of $t - 1$ dimensions, and thereby to obtain the different types of such maximal groups.]

§ 1. For the proof of Theorem I we require two lemmas.

LEMMA 1. *If $t = \tau \neq 0$ mod n, then the maximal number of matrices in an E-set is one.*

For, if the set contains at least two members E_1 and E_2 such that $E_1 E_2 = \omega E_2 E_1$, by taking the determinants of both sides of this last matrix equation we obtain

$$|E_1|\,|E_2| = \omega^t\,|E_2|\,|E_1|\,;$$

and, since by (2) both E_1 and E_2 are non-singular, ω^t must be equal to unity. This result contradicts the fact that t is not a multiple of n; and accordingly the lemma is proved. It is worth noticing that, since two matrices, which are both nth roots of the unit matrix, are not necessarily similar, so in the present case, if $t = \tau \neq 0$ mod n, two maximal E-sets, since each consists of a single member, are not necessarily similar.

LEMMA 2. *If E_1 is a member of an E-set, where $q > 1$ and $t = kn$, then there exists a non-singular matrix A such that*

$$(3) \qquad A^{-1} E_1 A = F_1, \quad \text{where } F_1 = \begin{bmatrix} e & 0 & 0 & . & . & 0 \\ 0 & \omega e & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & \omega^{n-1} e \end{bmatrix}.$$

and $e, \omega e, \ldots, \omega^{n-1} e$ are all scalar matrices of order k, e being the unit matrix of order k.

For, since E_1 satisfies the characteristic equation $E_1^n - E = 0$, the latent roots of E_1 are all powers of ω, and there must also exist a non-singular matrix B such that

$$(4) \qquad B E_1 B^{-1} = G_1,$$

where G_1 is a diagonal matrix having these powers of ω in the

https://doi.org/10.1017/S0013091500007914 Published online by Cambridge University Press

diagonal. Let the latent root ω^j, $j = 0, 1, \ldots, n-1$, appear exactly t_j times in G_1 so that we have the equality

$$kn = \sum_{j=0}^{n-1} t_j. \tag{5}$$

Now, if E_2 is a second non-singular matrix such that

$$E_1 E_2 = \omega E_2 E_1,$$

then we have the result

$$E_2^{-1} E_1 E_2 = \omega E_1.$$

Accordingly the latent roots of E_1 are the same as the latent roots of ωE_1; and, as the latent roots of ωE_1 are ω times the latent roots of E_1, multiplication by ω merely permutes the latent roots of E_1 amongst themselves. Now, if ω^s is the latent root of E_1, for which

$$t_s = t \geqq t_j, \quad j = 0, 1, 2, \ldots, n-1, \tag{6}$$

then ω^{s+1} appears at least t times amongst the latent roots of ωE_1, and therefore at least t times amongst the latent roots of E_1. Hence $t_{s+1} \geqq t$; and so, by (6), $t_{s+1} = t$. Similarly we can show that

$$t_s = t_{s+1} = \ldots = t_{s+n-1} = t,$$

where the subscripts must be reduced modulo n. From (5) it follows that $k = t$ and hence that (3) is true. An alternative statement of this lemma is as follows. *The latent roots of any matrix of an E-set, which consists of more than one member, are the roots of unity* $1, \omega, \omega^2, \ldots, \omega^{n-1}$, *each repeated the same number of times.*

It will now be shown by actual examples that matrices of the type postulated in Theorem I exist. If $t = n$, it may be verified without difficulty that the following three matrices satisfy (2):

$$\Omega_1 = \begin{bmatrix} 1 & 0 & . & . & 0 \\ 0 & \omega & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & \omega^{n-1} \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} 0 & 0 & 0 & . & . & 1 \\ 1 & 0 & 0 & . & . & 0 \\ . & 1 & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 0 \\ 0 & 0 & 0 & . & 1 & 0 \end{bmatrix}, \quad \Omega_3 = \lambda\, \Omega_1^{-1} \Omega_2, \tag{7}$$

where $\lambda = 1$, if n is odd, and $\lambda = \sqrt{\omega}$, if n is even. Further, if $E_1, E_2, \ldots, E_f$ form a set of f matrices of order m satisfying (2), then the matrices

$$E_1 \cdot \Omega, \quad E_2 \cdot \Omega, \quad \ldots, \quad E_{f-1} \cdot \Omega, \quad E_f \cdot \Omega_1, \quad E_f \cdot \Omega_2, \quad E_f \cdot \Omega_3, \tag{8}$$

where $\cdot$ denotes direct product[1] and Ω is the unit matrix of order n,

---

[1] See L. E. Dickson, *Algebras and their Arithmetics*, p. 72.

