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ON THE "THIRD DEFINITION" OF THE TOPOLOGY ON THE SPECTRUM OF A C^*-ALGEBRA

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0. In [**3**], Fell introduced a topology on $\mathrm{Rep}(A, H)$, the collection of all non-null but possibly degenerate *-representations of the C^*-algebra A on the Hilbert space H. This topology, which we will call the Fell topology, can be described by giving, as basic open neighbourhoods of $\pi_0 \in \mathrm{Rep}(A, H)$, sets of the form

$$\{\pi \in \mathrm{Rep}(A, H): \|\pi(a_i)\xi_j - \pi_0(a_i)\xi_j\| < \epsilon, 1 \leqq i \leqq n, 1 \leqq j \leqq m\},$$

where the $a_i \in A$, and the $\xi_j \in H(\pi_0)$, the essential space of π_0 [**4**].

A principal result of [**3**, Theorem 3.1] is that if the Hilbert dimension of H is large enough to admit all irreducible representations of A, then the quotient space $\mathrm{Irr}(A, H)/\sim$ can be identified with the spectrum (or "dual") $\hat{A}$ of A, in its hull-kernel topology. (Here, $\mathrm{Irr}(A, H)$ is the subset of $\mathrm{Rep}(A, H)$ consisting of representations π which are irreducible on $H(\pi)$, with the relativized Fell topology. Equivalence $\sim$ is via partial isometries in $\mathscr{L}(H)$. Unitary equivalence will be denoted by $\simeq$.)

Recently, another topology, which we will call the *strong* topology, has been found useful on $\mathrm{Rep}(A, H)$ [**2**; **5**; **1**]. Strong basic open neighbourhoods are defined as for the Fell topology, except that the ξ_j are now arbitrary in H. The strong topology is Hausdorff, and clearly stronger than the Fell topology (which is usually not Hausdorff).

Our aim in this note is to show that the strong and Fell topologies confer the same topology on the quotient $\mathrm{Irr}(A, H)/\sim$, at least if H is infinite-dimensional and "large enough" in the sense made precise above. Thus, our main result is the following.

THEOREM. *Let A be a C^*-algebra, and H an infinite-dimensional Hilbert space such that irreducible representations of every equivalence class in $\hat{A}$ can be realized on* (*subspaces of*) *H. Let* $\mathrm{Irr}(A, H)$ *be given the relative strong topology. Then the natural mapping of* $\mathrm{Irr}(A, H)$ *onto $\hat{A}$ is continuous and open, making* $\mathrm{Irr}(A, H)/\sim$ *homeomorphic to $\hat{A}$.*

Remark A. This result is intimated in [**2**, § 3.5], which gives the paper its title. The discussion there takes another direction, however, and seems to leave this result open.

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In § 1, we prove the theorem only for H "too large" for any irreducible representation. In § 2, we lift this restriction.

1. We begin with a trivial lemma.

LEMMA 1. *Let $\mathscr{V}$ be the strong, open neighbourhood of π_0 in* $\mathrm{Rep}(A, H)$ *defined by the conditions*

$$||\pi(a_i)\xi_j - \pi_0(a_i)\xi_j|| < \epsilon, \qquad 1 \leqq i \leqq n, \quad 1 \leqq j \leqq m.$$

Let K be any closed linear subspace of H containing $H(\pi_0)$ and let $\xi_j = \xi_j^{(1)} + \xi_j^{(2)}$, where $\xi_j^{(1)} \in K$, $\xi_j^{(2)} \in K^\perp$, $1 \leqq j \leqq m$. If $\mathscr{W}$ is the set of all $\pi \in$ $\mathrm{Rep}(A, H)$ *satisfying*

$$||\pi(a_i)\xi_j^{(1)} - \pi_0(a_i)\xi_j^{(1)}|| < \tfrac{1}{2}\epsilon \quad \text{and} \quad ||\pi(a_i)\xi_j^{(2)}|| < \tfrac{1}{2}\epsilon,$$
$$1 \leqq i \leqq n, \quad 1 \leqq j \leqq m,$$

then $\mathscr{W}$ is a strong, open neighbourhood of π_0 in $\mathrm{Rep}(A, H)$, *and $\mathscr{W} \subset \mathscr{V}$.*

