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A COHERENCE THEOREM ON SCHÜTZENBERGER GROUPS

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It is well-known that, in any semigroup S, all the left and right Schützenberger groups of the $\mathscr{H}$-classes contained in a fixed $\mathscr{D}$-class D of S are isomorphic to one group. We prove a sharper result: that, up to action-preserving isomorphisms, all these Schützenberger groups and their classical isomorphisms are but one group and its identity isomorphism; thus, one group is essentially sufficient to describe not only all the Schützenberger groups of $\mathscr{H}$-classes in D, but their action on these $\mathscr{H}$-classes and classical isomorphisms as well.

The paper is organized as follows. In Section 1 we recall the construction of Schützenberger groups and their classical isomorphisms; this essentially follows Clifford and Preston (1961), but we find it more convenient to have left Schützenberger groups operate on the left rather than on the right, as is now usually done. The main result is in Section 3. Its proof is rather more complex than the statement would suggest. Also we need a more precise statement of the main result; and for this it is helpful to consider a slightly more general situation which still accounts for all the basic properties of Schützenberger groups. This leads us to define *Schützenberger arrays* in Section 1; the bulk of the proof of the main result then yields, with little additional trouble, two necessary and sufficient conditions that a Schützenberger array have the property we seek, and will be found in Section 2 with these conditions.

Our main result can then be construed as giving another basic, hitherto overlooked property of Schützenberger groups.

In Section 4 we give some minor applications of the main result, to congruences contained in $\mathscr{H}$, to the location of inverses and to orthodox semigroups.

1. Schützenberger groups and Schützenberger arrays

1. Let S be a semigroup and H be any $\mathcal{H}$-class of S. Put $T(H) = \{x \in S^1; xH \subseteq H\}$ (recall that $xH \cap H \neq \emptyset$ implies $xH = H$) and $T'(H) = \{y \in S^1; Hy \subseteq H\}$. Each $x \in T(H)$ induces a mapping $\pi(x)$: $H \mapsto H$, which we denote as a left operator $a \mapsto \pi(x) \cdot a$; these mappings form a semigroup $G(H)$ under composition. The main property of $G(H)$ is that it is a group of permutations of H and acts simply and transitively on H; $G(H)$ is the *left Schützenberger group* of H. The *right Schützenberger group* $G'(H)$ of H is defined dually as the group of all mappings $\pi'(y)$: $a \mapsto a \cdot \pi'(y) = ay$ of H into itself, where $y \in T'(H)$; it is also a group of permutations of H and acts on H (on the right) simply and transitively. Furthermore, associativity in S^1, $x(ay) = (xa)y$, shows that $g \cdot (a \cdot g') = (g \cdot a) \cdot g'$ for all $a \in H$, $g \in G(H)$, $g' \in G'(H)$. These are the basic properties of $G(H)$, $G'(H)$; they imply that, for any $p \in H$, an isomorphism θ_p: $G(H) \mapsto G'(H)$ can be defined by: $g \cdot p = p \cdot \theta_p(g)$ (it is clear that θ_p is indeed well-defined and bijective; furthermore, $g \cdot p = p \cdot g'$, $h \cdot p = p \cdot h'$ imply $g \cdot (h \cdot p) = g \cdot (p \cdot h') = (g \cdot p) \cdot h' = (p \cdot g') \cdot h'$, which shows that θ_p is a homomorphism); in particular, $G(H)$ and $G'(H)$ are isomorphic (though not canonically isomorphic since θ_p usually depends on p).

2. Let now D be a fixed $\mathcal{D}$-class of S. We denote, as usual, by $(R_i)_{i \in I}$ the family of all $\mathcal{R}$-classes of D and by $(L_\lambda)_{\lambda \in \Lambda}$ the family of all $\mathcal{L}$-classes of D; the typical $\mathcal{H}$-class of D is then $H_{i\lambda} = R_i \cap L_\lambda$. Thus we obtain all Schützenberger groups $G(H_{i\lambda})$, $G'(H_{i\lambda})$ of D. There are three kinds of classical isomorphisms between these. The first arise by selecting one element $p_{i\lambda}$ in each $H_{i\lambda}$; we denote $\theta_{p_{i\lambda}}$ by $\theta_{i\lambda}$ (sometimes by $\theta^p_{i\lambda}$ if we wish to emphasize the dependence on $p = (p_{i\lambda})_{i \in I, \lambda \in \Lambda}$ or on $p_{i\lambda}$).

The second kind are canonical isomorphisms between $G(H_{i\lambda})$ and $G(H_{i\mu})$ within one $\mathcal{R}$-class R_i. We note that there exist $q, q' \in S^1$ such that $H_{i\lambda}q = H_{i\mu}$, $H_{i\mu}q' = H_{i\lambda}$ (arising from the usual application of Green's lemma); multiplication by q, q' then easily shows that $T(H_{i\lambda}) = T(H_{i\mu})$. We now use the homomorphisms $\pi_{i\lambda}$: $T(H_{i\lambda}) \to G(H_{i\lambda})$, $\pi_{i\mu}$: $T(H_{i\mu}) \to G(H_{i\mu})$; picking $a \in H_{i\lambda}$, we see that, for all $x, y \in T(H_{i\lambda})$, $xa = ya$ implies $xaq = yaq$, and hence $\pi_{i\lambda}(x) = \pi_{i\lambda}(y)$ implies $\pi_{i\mu}(x) = \pi_{i\mu}(y)$; the converse implication is proved by using q', so that the surjective homomorphisms $\pi_{i\lambda}$, $\pi_{i\mu}$ induce the same congruence on $T(H_{i\lambda}) = T(H_{i\mu})$. It follows that there exists an isomorphism $\phi^{i\lambda}_{i\mu}$: $G(H_{i\lambda}) \to G(H_{i\mu})$ unique such that $\phi^{i\lambda}_{i\mu} \circ \pi_{i\lambda} = \pi_{i\mu}$.

The isomorphism $\phi^{i\lambda}_{i\mu}$ is canonical and may be as well defined by any of the following two properties. First, for any $g \in G(H_{i\lambda})$, g and $\phi^{i\lambda}_{i\mu}g$ are induced by the same elements of $T(H_{i\lambda})$. Next, let $g = \pi_{i\lambda}(x) \in G(H_{i\lambda})$, $a \in H_{i\lambda}$, and $t \in S^1$ be such that $H_{i\lambda}t = H_{i\mu}$; then $\phi^{i\lambda}_{i\mu}g = \pi_{i\mu}(x)$ and hence $\phi^{i\lambda}_{i\mu}g \cdot at = xat = (g \cdot a)t$, which proves

(1) $\quad \phi^{i\lambda}_{i\mu}g \cdot at = (g \cdot a)t \quad$ whenever $\quad g \in G(H_{i\lambda}), a \in H_{i\lambda}, H_{i\lambda}t = H_{i\mu}$.

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The definition immediately yields the basic property of these isomorphisms; namely, $\phi^{i\lambda}_{i\lambda}$ is the identity on $G(H_{i\lambda})$, and $\phi^{i\mu}_{i\nu} \circ \phi^{i\lambda}_{i\mu} = \phi^{i\lambda}_{i\nu}$ for all i, λ, μ, ν. These composition properties essentially tell us that, up to action-preserving isomorphisms, all groups $G(H_{i\lambda})$ (with a fixed i) and their canonical isomorphisms are but one group with its identity isomorphism. In fact, this is precisely what happens in the Hunter-Anderson presentation (see, e.g., Hunter and Anderson (1965)), where $G(H)$ is defined as the quotient of $T(H)$ by the congruence induced by π; it is easy to check that all $\phi^{i\lambda}_{i\mu}$ can be redefined by (1) and are but the identity. (We do not use this presentation, despite its obvious advantages, because it would create confusion in section 2).