form a set of $f+2$ matrices of order mn satisfying (2). For

$$(E_i\,\Omega)^n = E_i^n \cdot \Omega^n = E, \qquad i = 1, 2, \ldots, f-1;$$

$$(E_f \cdot \Omega_j)^n = E_f^n \cdot \Omega_j^n = E, \qquad j = 1, 2, 3;$$

$$(E_i \cdot \Omega)(E_j \cdot \Omega) = E_i E_j \cdot \Omega^2 = \omega E_j E_i \cdot \Omega^2 = \omega (E_j \cdot \Omega)(E_i \cdot \Omega), \quad i<j;\ i, j = 1, 2, \ldots, f-1;$$

$$(E_i \cdot \Omega)(E_f \cdot \Omega_j) = E_i E_f \cdot \Omega\,\Omega_j = \omega E_f E_i \cdot \Omega_j\,\Omega, = \omega (E_f \cdot \Omega_j)(E_i \cdot \Omega), \qquad i = 1, 2, \ldots, f-1;\ j = 1, 2, 3;$$

$$(E_f \cdot \Omega_i)(E_f \cdot \Omega_j) = E_f^2 \cdot \Omega_i\,\Omega_j = \omega E_f^2 \cdot \Omega_j\,\Omega_i = \omega (E_f \cdot \Omega_j)(E_f \cdot \Omega_i), \qquad i<j;\ i, j = 1, 2, 3.$$

Thus, if there exists an E-set of matrices of order m containing f members, there exists an E-set of matrices of order mn containing $f+2$ members. But, by Lemma I, there exists an E-set of matrices of order $r \not\equiv 0 \bmod n$ containing one member; therefore, by induction there exists an E-set of matrices of order $t = n^p r$ containing $2p+1$ matrices. This proves the last part of Theorem I.

If $m = kn$, by Lemma 2 there exists a non-singular matrix A such that $A^{-1} E_i A = F_i$, where F_1 is given by (3). The matrices F_i so defined also form an E-set; and, if we write $F_s = (f_{ij})$, $i, j = 1, 2, \ldots, n$, where each f_{ij} is a matrix of order k, then since $F_1 F_s = \omega F_s F_1$, we have $f_{ij}(\omega^{i-1} - \omega^j) = 0$, or $f_{ij} = 0$, if $i \not\equiv j+1 \bmod n$. Accordingly F_s has the form

$$(9) \qquad \begin{bmatrix} 0 & 0 & . & . & . & 0 & F_{1s} \\ F_{2s} & 0 & . & . & . & 0 & 0 \\ 0 & F_{3s} & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & F_{ns} & 0 \end{bmatrix},$$

where each F_{is} is a matrix of order k, and 0 denotes the zero matrix of order k. Since $F_s^n = E$, it follows that

$$(10) \quad F_{ns} F_{n-1,s} \ldots F_{2s} F_{1s} = F_{1s} F_{ns} \ldots F_{2s} = \ldots = F_{n-1,s} \ldots F_{1s} F_{ns} = e,$$

and, if $s < u$, since $F_s F_u = \omega F_u F_s$, that

$$(11) \qquad F_{is} F_{i-1,u} = \omega F_{iu} F_{i-1,s},$$

where $F_{0u} = F_{nu}$. But by (10) F_{12} is non-singular and so the matrices

$$(12) \qquad G_s = \lambda F_{12}^{-1} F_{1s},$$

https://doi.org/10.1017/S0013091500007914 Published online by Cambridge University Press

of order k, exist for $s = 3, 4, \ldots, f$. We now proceed to show that, if $\lambda = 1$ when n is odd, and $\lambda = \sqrt{\omega}$ when n is even, the matrices G_s form an E-set containing $f - 2$ members. For we have