Proof. The first statement follows immediately from the fact that $\pi_0(a_i)\xi_j^{(2)} = 0$, which implies also that

$$\begin{aligned} ||\pi(a_i)\xi_j - \pi_0(a_i)\xi_j|| &= ||\pi(a_i)\xi_j^{(1)} - \pi_0(a_i)\xi_j^{(1)} + \pi(a_i)\xi_j^{(2)}|| \\ &\leqq ||\pi(a_i)\xi_j^{(1)} - \pi_0(a_i)\xi_j^{(1)}|| + ||\pi(a_i)\xi_j^{(2)}|| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon \end{aligned}$$

if $\pi \in \mathscr{W}$, whence the second statement.

Definition. A Hilbert space H will be called *large enough* (for A) if its Hilbert dimension $\dim H$ satisfies $\dim H \geqq \dim \pi$ for all irreducible representations π of A, and *too large* (for A) if the inequality is strict. For example, if $A \neq 0$, and $\dim H > \mathrm{Card}\, A$, then H is too large for A.

Note that if H is too large for A and infinite-dimensional, then $\dim(H \ominus H(\pi)) = \dim H$ for all $\pi \in \mathrm{Irr}(A, H)$. Thus in $\mathrm{Irr}(A, H)$, equivalence and unitary equivalence coincide. (This useful remark was made to me by J. Witol, in conversation.)

LEMMA 2. *If H is infinite-dimensional and too large for A, then a strongly closed, saturated subset S of* $\mathrm{Irr}(A, H)$ *is Fell closed in* $\mathrm{Irr}(A, H)$.

Recall that if $\sim$ is an equivalence relation on a set X, and $S \subset X$, then the saturation $S^\sim$ of S is the set

$$\{x \in X: x \sim y \text{ for some } y \in S\}.$$

S is saturated if and only if $S = S^\sim$.

Proof. Assuming the contrary, choose $\pi_0 \notin S$ in the Fell closure of S, and let $\mathscr{V}$ be a strong basic neighbourhood of π_0 such that $\mathscr{V} \cap S = \emptyset$. We may suppose (Lemma 1) that

$$\mathscr{V} = \{\pi \in \mathrm{Irr}(A, H): ||\pi(a_i)\xi_j - \pi_0(a_i)\xi_j|| < \epsilon, 1 \leqq i \leqq n, 1 \leqq j \leqq m\},$$

where

$$\xi_j \in H(\pi_0) \quad \text{for } 1 \leqq j \leqq r,$$
$$\xi_j \in H(\pi_0)^{\perp} \quad \text{for } r+1 \leqq j \leqq m.$$

Let F be the linear span of the ξ_j, $r + 1 \leqq j \leqq m$, and let

$$\mathscr{V}_F = \{\pi \in \operatorname{Irr}(A, H) \colon \|\pi(a_i)\xi_j - \pi_0(a_i)\xi_j\| < \epsilon,\ 1 \leqq i \leqq n,\ 1 \leqq j \leqq r\}.$$

Then $\mathscr{V}_F$ is a Fell neighbourhood of π_0, and so by hypothesis there is a $\pi \in S \cap \mathscr{V}_F$.

Now let K be the closed subspace $H(\pi_0) \vee H(\pi) \vee F$ of H. Then $H \ominus K$ is infinite-dimensional, and so there exists a partial isometry V in $\mathscr{L}(H)$ with K as initial space such that V is the identity on $K \ominus F$, and $V(F) \perp K$. Define $\pi_1(a)$ to be $V\pi(a)V^*$ $(a \in A)$ (or, for brevity, $\pi_1 = V\pi V^*$). Then $\pi_1 \in \operatorname{Irr}(A, H)$, since $H(\pi) \subset K =$ support $V =$ range V^*. Also, $\pi_1 \sim \pi$ and S saturated imply $\pi_1 \in S$, while

$$H(\pi_1) \subset \text{support } V^* = \text{range } V \perp F.$$

Therefore, $\|\pi_1(a_i)\xi_j - \pi_0(a_i)\xi_j\| = 0$ for $r + 1 \leqq j \leqq m$, while for $1 \leqq j \leqq r$,