The last classical isomorphisms arise by duality. For each λ, i, j, there is an isomorphism $\psi^{i\lambda}_{j\lambda}: G'(H_{i\lambda}) \to G'(H_{j\lambda})$, unique such that $\psi^{i\lambda}_{j\lambda} \circ \pi'_{i\lambda} = \pi'_{j\lambda}$ (and hence canonical); it can be also defined by

$$\text{(2)} \qquad ua \cdot \psi^{i\lambda}_{j\lambda} g' = u(a \cdot g') \quad \text{whenever} \quad a \in H_{i\lambda}, g' \in G'(H_{i\lambda}), uH_{i\lambda} = H_{j\lambda}$$

These have the composition properties: $\psi^{i\lambda}_{i\lambda}$ is the identity on $G'(H_{i\lambda})$ and $\psi^{j\lambda}_{k\lambda} \circ \psi^{i\lambda}_{j\lambda} = \psi^{i\lambda}_{k\lambda}$ for all i, j, k, λ.

The very existence of all classical isomorphisms yields the well-known result that all $G(H_{i\lambda})$, $G'(H_{i\lambda})$ are isomorphic.

3. We now formalize all the basic properties obtained so far, as follows. Let $\mathscr{C} = (C_{i\lambda})_{i \in I, \lambda \in \Lambda}$ be a family of pairwise disjoint non-empty sets, indexed by a non-empty cartesian product $I \times \Lambda$. Let $\mathscr{G} = (G_{i\lambda}, G'_{i\lambda})_{i \in I, \lambda \in \Lambda}$ be a family, indexed by the same set, of pairs of groups and group actions, so that $G_{i\lambda}$ acts simply and transitively on $C_{i\lambda}$ on the left, $G'_{i\lambda}$ acts simply and transitively on $C_{i\lambda}$ on the right, and $g \cdot (a \cdot g') = (g \cdot a) \cdot g'$ holds for all $g \in G_{i\lambda}$, $a \in C_{i\lambda}$, $g' \in G'_{i\lambda}$. Let $\phi = (\phi^{i\lambda}_{i\mu})_{i \in I, \lambda, \mu \in \Lambda}$ and $\psi = (\psi^{i\lambda}_{j\lambda})_{i,j \in I, \lambda \in \Lambda}$ be families of isomorphisms $\phi^{i\lambda}_{i\mu}: G_{i\lambda} \to G_{i\mu}$, $\psi^{i\lambda}_{j\lambda}: G'_{i\lambda} \to G'_{j\lambda}$, such that $\phi^{i\lambda}_{i\lambda}$ is the identity on $G_{i\lambda}$, $\phi^{i\mu}_{i\nu} \circ \phi^{i\lambda}_{i\mu} = \phi^{i\lambda}_{i\nu}$, $\psi^{i\lambda}_{i\lambda}$ is the identity on $G'_{i\lambda}$ and $\psi^{j\lambda}_{k\lambda} \circ \psi^{i\lambda}_{j\lambda} = \psi^{i\lambda}_{k\lambda}$, for all $i, j, k, \lambda, \mu, \nu$. We call such a quadruple $\mathfrak{A} = (\mathscr{C}, \mathscr{G}, \phi, \psi)$ a *Schützenberger array*. We have just seen that each $\mathscr{D}$-class of any semigroup yields a canonical Schützenberger array.

The definition of a Schützenberger array includes only two of the three kinds of classical isomorphisms. The reason is that the θ isomorphisms are available on the rest of the data. Namely, let $p = (p_{i\lambda})_{i \in I, \lambda \in \Lambda}$ be a family consisting of one element $p_{i\lambda}$ in each $C_{i\lambda}$ (we call p a *basis*). As in the $\mathscr{D}$-class case we see that an isomorphism $\theta_{i\lambda} = \theta^{p}_{i\lambda}: G_{i\lambda} \to G'_{i\lambda}$ is well-defined for each i, λ by: $g \cdot p_{i\lambda} = p_{i\lambda} \cdot \theta_{i\lambda} g$ for all $g \in G_{i\lambda}$. These isomorphisms have of course no place in the definition of a Schützenberger array but are nevertheless an important part of it: we need them to see that all $G_{i\lambda}$, $G'_{i\lambda}$ are isomorphic and we must therefore take them into consideration if we wish to replace all $G_{i\lambda}$, $G'_{i\lambda}$ by just one group.

2. Representation of general Schützenberger arrays

1. We are now in position to define with precision the idea that a Schützenberger array with all its isomorphisms is essentially (= up to action-preserving isomorphisms) but one group and its identity isomorphism.

First we define isomorphic Schützenberger arrays. If $\mathfrak{A} = (\mathscr{C}, \mathscr{G}, \phi, \psi)$ and $\bar{\mathfrak{A}} = (\mathscr{C}, \bar{\mathscr{G}}, \bar{\phi}, \bar{\psi})$ are Schützenberger arrays (with the same $\mathscr{C}$), an *isomorphism* $\iota: \mathfrak{A} \to \bar{\mathfrak{A}}$ is a family $\iota = (\iota_{i\lambda}, \iota'_{i\lambda})_{i \in I, \lambda \in \Lambda}$ of pairs of isomorphisms $\iota_{i\lambda}: G_{i\lambda} \to \bar{G}_{i\lambda}$, $\iota'_{i\lambda}: G'_{i\lambda} \to \bar{G}'_{i\lambda}$ with the following properties:

i) $\iota_{i\lambda} g \cdot a = g \cdot a$, $a \cdot \iota'_{i\lambda} g' = a \cdot g'$ whenever $a \in C_{i\lambda}$, $g \in G_{i\lambda}$, $g' \in G'_{i\lambda}$;

ii) $\iota_{i\mu} \circ \phi^{i\lambda}_{i\mu} = \bar{\phi}^{i\lambda}_{i\mu} \circ \iota_{i\lambda}$, $\iota'_{j\lambda} \circ \psi^{i\lambda}_{j\lambda} = \bar{\psi}^{i\lambda}_{j\lambda} \circ \iota'_{i\lambda}$ for all i, j, λ, μ (in other words, ι preserves all the existing structure). Note that (relative to the same basis p), the isomorphisms ϑ are also preserved, i.e. $\iota'_{i\lambda} \circ \theta_{i\lambda} = \bar{\theta}_{i\lambda} \circ \iota_{i\lambda}$: indeed, for all $g \in G_{i\lambda}$,

$$p \cdot \iota'\theta g\, [= p_{i\lambda} \cdot \iota'_{i\lambda}\theta_{i\lambda} g] = p \cdot \theta g = g \cdot p = \iota g \cdot p = p \cdot \bar{\theta}\iota g.$$

[N.B. Clearly we could consider the more general problem of classifying all Schützenberger arrays up to isomorphism. Isomorphisms could also be defined in the obvious way between Schützenberger arrays on different underlying families of sets. Such generalizations, however, would be of no benefit for our main result.]

Next we define 'essentially-one-group' Schützenberger arrays. We say that a Schützenberger array $\mathfrak{A}$ is *simple* in case all the groups $G_{i\lambda}, G'_{i\lambda}$ are the same group G, all the $\phi^{i\lambda}_{i\mu}$ and $\psi^{i\lambda}_{j\lambda}$ are the identity on G, and furthermore there exists a basis such that all $\theta_{i\lambda}$ are also the identity on G. A Schützenberger array is *coherent* in case it is isomorphic to a simple one, and this is the property we wish to establish for Schützenberger arrays of $\mathscr{D}$-classes.