$$G_s G_u = \lambda F_{12}^{-1} F_{1s} \lambda F_{12}^{-1} F_{1u},$$

$$= \lambda^2 F_{n2} F_{n-1,2} \ldots F_{22} F_{1s} F_{n2} \ldots F_{22} F_{1u} \quad \text{by (10)},$$

$$= \lambda^2 \omega F_{n2} F_{n-1,2} \ldots F_{32} F_{2s} F_{12} F_{n2} \ldots F_{22} F_{1u} \quad \text{by (11)},$$

$$= \lambda^2 \omega F_{n2} F_{n-1,2} \ldots F_{32} F_{2s} F_{1u} \quad \text{by (10)}.$$

Similarly

$$G_u G_s = \lambda^2 \omega F_{n2} F_{n-1,2} \ldots F_{32} F_{2u} F_{1s};$$

and, since $F_{2s} F_{1u} = \omega F_{2u} F_{1s}$, we obtain $G_s G_u = \omega G_u G_s$.

Moreover

$$G_s^n = \lambda^n F_{n2} \ldots F_{22} F_{1s} (F_{12}^{-1} F_{1s})^{n-1} \quad \text{by (10)},$$

$$= \lambda^n \omega^{n-1} F_{ns} F_{n-1,2} \ldots F_{22} F_{12} (F_{12}^{-1} F_{1s})^{n-1} \quad \text{by (11)},$$

$$= \lambda^n \omega^{n-1} F_{ns} F_{n-1,2} \ldots F_{22} F_{1s} (F_{12}^{-1} F_{1s})^{n-2}$$

$$= \lambda^n \omega^{n-1+n-2} F_{ns} F_{n-1,s} F_{n-2,2} \ldots F_{22} F_{12} (F_{12}^{-1} F_{1s})^{n-2}.$$

By repeating this process $n - 1$ times we finally arrive at the result that

$$G_s^n = \lambda^n \omega^d F_{ns} F_{n-1,s} \ldots F_{2s} F_{1s} = \lambda^n \omega^d e,$$

where $d = n - 1 + n - 2 + \ldots + 2 + 1 = n(n-1)/2$. If n is odd, $\omega^d = 1$, while if n is even, $\omega^d = \omega^{n/2}$. In either case $\lambda^n \omega^d = 1$. *Hence, if there exists an E-set of matrices of order kn containing f members, there also exists an E-set of matrices of order k containing $f - 2$ members.* Thus, if there existed an E-set containing more than $2p + 1$ matrices of order $t = 2^p r$, $r \not\equiv 0 \mod n$, there would exist more than one member of an E-set of matrices of order r. But by Lemma I this last result is impossible and so Theorem I is proved.

Now let $R(\omega)$ denote the field obtained by adjoining ω to the field of all rational numbers. Then, if n is even, $\sqrt{\omega}$ does not belong to the field[1] $R(\omega)$ and so there exist at least two distinct types of

---

[1] If ω is a primitive nth root of unity, $\sqrt{\omega}$ is a primitive $2n$th root of unity. A primitive nth root of unity satisfies an equation of degree $\phi(n)$, irreducible in the field of rational numbers, where $\phi(n)$ is the Euler ϕ-function. If $n = 2^s k$, where k is odd, $\phi(n) = 2^{s-1}\phi(k)$ and $\phi(2n) = 2^s \phi(k)$. Thus the degrees of the irreducible equations satisfied by ω and $\sqrt{\omega}$ are different. Hence the fields $R(\omega)$ and $R(\sqrt{\omega})$ cannot coincide. This is no longer true if n is odd, since, if $n = 2f + 1$, $\sqrt{\omega} = \omega^{f+1}$.

matrices, R-matrices and I-matrices, which are defined in the following manner. *A matrix is said to be an R-matrix, when each element of the matrix lies in the field* $R(\omega)$; *a matrix is said to be an I-matrix, when each element of the matrix is a product of a number of* $R(\omega)$ *and* $\sqrt{\omega}$. We shall now consider E-sets, whose members are either R-matrices or else I-matrices, and shall accordingly assume n to be even. That this restriction does not lead to a triviality is apparent from the consideration of the matrices (7), of which two are R-matrices and one is an I-matrix. Further, if the number of R-matrices and the number of I-matrices in one E-set are equal respectively to the number of R-matrices and the number of I-matrices in a second E-set, we shall call the two E-sets *R-congruent*. Similarly if a matrix A is an R-matrix or an I-matrix, according as a matrix B is an R-matrix or an I-matrix, we shall call the two matrices *R-congruent*.