$$\begin{aligned}\|\pi_1(a_i)\xi_j - \pi_0(a_i)\xi_j\| &= \|V\pi(a_i)V^*\xi_j - VV^*\pi_0(a_i)\xi_j\| \\ &\leqq \|\pi(a_i)\xi_j - V^*\pi_0(a_i)\xi_j\| \\ &= \|\pi(a_i)\xi_j - \pi_0(a_i)\xi_j\| \\ &< \epsilon,\end{aligned}$$

since $\pi \in \mathscr{V}_F$. But then $\pi_1 \in \mathscr{V} \cap S$, contrary to the choice of $\mathscr{V}$. This completes the proof of the lemma.

Since the strong topology is stronger than the Fell topology on $\operatorname{Irr}(A, H)$, the lemma shows that the two topologies have the same closed, saturated sets. This establishes our Theorem for H too large, except for the openness of the natural map, which appears in Corollary 6 below.

2. Now let H be infinite-dimensional and too large, as before, and let H_1 be a closed subspace of H which is infinite-dimensional and large enough. We consider the natural injection $j \colon \operatorname{Irr}(A, H_1) \to \operatorname{Irr}(A, H)$.

LEMMA 3. *j is strongly continuous; $j(\pi_1) \sim j(\pi_2)$ if and only if $\pi_1 \sim \pi_2$.*

The proof is trivial.

COROLLARY 4. *For any subset $S \subset \operatorname{Irr}(A, H_1)$,*

$$\begin{aligned}S^{\sim} &= (j(S))^{\sim} \cap \operatorname{Irr}(A, H_1) \\ &= (j(S))^{\approx} \cap \operatorname{Irr}(A, H_1) \\ &= \Bigg(\bigcup_{\substack{W \text{ unitary,} \\ W \in \mathscr{L}(H)}} Wj(S)W^*\Bigg) \cap \operatorname{Irr}(A, H_1).\end{aligned}$$

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Here, of course, $S^\sim$ is the saturation of S in $\operatorname{Irr}(A, H_1)$.

The proof is an easy consequence of Lemma 3 together with Witol's remark (§ 1).

LEMMA 5. *If S is a strongly open subset of* $\operatorname{Irr}(A, H_1)$, *then* $(j(S))^\sim$ *is open in* $\operatorname{Irr}(A, H)$.

Proof. We may suppose that $\pi_0 \in \operatorname{Irr}(A, H_1)$, and that

$$S = \{\pi \in \operatorname{Irr}(A, H_1): \|\pi(a_i)\xi_j - \pi_0(a_i)\xi_j\| < \epsilon,\ 1 \leqq i \leqq n,\ 1 \leqq j \leqq m\},$$

where $\xi_j \in H_1$. Let

$$S' = \{\pi \in \operatorname{Irr}(A, H): \|\pi(a_i)\xi_j - \pi_0(a_i)\xi_j\| < \epsilon,\ 1 \leqq i \leqq n,\ 1 \leqq j \leqq m\}.$$

This is a strong basic neighbourhood of π_0 in $\operatorname{Irr}(A, H)$. We will show that $S' \subset (j(S))^\sim$.

In fact, let $\pi' \in S'$, and let G be the finite-dimensional linear span of the ξ_j and the $\pi_0(a_i)\xi_j$, $1 \leqq i \leqq n$, $1 \leqq j \leqq m$. Then $G \subset H_1$, and $H_1 \ominus G$ (like H_1) is infinite-dimensional and large enough for A. Let W be a partial isometry in $\mathscr{L}(H)$ which is the identity on G, which maps $(G \vee H(\pi')) \ominus G$ into $H_1 \ominus G$, and such that $W^*W = G \vee H(\pi')$.