Coherence can be expressed in terms of bases. A basis p of a Schützenberger array $\mathfrak{A}$ will be called *coherent* in case there exists a simple Schützenberger array $\bar{\mathfrak{A}}$ isomorphic to $\mathfrak{A}$ in which all $\theta_{i\lambda}$ are the identity when calculated relative to p. (Thus $\mathfrak{A}$ is coherent if and only if it has a coherent basis.) We want to find coherent bases, and if possible describe how they can all be obtained (in case there is one).

2. The first result is expressed in terms of *array isomorphisms*, i.e. those isomorphisms that can be obtained by composing finitely many $\phi^{i\lambda}_{i\mu}, \psi^{i\lambda}_{j\lambda}, \theta_{i\lambda}$ and $\theta^{-1}_{i\lambda}$'s (note that $(\phi^{i\lambda}_{i\mu})^{-1} = \phi^{i\mu}_{i\lambda}$, $(\psi^{i\lambda}_{j\lambda})^{-1} = \psi^{j\lambda}_{i\lambda}$, by the composition properties).

LEMMA 1. *A basis is coherent if and only if every diagram of array isomorphisms is commutative.*

PROOF. First assume that p is a coherent basis of the given Schützenberger array $\mathfrak{A}$. Hence, $\mathfrak{A} \cong \bar{\mathfrak{A}}$, where all $\bar{\phi}, \bar{\psi}$ and $\bar{\theta}$'s (relative to the given p) are the identity on some group $\bar{G}$. We now observe that the isomorphism $\mathfrak{A} \to \bar{\mathfrak{A}}$ comes

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with a number of commutative squares whose sides are ϕ, ψ, θ's and hence any diagram $\mathscr{D}$ of array isomorphisms of $\mathfrak{A}$ is isomorphic to a diagram $\bar{\mathscr{D}}$ of corresponding array isomorphisms of $\bar{\mathfrak{A}}$. Now $\bar{\mathscr{D}}$ is made up of identity isomorphisms on $\bar{G}$ and hence is (trivially) commutative. It follows that $\mathscr{D}$ is also commutative.

Conversely, assume that every diagram of array isomorphisms (relative to the given basis p) is commutative. Let $\bar{G}$ be any one of the groups $G_{i\lambda}$, $G'_{i\lambda}$. Any two such groups are always connected by at least one array isomorphism and so there exists an array isomorphism $\iota_{i\lambda}: G_{i\lambda} \to \bar{G}$ for each i, λ; but the hypothesis implies that any two such isomorphisms must in fact be equal, and so $\iota_{i\lambda}$ is uniquely determined. Similarly, there is for each i, λ a unique array isomorphism $\iota'_{i\lambda}: G'_{i\lambda} \to \bar{G}$. We now define a left action of $\bar{G}$ on each $C_{i\lambda}$ by: $\bar{g} \cdot a = g \cdot a$ whenever $a \in C_{i\lambda}$, $g \in G_{i\lambda}$ and $\bar{g} = \iota_{i\lambda}(g)$; this is clearly well-defined and makes each $\iota_{i\lambda}$ action-preserving. Similarly, there is a well-defined right action of $\bar{G}$ on each $C_{i\lambda}$ such that each $\iota'_{i\lambda}$ is action-preserving: it is given by $a \cdot \bar{g} = a \cdot g'$ whenever $a \in C_{i\lambda}$, $g' \in G'_{i\lambda}$ and $\bar{g} = \iota'_{i\lambda}(g')$. Obviously, $\bar{g} \cdot (a \cdot \bar{h}) = (\bar{g} \cdot a) \cdot \bar{h}$ for all $\bar{g}, \bar{h} \in \bar{G}$, $a \in C_{i\lambda}$. Hence we obtain a Schützenberger array $\bar{\mathfrak{A}}$ in which $\bar{G}_{i\lambda} = \bar{G}'_{i\lambda} = \bar{G}$ for all i, λ, and $\bar{\phi}^{i\lambda}_{i\mu}, \bar{\psi}^{i\lambda}_{j\lambda}$ are the identity on $\bar{G}$ for all i, j, λ, μ. Furthermore the hypothesis also tells us that $\iota'_{i\lambda} \circ \theta_{i\lambda} = \iota_{i\lambda}$; hence, for all $g \in G_{i\lambda}$, $\iota g \cdot p[= \iota_{i\lambda} g \cdot p_{i\lambda}] = g \cdot p = p \cdot \theta g = p \cdot \iota' \theta g = p \cdot \iota g$, which shows that (relative to p), $\bar{\theta}_{i\lambda}$ is also the identity on $\bar{G}$. It follows that $\bar{\mathfrak{A}}$ is simple. Finally, we use the hypothesis one more time to obtain $\iota_{i\mu} \circ \phi^{i\lambda}_{i\mu} = \iota_{i\lambda}$, $\iota'_{j\lambda} \circ \psi^{i\lambda}_{j\lambda} = \iota'_{i\lambda}$; it follows that ι is an isomorphism. Therefore p is a coherent basis.

3. If we are to produce coherent bases, we shall have to build them from (presumably) other bases, and hence we need the following trivial result on change of bases.

LEMMA 2. *Let p, q be bases. Then $\theta^q_{i\lambda} g = h'^{-1}\, \theta^p_{i\lambda} g\, h'$, where $h' \in G'_{i\lambda}$ is such that $q_{i\lambda} = p_{i\lambda} \cdot h'$.*

PROOF. For all $g \in G_{i\lambda}$, $q \cdot \theta^q g\,[= q_{i\lambda} \cdot \theta^q_{i\lambda} g] = g \cdot q = g \cdot (p \cdot h') = (g \cdot p) \cdot h' = (p \cdot \theta^p g) \cdot h' = q \cdot h'^{-1}\, \theta^p g\, h'$.

This immediately yields necessary conditions for the existence of a coherent basis. If α is any array isomorphism (relative to p) then the corresponding array isomorphism, relative to another basis q, can be obtained from α by composing α with an inner automorphism of its codomain. Hence:

COROLLARY. *If there exists a coherent basis, then (relative to any basis) every diagram of array isomorphisms commutes up to inner automorphisms.*

At the start there was no evidence that the property we try to prove is not always true (and even trivial to boot). Lemma 1, however, suggests that it might not be as trivial as it seems. Furthermore we can now give an example of a Schützenberger array which is not coherent.

Let $I = \{j, k\}$, $\Lambda = \{\mu, \nu\}$, and G be a group possessing an automorphism α that is not inner (e.g. the Klein group V_4). Select four pairwise disjoint sets $C_{i\lambda}$ with same cardinal as G; they come with bijections $\beta_{i\lambda}: G \to C_{i\lambda}$. We let $G_{i\lambda} = G'_{i\lambda} = G$ for all i, λ, and all ϕ, ψ's be the identity on G; the left actions of G on each $C_{i\lambda}$ are defined by $g \cdot \beta_{i\lambda}(h) = \beta_{i\lambda}(gh)$, while the right actions are defined by: $\beta_{i\lambda}(h) \cdot l = \beta_{i\lambda}(hl)$ if $(i, \lambda) \neq (k, \nu)$, and $\beta_{k\nu}(h) \cdot l = \beta_{k\nu}(h\alpha^{-1}(l))$. It is immediate that this indeed defines a Schützenberger array. A basis is given by $p_{i\lambda} = \beta_{i\lambda}(1)$; in that basis, one finds $\theta_{i\lambda}$ is the identity on G if $(i, \lambda) \neq (k, \nu)$, while $\theta_{k\nu} = \alpha$: e.g. $g \cdot p_{k\nu} = g \cdot \beta_{k\nu}(1) = \beta_{k\nu}(1) \cdot \alpha(g) = p_{k\nu} \cdot \alpha(g)$. Then consider the two following array isomorphisms $G_{k\nu} \to G_{k\nu}$: $\phi^{k\nu}_{k\nu}$ (the identity on G) and

$$\phi^{k\mu}_{k\nu} \circ \theta^{-1}_{k\mu} \circ \psi^{j\mu}_{k\mu} \circ \theta_{j\mu} \circ \phi^{j\nu}_{j\mu} \circ \theta^{-1}_{j\nu} \circ \psi^{k\nu}_{j\nu} \circ \theta_{k\nu} = \alpha;$$

these form a diagram which only commutes up to α, and hence does not commute up to inner automorphism. Hence this Schützenberger array is not coherent.