In order to determine the number of R-matrices and I-matrices, which may occur in a maximal E-set, we require two Lemmas.

LEMMA 3. *If the matrices* E_i, $i = 1, 2, \ldots, f$, *form an E-set,* (E), *all of whose members are either R-matrices or else I-matrices, and if* $r_1, r_2, \ldots, r_t$ *are t integers such that* $1 \leqq r_1 < r_2 < \ldots < r_t \leqq f$, *then there exists an E-set R-congruent to* (E), *whose first t members are the matrices* $E_{r_1}, E_{r_2}, \ldots E_{r_t}$.

It is easily verified that the set of matrices (T), where

$$T_i = E_i, \quad i \neq j \text{ or } j-1, \quad T_{j-1} = E_j, \quad T_j = E_{j-1}^{-1} E_j^2,$$

form an E-set. But, since the matrices E_{j-1} and T_j are R-congruent, the sets (E) and (T) are also R-congruent. In the same manner, if j is replaced by $j-1$, from (T) a set (S) can be formed such that

$$S_{j-2} = T_{j-1} = E_j,$$

and such that the sets (E) and (S) are R-congruent. By repeating this process $j-1$ times we finally arrive at a set (K), R-congruent to (E), and such that its first member is E_j and its kth member is E_k, if $k > j$. If $j = r_1$, the set (K) has for its first member E_{r_1} and for its r_ith member E_{r_i}, $i > 1$. By applying the same process, with $j = r_2$, $r_2 - 2$ times to the set (K), we obtain a set (P), R-congruent to (E), which has for its first two members E_{r_1} and E_{r_2}. Finally in $r_1 - 1 + r_2 - 2 + \ldots + r_t - t$ steps we arrive at an E-set R-congruent to (E), whose first t members are the matrices E_{r_i}, $i = 1, 2, \ldots t$, and so the lemma is proved.

https://doi.org/10.1017/S0013091500007914 Published online by Cambridge University Press

LEMMA 4. *If in an E-set consisting of f matrices of order $t = kn$, g of the members are R-matrices and $h = f - g$ are I-matrices and both g and h are different from zero, then there exists an E-set of matrices of order k, of which $g - 1$ are R-matrices and $h - 1$ are I-matrices.*

Since in an E-set, satisfying the above hypotheses, at least one matrix is an R-matrix and at least one an I-matrix, there exists, by Lemma 3, an E-set, E_i, $i = 1, 2, \ldots . f$, which is R-congruent to the original set, and such that E_1 is an R-matrix and E_2 an I-matrix. Now, if A is an R-matrix, the set $F_i = A^{-1} E_i A$, $i = 1, 2, \ldots . f$, and the set E_i are R-congruent. But, since E_1 is an R-matrix, the matrix A in (3) must be an R-matrix, so that the set $F_i = A^{-1} E_i A$, where F_1 is defined by (3) and F_i, $i > 1$, by (9), is R-congruent to the set E_i. As E_2 is an I-matrix, so is F_2, and accordingly F_{12}, being a sub-matrix of F_2, is also an I-matrix. Hence the set of matrices G_s, $s = 3, 4, \ldots . f$, defined by (11), since λ now has the value $\sqrt{\omega}$, and the set F_s, $s = 3, 4, \ldots . f$, are R-congruent. But the set F_s and the set E_s, $s = 3, 4, \ldots . f$, are R-congruent and so the set G_s and the set E_s are R-congruent. Since the set E_s, $s = 3, 4, \ldots, f$ contains exactly $(g - 1)$ R-matrices and exactly $(h - 1)$ I-matrices, the lemma is proved.