Then $\pi = W\pi'W^*$ is defined in $\operatorname{Irr}(A, H)$, since $H(\pi') \subset G \vee H(\pi') =$ range W^*, and $\pi \sim \pi'$. But also, $\pi \in j(S)$. For $H(\pi) \subset$ range $W \subset H_1$; while, just as in the proof of Lemma 2,

$$\begin{aligned}\|\pi(a_i)\xi_j - \pi_0(a_i)\xi_j\| &= \|W\pi'(a_i)W^*\xi_j - WW^*\pi_0(a_i)\xi_j\| \\ &\leqq \|\pi'(a_i)\xi_j - W^*\pi_0(a_i)\xi_j\| \\ &= \|\pi'(a_i)\xi_j - \pi_0(a_i)\xi_j\| \\ &< \epsilon, \qquad 1 \leqq i \leqq n, \quad 1 \leqq j \leqq m.\end{aligned}$$

Thus $\pi' \in (j(S))^\sim$, as claimed. Now if $\pi_1 \in (j(S))^\sim$ is arbitrary, then $\pi_1 \sim \pi_2$ for some $\pi_2 \in j(S)$, and so $\pi_1 \simeq \pi_2$. Let $\pi_1 = V\pi_2V^*$, V unitary in $\mathscr{L}(H)$. We have just seen that $(j(S))^\sim$ contains a strong neighbourhood $\mathscr{N}$ of π_2 in $\operatorname{Irr}(A, H)$. Therefore, $V(j(S))^\sim V^* = (j(S))^\sim$ contains the strong neighbourhood $V\mathscr{N}V^*$ of π_1. This proves the lemma.

COROLLARY 6. *In* $\operatorname{Irr}(A, H_1)$, *the saturation of an open set is open. Thus, the quotient map* q: $\operatorname{Irr}(A, H_1) \to \operatorname{Irr}(A, H_1)/\sim$ *is open.*

Proof. Let S be open in $\operatorname{Irr}(A, H_1)$. Then by Corollary 4,

$$S^\sim = (j(S))^\sim \cap \operatorname{Irr}(A, H_1),$$

while by Lemma 5, $(j(S))^\sim$ is open, and so by Lemma 3, $j(S)^\sim \cap \operatorname{Irr}(A, H_1)$ is open in $\operatorname{Irr}(A, H_1)$.

Now we have only to fit the pieces together. In the following diagram, the vertical arrows are quotient maps, and the horizontal ones exist, top to bottom, respectively, by virtue of the two statements of Lemma 3.

$$\begin{array}{ccccc} \mathrm{Irr}(A, H_1) = X & \xrightarrow{j} & Y = \mathrm{Irr}(A, H) \\ q \downarrow & & \downarrow r \\ \mathrm{Irr}(A, H_1)/\sim\; = Q & \xrightarrow{\tilde{j}} & R = \mathrm{Irr}(A, H)/\sim \end{array}$$

$\tilde{j}$ is a continuous bijection, and

(1) $$\tilde{j} \circ q = r \circ j.$$

Then, for any subset U of Q,

(2) $$r^{-1}(\tilde{j}(U)) = (j(q^{-1}(U)))^{\sim},$$

for each member represents a saturated subset of Y, and the application of r to each member yields, with (1) (since $r(T) = r(T^{\sim})$ for any T) simply $\tilde{j}(U)$.

Now, if U is open in Q, Lemma 5 shows that the right member is open, whence $\tilde{j}(U)$ is open in R. Thus $\tilde{j}$ is an *open* continuous bijection, and hence a homeomorphism.

The above discussion is summarized in the following result.

PROPOSITION 7. *The strong quotient topology on* $\mathrm{Irr}(A, H)/\sim$ *does not depend on the choice of H in the class of infinite-dimensional Hilbert spaces large enough for A.*

In [**3**], Fell proved the analogue of Proposition 7 for the Fell quotient topology (without insisting on the infinite-dimensionality of H).

Proposition 7, with Corollary 6 and the results of § 1, establishes the Theorem.

Remark B. If the theorem were modified by deleting "infinite-dimensional", the statement would be false. This is seen by considering the C^*-algebra A of [**2**, 4.7.19]: Sequences x of 2×2 complex matrices with diagonal limits

$$\begin{pmatrix} \lambda(x) & 0 \\ 0 & \mu(x) \end{pmatrix}.$$

It is well known that λ and μ cannot be separated by open sets in $\hat{A}$, but if we take $\dim H = 2$, it is easy to see that in the strong quotient space, $\mathrm{Irr}(A, H)/\sim$, λ and μ are isolated points.

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