4. The main result of this section gives a simple criterion for coherence.

THEOREM 1. *A Schützenberger array $\mathfrak{A}$ is coherent if and only if (relative to any given basis) all array isomorphism of the form*

$$\theta_{i\lambda} \circ \phi^{i\mu}_{i\lambda} \circ \theta^{-1}_{i\mu} \circ \psi^{j\mu}_{i\mu} \circ \theta_{j\mu} \circ \phi^{j\lambda}_{j\mu} \circ \theta^{-1}_{j\lambda} \circ \psi^{i\lambda}_{j\lambda}$$

(of $G'_{i\lambda}$ into itself) are inner automorphisms, for all i, j, λ, μ.

PROOF. The condition is necessary by the corollary of Lemma 2 (since the diagram formed by the given array isomorphism and the identity $\psi^{i\lambda}_{i\lambda}$ on $G'_{i\lambda}$ must commute up to inner automorphism). The proof that the condition is sufficient will be done in several steps.

The first step is to replace the given basis p by one with the property that all compositions in the statement, in which j and μ are fixed indices x and ξ, actually yield the identity on $G'_{i\lambda}$. (Then we show that the new basis is coherent). Thus, let $x \in I$ and $\xi \in \Lambda$ be fixed; put

$$\alpha^p_{i,\lambda} = \alpha_{i,\lambda} = \theta_{i\lambda} \circ \phi^{i\xi}_{i\lambda} \circ \theta^{-1}_{i\xi} \circ \psi^{x\xi}_{i\xi} \circ \theta_{x\xi} \circ \phi^{x\lambda}_{x\xi} \circ \theta^{-1}_{x\lambda} \circ \psi^{i\lambda}_{x\lambda}: G'_{i\lambda} \to G'_{i\lambda}.$$

By the composition properties of ϕ and ψ, we see that $\alpha_{i,\lambda}$ is already the identity on $G'_{i\lambda}$ when $i = x$ and when $\lambda = \xi$. We now assume $i \neq x$ and $\lambda \neq \xi$. Then $\alpha_{i,\lambda}$ is by the hypothesis the inner automorphism of $G'_{i\lambda}$ determined by some $h' \in G'_{i\lambda}$: $\alpha_{i,\lambda}(g') = h'g'h'^{-1}$ for all $g' \in G'_{i\lambda}$. We then replace $p_{i\lambda}$ by $q_{i\lambda} = p_{i\lambda} \cdot h'$; since $i \neq x$ and $\lambda \neq \xi$, this does not affect $p_{x\xi}$, $p_{x\lambda}$, $p_{i\xi}$; hence all terms of $\alpha_{i,\lambda}$ remain unchanged except for $\theta^p_{i\lambda}$, which is replaced by $\theta^q_{i\lambda}$, where $\theta^q_{i\lambda}(g) = h'^{-1}\theta^p_{i\lambda}(g)h'$ for all $g \in G_{i\lambda}$, by Lemma 2. Then $\alpha^q_{i,\lambda}(g') = h'^{-1}\alpha^p_{i,\lambda}(g')\, h' = g'$ for all $g' \in G'_{i\lambda}$.

Now we have only changed one $p_{i\lambda}$ and it follows that the operation can be done simultaneously on all $p_{i\lambda}$ $(i \neq x, \lambda \neq \xi)$, yielding a new basis $q_{i\lambda}$ with the property that $\alpha_{i,\lambda}$ is the identity for all i, λ.

We note that $q_{i\lambda} = p_{i\lambda}$ whenever $i = x$ or $\lambda = \xi$, and this eventually implies that all $p_{x\lambda}, p_{i\xi}$ (with x, ξ fixed) can be chosen arbitrarily to build a coherent basis, in case such exists.

5. For the rest of the proof it will be convenient to visualize the groups $G_{i\lambda}$ as forming an $I \times \Lambda$ array of groups, and the groups $G'_{i\lambda}$ as forming another $I \times \Lambda$ array of groups lying immediately above the previous one (so that each $G'_{i\lambda}$ lies above $G_{i\lambda}$). Then the maps $\phi^{i\lambda}_{i\mu}$ lie along the rows of the lower level, while the maps $\psi^{i\lambda}_{j\lambda}$ lie along the columns of the upper level; the maps $\theta_{i\lambda}$ are one-way elevators: to go down, one must use a θ^{-1}

From this we see that there are only two simplest ways of going from $G'_{i\lambda}$ to $G'_{j\mu}$: these are the compositions

$$\theta_{j\mu} \circ \phi^{j\lambda}_{j\mu} \circ \theta^{-1}_{j\lambda} \circ \psi^{i\lambda}_{j\lambda}; \quad \psi^{i\mu}_{j\mu} \circ \theta_{i\mu} \circ \phi^{i\lambda}_{i\mu} \circ \theta^{-1}_{i\lambda}$$

indicated on fig. 1 below. Together, they constitute the diagram in fig. 1, which is uniquely determined by i, j, λ, μ; we denote this diagram by $(i, \lambda; j, \mu)$. It is convenient to have a more compact schematic picture of $(i, \lambda; j, \mu)$ and this is given by fig. 2 below. We refer to the first of the two compositions above ($i\lambda \to j\lambda \to j\mu$ on fig. 2) as the *column-starting* side of $(i, \lambda; j, \mu)$; the other one is the *row-starting* side.

Because of the composition properties of ϕ, the juxtaposition of $(i, \lambda; j, \mu)$ and $(i, \mu; j, \nu)$, as depicted below:

$$\begin{array}{ccccc} i\lambda & \to & i\mu & \to & i\nu \\ \downarrow & & \downarrow & & \downarrow \\ j\lambda & \to & j\mu & \to & j\nu \end{array}$$

yields a "square" which is but $(i, \lambda; j, \nu)$. Similarly, juxtaposing $(i, \lambda; j, \mu)$ and $(j, \lambda; k, \mu)$ yields $(i, \lambda; k, \mu)$.

$$\begin{tikzcd}
G'_{i\lambda} \arrow[d, "\theta^{-1}"'] \arrow[ddr, "\psi"] & & G'_{i\mu} \arrow[ddr, "\psi"] & \\
G_{i\lambda} \arrow[rr, "\phi"] & & G_{i\mu} \arrow[u, "\theta"] & \\
& G'_{j\lambda} \arrow[d, "\theta^{-1}"'] & & G'_{j\mu} \\
& G_{j\lambda} \arrow[rr, "\phi"] & & G_{j\mu} \arrow[u, "\theta"']
\end{tikzcd}$$

Figure 1

$$\begin{tikzcd}
i\lambda \arrow[r] \arrow[d] & i\mu \arrow[d] \\
j\lambda \arrow[r] & j\mu
\end{tikzcd}$$

Figure 2

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6. We can now resume the proof of Theorem 1. We stopped after showing that there exists a basis such that

$$\theta_{i\lambda} \circ \phi^{i\xi}_{i\lambda} \circ \theta^{-1}_{i\xi} \circ \psi^{x\xi}_{i\xi} \circ \theta_{x\xi} \circ \phi^{x\lambda}_{x\xi} \circ \theta^{-1}_{x\lambda} \circ \psi^{i\lambda}_{x\lambda}$$

is the identity on $G'_{i\lambda}$ for all i, λ (where x, ξ are fixed). This means that (x, ξ, i, λ) commutes for all i, λ. The theorem follows from the following

LEMMA 3. *A basis is coherent if and only if there exist $x \in I$, $\xi \in \Lambda$ such that $(x, \xi; i, \lambda)$ commutes for all $i \neq x$, $\lambda \neq \xi$.*

PROOF. The condition is necessary by lemma 1, since $(x, \xi; i, \lambda)$ is a diagram of array isomorphisms. We now assume that $(x, \xi; i, \lambda)$ commutes for all $i \neq x$, $\lambda \neq \xi$, relative to a given basis p, and prove that p is coherent. Note that $(x, \xi; i, \lambda)$ commutes (trivially) if $i = x$, or if $\lambda = \xi$, and hence commutes in fact for all i, λ.