We have already proved in Theorem I that maximal E-sets of matrices of order $t = n^p r$, where r is not divisible by n, exist and that the number of matrices in such a set is $2p + 1$. We now suppose that the number of R-matrices in such a maximal E-set has one of the values

(i) $p - 1$;
(ii) p;
(iii) $p + 1$;
(iv) $p + 2$;

and proceed to show that in some cases we are led to a contradiction.

By repeated applications of Lemma 4 we deduce the existence of E-sets consisting respectively of the following matrices:

(i) three I-matrices of order nr;
(ii) one I-matrix of order r;
(iii) one R-matrix of order r;
(iv) three R-matrices of order nr.

But an E-set of matrices of order nr contains the three members E_1, E_2, E_3, where as in (7), since n is even, $E_3 = \sqrt{\omega}\, E_1^{-1} E_2$, so that

https://doi.org/10.1017/S0013091500007914 Published online by Cambridge University Press

E_1, E_2, E_3 cannot all be R-matrices or all I-matrices. Moreover when r is odd, an I-matrix of order r cannot be a member of an E-set, for the determinant of an I-matrix of order r is of the form $\sqrt{\omega}\, k$, where k is a number of $R(\omega)$, while the determinant of a member of an E-set, being the product of nth roots of unity, must lie in $R(\omega)$. If, however, r is even, a matrix of order r which is a member of an E-set may be an I-matrix; for the matrix

$$H = \sqrt{\omega}\,(h_{ij}), \quad i, j = 1, 2, \ldots . \tau,$$

where $$h_{ij} = 0, \text{ if } j \neq i + 1 \bmod n;$$

$$h_{i,\,i+1} = \omega^{-1}, \; i = 1, 2, \ldots, r/2; \; h_{i,\,i+1} = 1, \; i = r/2 + 1, \ldots, r,$$

is an I-matrix and, since $H^r = E$, it is a member of an E-set. As the unit matrix of order r is an R-matrix and at the same time a member of an E-set, whether r is even or odd, we have shown that of the four possibilities (i), (ii), (iii), (iv) only (ii) and (iii) may occur, when r is even, and (iii) alone, when r is odd. Further by repeated applications of (8) we see that maximal E-sets of matrices, of order $t = n^p r$, r not divisible by n, exist, in which the number of R-matrices is p or $p + 1$ when r is even, and $p + 1$ when r is odd.

We are now in a position to prove the following theorem.

THEOREM 2. *If in a maximal E-set of matrices, of order $t = n^p r$, $r \neq 0$ mod n, the members are restricted to be either R-matrices or else I-matrices, then the number of R-matrices in the set is u, where u satisfies*

$$0 \leqq u \leqq 2p + 1, \quad u \equiv p + 1 \bmod 4, \tag{13}$$

or

$$0 \leqq u \leqq 2p + 1, \quad u \equiv p + 1 \text{ or } p \bmod 4, \tag{14}$$

according as r is odd or even. Sets exist for every admissible value of u.

Let $E_1, E_2, E_3, \ldots, E_q = E_{2p+1}$ be a set (E) of matrices of order t satisfying the hypotheses of the theorem, and let g of the matrices be R-matrices, and $h = 2p + 1 - g$ be I-matrices. Then the matrices in the set (F), defined by

$$\begin{cases} F_i = E_i, & 1 \leqq i < 2k, \\ F_i = \mu\, E_1^{-1} E_2 E_3^{-1} E_4 \ldots . E_{2k-1}^{-1} E_i = S_k E_i, & 2k \leqq i \leqq q, \end{cases} \tag{15}$$

where $\mu = 1$, if k is even, and $\mu = \sqrt{\omega}$, if k is odd, form an E-set for all values of k, where $1 \leqq k \leqq p$. For it is easily verified that $E_i S_k = S_k E_i$, if $i < 2k$, and that $E_i S_k = \omega S_k E_i$, if $i > 2k$.