We now show that every $(i, \lambda; j, \mu)$ commutes. For this, observe that $(x, \xi; j, \mu)$ can be obtained by juxtaposition, as follows:

$$\begin{array}{ccccc} x\xi & \to & x\lambda & \to & x\mu \\ \downarrow & & \downarrow & & \downarrow \\ i\xi & \to & i\lambda & \to & i\mu \\ \downarrow & & \downarrow & & \downarrow \\ j\xi & \to & j\lambda & \to & j\mu \end{array}$$

Now all maps in this diagram are (in particular) surjective. Since $(x, \xi; i, \lambda)$ and $(x, \xi; i, \mu)$ commute, it follows that $(x, \lambda; i, \mu)$ commutes. Since $(x, \xi; j, \mu)$ and $(x, \xi; j, \lambda)$ commute, it follows that $(x, \lambda; j, \mu)$ commute. Now $(x, \lambda; j, \mu)$ and $(x, \lambda; i, \mu)$ commute, and therefore $(i, \lambda; j, \mu)$ also commutes.

7. We now let i, λ, j, μ be arbitrary and consider any array isomorphism $\alpha: G'_{i\lambda} \to G'_{j\mu}$. Thus α arises by composing a sequence of finitely many $\phi, \psi, \theta, \theta^{-1}$'s. We may from the start, owing to the composition properties of ϕ and ψ, assume that this sequence is irredundant; in other words, we may assume that it does not contain any two consecutive ϕ's, nor any consecutive ψ's, and furthermore that we never consecutively ride up and down the same $\theta_{k\nu}$. This leaves little freedom: in fact we see that, from any $G'_{k\nu}$ (such as $G'_{i\lambda}$) we can only follow one of the two sides of some $(k, \nu; m, \sigma)$ until we encounter $G'_{m\sigma}$, and proceed again. Thus α can be described as follows: there is a sequence (m_t, σ_t) $(t = 1, 2, \cdots, r+1)$ with $(m_1, \sigma_1) = (i, \lambda)$, $(m_{r+1}, \sigma_{r+1}) = (j, \mu)$ such that α is the composition of $\alpha_1, \cdots, \alpha_r$, when α_t is the composition of the maps on one of the two sides of $D_t = (m_t, \sigma_t; m_{t+1}, \sigma_{t+1})$. In fact, we may not even change sides from one D_t to the next: if we use, say, the row-starting side of D_t, then we reach

$G'_{m_{t+1}\sigma_{t+1}}$ by means of some ψ, and since we have assumed we never use consecutive ψ's we can only leave $G'_{m_{t+1}\sigma_{t+1}}$ by means of a θ^{-1} and hence follow the row-starting side of D_{t+1} (cf. fig. 1). It follows that we are only using either row-starting sides throughout, or column-starting sides throughout. A schematic illustration of α may then be as follows (in the row-starting case):

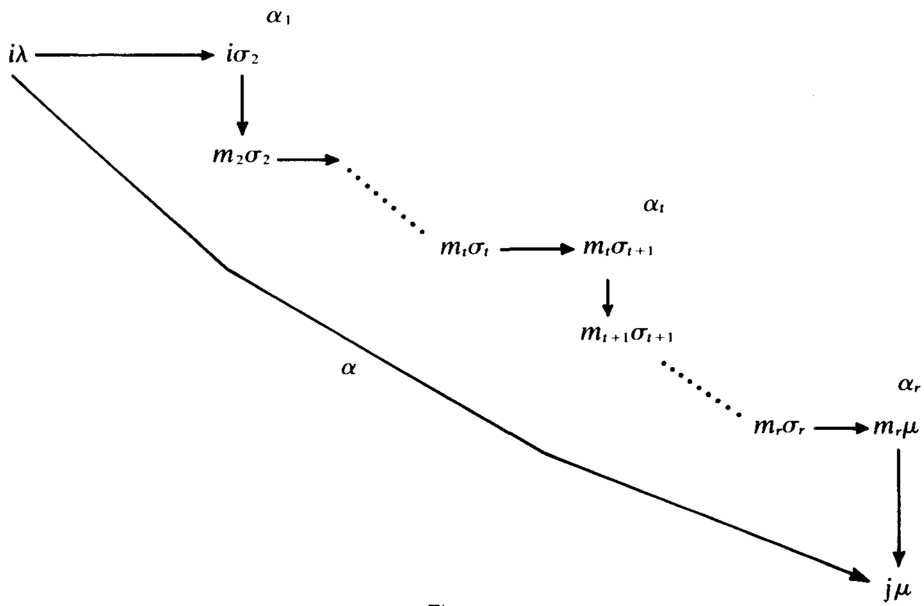

Figure 3

We now prove, by induction on r, that we would obtain the same α in just one step, by following one side of $(i, \lambda; j, \mu)$. This is trivial if $r = 1$. Assume that $r > 1$ and α is obtained by using, say, row-starting sides only. Write $\alpha = \alpha_r \circ \alpha'$, where $\alpha' = \alpha_{r-1} \circ \cdots \circ \alpha_1$: $G'_{i\lambda} \to G'_{m_r\sigma_r}$. By the induction hypothesis, α' is the composition of the maps on the row-starting side of $(i, \lambda; m_r, \sigma_r)$. When we compose α' and α_r we then see on fig. 4 that we use the column-starting side of $(i, \sigma_r; m_r, \mu)$. Since $(i, \sigma_r; m_r, \mu)$ commutes, we shall obtain the same map α by travelling along its row-starting side instead, and then the composition properties of ϕ and ψ show that α can also be obtained by composing along the row-starting side of $(i, \lambda; j, \mu)$.

Since $(i, \lambda; j, \mu)$ commutes, this part of the proof shows that there exists only one array isomorphism $G'_{i\lambda} \to G'_{j\mu}$.

8. This immediately implies that, if α, β are array isomorphisms $G'_{i\lambda} \to G_{j\mu}$, then $\theta_{j\mu} \circ \alpha = \theta_{j\mu} \circ \beta$ since these are array isomorphisms $G'_{i\lambda} \to G'_{j\mu}$, and hence

$\alpha = \beta$. In a similar manner we see that there is only one array isomorphism $G_{i\lambda} \to G'_{j\mu}$ and only one array isomorphism $G_{i\lambda} \to G_{j\mu}$ (when i, j, λ, μ are given).

Therefore, every diagram of array isomorphisms has to commute. By Lemma 1, this says that the given basis is coherent, which completes the proof.

3. The semigroup case

1. We have seen in Section 1 that any $\mathcal{D}$-class D of a semigroup gives rise, canonically, to a Schützenberger array. Our main result states:

THEOREM 2. *The canonical Schützenberger array of a $\mathcal{D}$-class is always coherent.*

$$\begin{tikzcd}
i\lambda \arrow[r] \arrow[dr, "\alpha'"] \arrow[ddrr, "\alpha"'] & i\sigma_r \arrow[r, dashed] \arrow[d] & i\mu \arrow[d, dashed] \\
& m_r\sigma_r \arrow[r] \arrow[dr, "\alpha_r"] & m_r\mu \arrow[d] \\
& & j\mu
\end{tikzcd}$$

Figure 4

PROOF. We use the notation in Section 1. Pick $1 \in I$, $1 \in \Lambda$, and elements $q_\lambda, r_i \in S^1$ $(\lambda \in \Lambda, i \in I)$ such that $H_{i1}q_\lambda = H_{i\lambda}$, $r_iH_{1\lambda} = H_{i\lambda}$ for all i, λ. [For instance, we could pick $a_\lambda \in H_{1\lambda}$; then there exist $q_\lambda, q'_\lambda \in S^1$ such that $a_\lambda = a_1q_\lambda$, $a_1 = a_\lambda q'_\lambda$, and Green's lemma says that q_λ serves; r_i can be chosen dually. This really says that the elements $p_{11}, p_{1\lambda}, p_{i1}$ below can be chosen arbitrarily.] Now select $p_{11} \in H_{11}$ and define $p_{1\lambda} = p_{11}q_\lambda$, $p_{i1} = r_ip_{11}$ and $p_{i\lambda} = r_ip_{11}q_\lambda$ for all $i \neq 1$, $\lambda \neq 1$. Note that $p_{i\lambda} \in H_{i\lambda}$ for all i, λ and hence this defines a basis.

For every $g' \in G'_{11}$, $i \neq 1$, $\lambda \neq 1$, we calculate, using formulae (1) and (2) in Section 1:

$$\begin{aligned}
p_{i\lambda} \cdot \psi^{i\lambda}_{i\lambda}\theta_{1\lambda}\phi^{11}_{1\lambda}\theta^{-1}_{11}g' &= r_ip_{1\lambda} \cdot \psi^{1\lambda}_{i\lambda}(\theta_{1\lambda}\phi^{11}_{1\lambda}\theta^{-1}_{11}g') \\
&= r_i\,(p_{1\lambda} \cdot \theta_{1\lambda}\phi^{11}_{1\lambda}\theta^{-1}_{11}g') \\
&= r_i\,(\phi^{11}_{1\lambda}\theta^{-1}_{11}g' \cdot p_{1\lambda}) = r_i\,(\phi^{11}_{1\lambda}(\theta^{-1}_{11}g') \cdot p_{11}q_\lambda) \\
&= r_i\,((\theta^{-1}_{11}g' \cdot p_{11})q_\lambda) = r_i\,(p_{11} \cdot g')q_\lambda,
\end{aligned}$$

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$$
\begin{aligned}
p_{i\lambda} \cdot (\theta_{i\lambda}\phi^{i1}_{i\lambda}\theta^{-1}_{i1}\psi^{11}_{i1}g')
&= \phi^{i1}_{i\lambda}\theta^{-1}_{i1}\psi^{11}_{i1}g' \cdot p_{i\lambda} = \phi^{i1}_{i\lambda}(\theta^{-1}_{i1}\psi^{11}_{i1}g') \cdot p_{i1}q_\lambda \\
&= (\theta^{-1}_{i1}\psi^{11}_{i1}g' \cdot p_{i1})q_\lambda \\
&= (p_{i1} \cdot \psi^{11}_{i1}g')q_\lambda = (r_i p_{11} \cdot \psi^{11}_{i1}g')q_\lambda \\
&= r_i(p_{11} \cdot g')q_\lambda .
\end{aligned}
$$

This proves that $\psi^{1\lambda}_{i\lambda} \circ \theta_{1\lambda} \circ \phi^{11}_{1\lambda} \circ \theta^{-1}_{11} = \theta_{i\lambda} \circ \phi^{i1}_{i\lambda} \circ \theta^{-1}_{i1} \circ \psi^{11}_{i1}$, i.e. $(1,1;\, i,\lambda)$ commutes, for all $i \neq 1$, $\lambda \neq 1$. Then it follows from Lemma 3 that p is a coherent basis, q.e.d.

2. We conclude by some further remarks on coherent bases. We note that the proofs of Theorem 1 (for the general case) and Theorem 2 (for the semigroup case) actually produce coherent bases (when such exist), along with the information that one row and one column can always be chosen arbitrarily. This leaves the problem of finding all coherent bases when we know one of them. Since isomorphic Schützenberger arrays have the same coherent basis, it will suffice to consider the case of a simple Schützenberger array. In that case, we may as well assume that the given basis has the property that all θ's are the identity.

PROPOSITION 2. *Let $\mathfrak{A}$ be a simple Schützenberger array with group G and p be a basis of $\mathfrak{A}$ such that all $\theta_{i\lambda}$ are the identity on G. Let $(u_{i\lambda})_{i\in I, \lambda\in\Lambda}$ be a family of elements of G. Then $q = (u_{i\lambda} \cdot p_{i\lambda})_{i\in I, \lambda\in\Lambda}$ is a coherent basis if and only if, for some fixed $1 \in I$, $1 \in \Lambda$, $u_{i\lambda}u^{-1}_{1\lambda}u_{11}u^{-1}_{i1} \in Z(G)$ (the center of G) for all $i \neq 1$, $\lambda \neq 1$.*

PROOF. By Lemma 3, q is a coherent basis if and only if

$$\psi^{1\lambda}_{i\lambda} \circ \theta^q_{1\lambda} \circ \phi^{11}_{1\lambda} \circ (\theta^q_{11})^{-1} = \theta^q_{i\lambda} \circ \phi^{i1}_{i\lambda} \circ (\theta^q_{i1})^{-1} \circ \psi^{11}_{i1},$$

i.e. $\theta^q_{1\lambda}(\theta^q_{11})^{-1}g = \theta^q_{i\lambda}(\theta^q_{i1})^{-1}g$ for all $g \in G$, for all $i \neq 1$, $\lambda \neq 1$. By Lemma 2, $\theta^q_{i\lambda}g = u^{-1}_{i\lambda}\theta^p_{i\lambda}g\,u_{i\lambda} = u^{-1}_{i\lambda}g\,u_{i\lambda}$ for all g, i, λ. Hence, for each $i \neq 1$, $\lambda \neq 1$, $\theta^q_{1\lambda}(\theta^q_{11})^{-1}g = \theta^q_{i\lambda}(\theta^q_{i1})^{-1}g$ for all g is successively equivalent to:

$$u^{-1}_{1\lambda}u_{11}gu^{-1}_{11}u_{1\lambda} = u^{-1}_{i\lambda}u_{i1}gu^{-1}_{i1}u_{i\lambda}$$

for all g; $u^{-1}_{i1}u_{i\lambda}u^{-1}_{1\lambda}u_{11}gu^{-1}_{11}u_{1\lambda}u^{-1}_{i\lambda}u_{i1} = g$ for all g;

$$
\begin{aligned}
&u^{-1}_{i1}u_{i\lambda}u^{-1}_{1\lambda}u_{11} \in Z(G);\ u_{i\lambda} \in u_{i1}Z(G)\,u^{-1}_{11}u_{1\lambda} \\
&\qquad = Z(G)\,u_{i1}u^{-1}_{11}u_{1\lambda};\ u_{i\lambda}u^{-1}_{1\lambda}u_{11}u^{-1}_{i1} \in Z(G).
\end{aligned}
$$

The result follows.