Accordingly

$$F_i F_j = E_i E_j = \omega E_j E_i = \omega F_j F_i, \quad i<j;\ i, j = 1, 2, \ldots, 2k-1;$$

$$F_i F_j = E_i S_k E_j = S_k E_i E_j = \omega S_k E_j E_i = \omega F_j F_i, \quad i < 2k,\ j \geqq 2k;$$

$$F_i F_j = S_k E_i S_k E_j = \omega S_k^2 E_i E_j = \omega^2 S_k^2 E_j E_i = \omega S_k E_j S_k E_i;$$
$$= \omega F_j F_i, \quad i, j \geqq 2k,\ i<j;$$

and

$$F_i^n = E_i^n = E, \quad i < 2k;$$

$$F_i^n = (S_k E_i)^n = \mu^n (E_1^{-1} E_2)^n (E_3^{-1} E_4)^n \ldots (E_{2k-1}^{-1} E_i)^n,$$
$$= \mu^n \omega^d E, \quad d = -kn(n-1)/2,\ i \geqq 2k,$$
$$= E,$$

if $\mu = 1$, when k is even, and $\sqrt{\omega}$, when k is odd. But if k is odd and the matrices $E_1, E_2, \ldots, E_{2k-1}$ are all R-matrices, then the matrix S_k, defined by (15), is an I-matrix and the number of I-matrices in the set F_i is

$$v = g - (2k-1) \equiv g - 1, \text{ mod } 4.$$

If $1 \leqq 2k-1 \leqq g$ we may assume that the matrices $E_1, E_2, \ldots E_{2k-1}$ are R-matrices, since otherwise, by Lemma (3), we can find a set (E'), R-congruent to the set (E), of which the first $2k-1$ members are R-matrices and so we can use the set (E') instead of the set (E) to define the set (F). Thus E-sets exist, in which the number of I-matrices is v for all values of v satisfying

(16) $$v \equiv g-1, \text{ mod } 4, \quad 0 \leqq v \leqq g-1.$$

On the other hand, if k is odd, and $E_1, E_2, \ldots, E_{2k-1}$ are all I-matrices, S_k is still an I-matrix so that the number of I-matrices in the set (F) is now

$$v = 2k - 1 + g \equiv g - 1, \text{ mod } 4.$$

Once again, by Lemma 3, for every value of k, $3 \leqq 2k-1 \leqq h$, we can find a set (E'), R-congruent to (E), such that its first $2k-1$ members are I-matrices, and so there exist E-sets in which the number of I-matrices is v, for all values of v satisfying

(17) $$v \equiv g-1, \text{ mod } 4, \quad g+3 \leqslant v \leqslant 2p+1.$$

Accordingly by (16) and (17), if an E-set of $2p+1$ members exists, in which g of the matrices are R-matrices, and $2p-g+1$ are

https://doi.org/10.1017/S0013091500007914 Published online by Cambridge University Press

I-matrices, then there exists an E-set, in which the number of I-matrices is v, where v satisfies

$$(18) \qquad v \equiv g - 1, \text{ mod } 4, \quad 0 \leqslant v \leqslant 2p + 1.$$

But, if r is odd, E-sets exist in which $g = p + 1$, while, if r is even, E-sets exist in which g is either p or $p + 1$. Accordingly E-sets do exist in which the number of R-matrices is u, for every value of u satisfying (13) if r is odd, and (14) if r is even. If there existed an E-set in which the number u of R-matrices did not satisfy (13) or (14), there would exist an E-set in which u had the value p, $p - 1$, or $p + 2$, in the one case, and $p + 2$ or $p - 1$ in the other. As it has already been shown that such E-sets cannot exist, Theorem 2 is proved.

No such theorem is true when n is odd, for, as already remarked, there is then no distinction between I-matrices and R-matrices. If $n = 2$, we have $\omega = -1$ and $\sqrt{\omega} = i$, so that R-matrices are real rational matrices, while I-matrices are pure imaginary matrices. $R(\omega)$ is now the field of all rational numbers, but in this particular case the argument would remain unaltered if the field of all real numbers were used instead. Since, if $n = 2$, r must be odd, in Theorem 2 only formula (13) is required.[1]

[1] As both Eddington and Newman consider matrices whose squares are $-E$, the number of imaginary matrices in a set of such matrices is the same as the number of real matrices in an E-set, satisfying (2) with $n = 2$.

https://doi.org/10.1017/S0013091500007914 Published online by Cambridge University Press