Since $1 \in I$, $1 \in \Lambda$ were arbitrary, the result tells us that when q is a coherent basis then $u_{i\lambda}u^{-1}_{j\lambda}u_{j\mu}u^{-1}_{i\mu} \in Z(G)$ for all i, j, λ, μ. The result also tells us that when all q_{i1}, $q_{1\lambda}$ are known then the remaining $q_{i\lambda}$'s are uniquely determined modulo the center of G.

4. Some applications

1. The applications of the main theorem come from the following remark. Given a $\mathscr{D}$-class D of any semigroup S and a coherent basis $p = (p_{i\lambda})_{i\in I,\lambda\in\Lambda}$ of D, we may let G be the one group G in any simple Schützenberger array isomorphic to the canonical array of D; then every element x of D can be written uniquely as $x = g \cdot p_{i\lambda}$ $(= p_{i\lambda} \cdot g)$, with $i \in I$, $\lambda \in \Lambda$, $g \in G$; this defines a mapping $x \mapsto g(x) = g$ of D into G (which of course depends on the chosen coherent basis). This mapping g can be used to make more precise some classical results on the location of products and inverses of elements of D, and this in turn has some applications.

This will use the following rules of calculation: $g \cdot p_{i\lambda} = p_{i\lambda} \cdot g$ always (since $\theta_{i\lambda}$ is the identity on G in our simple array); $g \cdot xy = (g \cdot x)y$ whenever x, $xy \in D$ and $x\mathscr{R}xy$ (since all $\phi^{i\lambda}_{i\mu}$ are the identity in our simple array); dually, $xy \cdot g = x(y \cdot g)$ whenever y, $xy \in D$ and $y\mathscr{L}xy$.

2. First we give the basic results on location of products and inverses.

Proposition 2. *Let $a, b \in D$ be such that $R_b \cap L_a$ contains an idempotent e (so that $ab \in R_a \cap L_b$); take $a = e$ if $a\mathscr{H}e$ and $b = e$ if $b\mathscr{H}e$. Then there exists a coherent basis p of D which contains a, b, e and ab. For any such basis, $g(xy) = g(x)g(y)$ for all $x \in H_a$, $y \in H_b$.*

Proof. The hypothesis that we take $a = e$ if $a\mathscr{H}e$ and $b = e$ if $b\mathscr{H}e$ insures that if two of the elements a, b, e, ab lie in the same $\mathscr{H}$-class then they are equal (this is clearly necessary for the conclusion). In what follows we assume that a, b, e, ab lie in different $\mathscr{H}$-classes and leave the degenerate cases to the reader. Put $H_a = H_{j\lambda}$, $H_b = H_{i\mu}$, so that $H_e = H_{i\lambda}$, $H_{ab} = H_{j\mu}$. We now follow the proof of Theorem 2. We can build a coherent basis from arbitrary choices of $p_{i\xi}$, $p_{x\lambda}$; choose $p_{i\lambda} = e$, $p_{j\lambda} = a$, $p_{i\mu} = b$. We now have $e\mathscr{L}a$, with $a = ae$ since e is idempotent, and $b\mathscr{R}e$ with $b = eb$; therefore we may choose $p_{j\mu} = a\ p_{i\lambda}\ b = aeb = ab$. Then a, b, e, ab are all part of the coherent basis p.

Now take any $x \in H_a$, $y \in H_b$; put $g(x) = g$, $g(y) = h$, so that $x = g \cdot a$, $y = h \cdot b$. Since $R_y \cap L_x = R_b \cap L_a$ contains an idempotent, we still have $xy \in H_{ab}(= R_x \cap L_y)$ [proposition 2 then pinpoints xy in H_{ab} in terms of the mapping g]; similarly, $ay \in H_{ab}$, so that $ay\mathscr{R}a$. Therefore

$$\begin{aligned} xy = (g \cdot a)y = g \cdot ay = g \cdot (a(h \cdot b)) = g \cdot (a(b \cdot h)) &= g \cdot ab \cdot h \\ &= g \cdot (h \cdot ab) = gh \cdot ab \end{aligned}$$

since b, ab are part of the coherent basis and $ab\mathscr{L}b$. Thus $g(xy) = gh = g(x)g(y)$.

COROLLARY. *Let b be an inverse of $a \in D$. There exists a coherent basis of D containing a, b, ab, ba. For any such basis, the inverse $y \in H_b$ of any $x \in H_a$ satisfies $g(y) = g(x)^{-1}$.*

PROOF. One has $g(xy) = 1$ since $xy = ab$ is part of the coherent basis.

3. Applications to congruences contained in $\mathcal{H}$ come from the following result, which in essence is a restatement of Leech's description of these congruences by normal subfunctors in Leech (1975).

PROPOSITION 3. *Let $\mathcal{C}$ be a congruence on S contained in $\mathcal{H}$. Relative to any coherent basis of D, there exists a normal subgroup K of G such that, for all $x, y \in D$, $x\mathcal{C}y$ if and only if $x\mathcal{H}y$ and $g(x) \in g(y)K$.*

PROOF. Let p be any coherent basis of D. Pick any $i \in I, \lambda \in \Lambda$ and define $K = \{g \in G;\ g = g(x)$ for some $x \in H_{i\lambda}$ with $x\mathcal{C}p_{i\lambda}\}$. Note that $1 \in K$. First we show that, for any $y \in D$, say $y \in H_{j\mu}$, then $y\mathcal{C}p_{j\mu}$ if and only if $g(y) \in K$ (i.e. K does not depend on i, λ). First take $y \in H_{i\mu}$, say $y = g \cdot p_{i\mu}$. Since $p_{i\lambda}\mathcal{R}p_{i\mu}$ there exist $t, t' \in S^1$ such that $p_{i\lambda} = p_{i\mu}t$, $p_{i\mu} = p_{i\lambda}t'$. Now $y\mathcal{C}p_{i\mu}$ implies $yt\mathcal{C}p_{i\lambda}$, and $yt = (g \cdot p_{i\mu})t = g \cdot p_{i\mu}t = g \cdot p_{i\lambda}$; hence $y\mathcal{C}p_{i\mu}$ implies $g \in K$. If conversely $g \in K$, then $yt\mathcal{C}p_{i\lambda}$, which implies $ytt'\mathcal{C}p_{i\mu}$; but $p_{i\mu}tt' = p_{i\mu}$ implies $ytt' = y$ for all $y \in H_{i\mu}$. Thus, $g \cdot p_{i\mu}\mathcal{C}p_{i\mu}$ if and only if $g \in K$, i.e. $g \cdot p_{i\lambda}\mathcal{C}p_{i\lambda}$; a dual argument shows that $g \cdot p_{j\mu}\mathcal{C}p_{j\mu}$ if and only if $g \cdot p_{i\mu}\mathcal{C}p_{i\mu}$.

Next we show that K is a subgroup of G. We have $1 \in K$. Let $g, h \in K$; there exist $s, t \in S^1$ such that $g \cdot x = sx$, $h \cdot x = tx$ for all $x \in H_{i\lambda}$. Now let $x = p_{i\lambda}$; from $g \in K$ we know that $sx\mathcal{C}x$; similarly $tx\mathcal{C}x$; since $\mathcal{C}$ is a congruence, it follows that $stx\mathcal{C}x$ and so $gh \in K$. Similarly let $s' \in S^1$ be such that $g^{-1} \cdot x = s'x$ for all $x \in H_{i\lambda}$; with $x = p_{i\lambda}$, we now have $x\mathcal{C}sx$, which implies $s'x\mathcal{C}s'sx = x$ and $g^{-1} \in K$.

Finally, take $x, y \in D$ with $x\mathcal{H}y$, say $x = g \cdot p_{j\mu}$, $y = h \cdot p_{j\mu}$; let $s, t, t' \in S^1$ be such that $g \cdot z = sz$, $h \cdot z = tz$, $h^{-1} \cdot z = t'z$ for all $z \in H_{j\mu}$. Now assume $x\mathcal{C}y$. Taking $z = p_{j\mu}$ we have $sz\mathcal{C}tz$, whence $t'sz\mathcal{C}t'tz = z$ and $h^{-1}g \in K$ by the first part of the proof. Conversely, $h^{-1}g \in K$ implies $t'sz\mathcal{C}z$ and $x = sz = tt'sz\mathcal{C}tz = y$. Thus $x\mathcal{C}y$ if and only if $g \in hK$, i.e. $g(x) \in g(y)K$. But we also have $x = z \cdot g$, $y = z \cdot h$ and there exist $u, v, v' \in S^1$ such that $a \cdot g = au$, $a \cdot h = av$, $a \cdot h^{-1} = av'$ for all $a \in Hj\mu$; hence we can argue as above, i.e. $x\mathcal{C}y$ if and only if: $zu\mathcal{C}zv$, $zuv'\mathcal{C}z$ and $gh^{-1} \in K$ (as $zuv' = z \cdot gh^{-1} = gh^{-1} \cdot z$). In particular $g \in hK$ if and only if $g \in Kh$ and the proof is now complete.

This proof is quite similar to Leech's argument; what coherence has contributed is merely a neater description.

Our first application which does not outwardly involve coherence is the following:

COROLLARY. *Let $\mathscr{C}$ be a congruence contained in $\mathscr{H}$. Assume $x\mathscr{H}y$ and let x', y' be inverses of x, y such that $x'\mathscr{H}y'$. Then $x\mathscr{C}y$ implies $x'\mathscr{C}y'$.*

PROOF. First note that any $a \in H_x$ then has an inverse $b \in H_{x'}$. If $x, x', xx', x'x$ do not lie in four different $\mathscr{H}$-classes, then we see that H_x, $H_{x'}$ are maximal subgroups of S and that if a is the identity of H_x then b is the identity of $H_{x'}$ (since in this case $x\mathscr{R}x'$ or $x\mathscr{L}x'$). Otherwise we may take $a = x$, $b = x'$. Pick a coherent basis of D_a as in the corollary to Prop. 1. The result then follows at once from the rest of the corollary and from Proposition 3.

An interesting fact about this result is that we lack a direct proof of it (= one that does not use coherence in an obvious way).

3. We now give some applications to orthodox semigroups (as defined in Hall (1969)). The basic fact here is:

PROPOSITION 4. *Any $\mathscr{D}$-class D of an orthodox semigroup has a coherent basis which contains all the idempotents.*

PROOF. Assume that S is orthodox; let D be a $\mathscr{D}$-class of S and $E = E_S \cap D$ be the set of idempotents of D. The $\mathscr{D}$ relation of the band E_S is contained in that of S and therefore E is a union of $\mathscr{D}$-classes $(E_\alpha)_{\alpha \in A}$ of the band E_S; each E_α is a rectangular band.

Select one e_α in each E_α; also select $0 \in A$, so that we have $e_0 \in E_0$. For each $\alpha \in A$, pick $r_\alpha \in R_{e_\alpha} \cap L_{e_0}$ and let q_α be the inverse of r_α in $R_{e_0} \cap L_{e_\alpha}$. When $\alpha = 0$ we let $r_0 = q_0 = e_0$.

For each $a \in E_\alpha$, let $r_a = ar_\alpha$. Since $R_{r_\alpha} \cap L_a = R_{e_\alpha} \cap L_a$ contains the idempotent $e_\alpha a$, we have $r_a \in R_a \cap L_{e_0}$. Furthermore $a\mathscr{R}b \in E$ implies $b \in E_\alpha$ and, in the rectangular band E_α, $ae_\alpha = be_\alpha$; hence also

$$ar_\alpha = ar_\alpha q_\alpha r_\alpha = ae_\alpha r_\alpha = be_\alpha r_\alpha = br_\alpha.$$

As a result, the elements r_a $(a \in E)$ yield but one element in every $\mathscr{H}$-class in L_{e_0}. Also note that $r_{e_0} = e_0$. Dually, the elements $q_a = q_\alpha a$ $(a \in E)$ yield just one element of each $\mathscr{H}$-class in R_{e_0}, with $q_{e_0} = e_0$. As a consequence, a coherent basis for D can be constructed as in Theorem 2 from these elements: we have $r_{e_0}\mathscr{L}r_a$, with $r_a = r_a r_{e_0}$ (since $r_{e_0} = e_0$) and $q_{e_0}\mathscr{R}q_b$, with $q_b = q_{e_0}q_b$, and hence the coherent basis element in $R_a \cap L_b$ may for all a, b be chosen to be $r_a r_{e_0} q_b = r_a q_b$ (this depends only on $R_a \cap L_b$). This coherent basis contains all r_a, q_b.

Now, for each $a \in E$, say $a \in E_\alpha$, the coherent basis element in $H_a = R_a \cap L_a$ is $r_a q_a = ar_\alpha q_\alpha a = ae_\alpha a = a$ (as $a\mathscr{D}e_\alpha$ in E_S), i.e. our coherent basis contains all the idempotents of D.

We express this result by saying that *the idempotents of an orthodox semigroup are coherent.* Coherence of idempotents is not characteristic of orthodox semigroups: for example, one can build a completely 0-simple semigroup from a sandwich matrix that has ones in a full row and a full column and zeroes everywhere else; it is fairly clear that the idempotents of this semigroup are coherent, yet (if I, Λ are not trivial) it is not an orthodox semigroup.

The coherent basis in Prop. 4. has another valuable property: the basis elements r_aq_b, r_bq_a in $R_a \cap L_b$, $R_b \cap L_a$ are always mutually inverse. To see this, first note that $q_ar_a \in H_{e_0}$ (since $R_{r_a} \cap L_{q_a}$ contains the idempotent a); since $q_ar_aq_ar_a = q_aar_a = q_at_a$, it follows that $q_ar_a = e_0$. Hence $r_aq_br_bq_ar_aq_b = r_ae_0q_b = r_aq_b$, and $r_bq_ar_aq_br_bq_a = r_bq_a$. In other words, when $x, y \in D$ are mutually inverse, the basis elements in H_x, H_y are mutually inverse too. Then it follows from the corollary to Proposition 2 that (in this coherent basis), x and y are mutually inverse if and only if x has an inverse in H_y and $g(y) = g(x)^{-1}$.

We can use this, and the fact that the idempotents of an orthodox semigroup S form a rectangular pattern (i.e. when $H_{i\lambda}$, $H_{i\mu}$ and $H_{j\lambda}$ all contain idempotents, then $H_{j\mu}$ also contains an idempotent—namely the inverse in $H_{j\mu}$ of the idempotent in $H_{i\lambda}$), to give an alternate proof that Yamada's relation ($x\mathcal{Y}y$ if and only if x and y have a common inverse) is the least inverse congruence on S. The difficult part of the proof is to show that $\mathcal{Y}$ is transitive (cf. Hall (1969), Schein (1965) and Yamada (1970)); equivalently, if s and y, y and t, t and z are mutually inverse, then so are s and z. We first note that $R_s \cap L_y$, $R_t \cap L_y$, $R_t \cap L_z$ all contain idempotents, so that $R_s \cap L_z$ contains an idempotent; similarly, $R_z \cap L_s$ contains an idempotent. Therefore s has an inverse in R_z; also $g(z) = g(t)^{-1} = g(y) = g(s)^{-1}$, and thus s and z are mutually inverse.

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