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BASIC COMMUTATORS AND MINIMAL MASSEY PRODUCTS

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The purpose of this paper is to continue the investigation into Massey products defined on two dimensional polyhedra, initiated in [**13**]. It will be shown that for many such spaces there is a hyperbolic model which can be used to study Massey products. More precisely, Massey products may be interpreted as intersections of geodesics in the Poincaré model. These elements are called minimal Massey products and are the analogue of Massey products over a system considered in Porter's paper. They enjoy the property of being uniquely defined (without indeterminacy) and of being multilinear and natural. Minimal products also satisfy symmetry properties generalising the symmetry properties enjoyed by cup products.

A device which will be useful in the proof of the main theorem, 7.4, is the introduction of a class of complexes called basic complexes. These generalise the notion of a surface and each one houses a standard copy of a Massey product. As an example consider the complementary space of the Borromean rings. This has a two dimensional spine which is the union of three basic complexes of weight three. In [**9**] it is shown that this space has a well defined three fold Massey product. This Massey product can be defined by the intersections of a suitable system of geodesics in the hyperbolic plane.

The value of the minimal product will be shown to be an appropriate element in the Magnus ring. The relation with the $\bar{\mu}$-invariants [**11**] has already been shown in [**13**] and [**15**].

All homology will be with integer coefficients. A later paper will deal with more general coefficients.

1. Preliminary notation. A presentation $K = \{x_1, \dots, x_n | r_1, \dots, r_m\}$ of a group will be consistently confused with the associated two dimensional cell complex with one 0-cell, n 1-cells and m 2-cells attached by the words r_i, $i = 1, \dots, m$.

Definition 1.1. The symbol $[a, b]$ in a group means the commutator $aba^{-1}b^{-1}$. A multi index or index for short $I = (i_1, \dots, i_k)$ is a finite sequence of indices $1 \leqq i_j \leqq n$. Two indices I and J can be multiplied by

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juxtaposition. If $I = I_1I_2I_3$ then I_1, I_2 and I_3 are called subindices of I. The empty index $\emptyset$ is allowed and is a subindex of every index. Write $J \to I$ if I can be obtained from J by deleting subindices and permuting the remainder. Rather more important is the negative $J \not\to I$. Note that if $J = J_1J_2$, $I = I_1I_2$ and $J \not\to I$ then either $J_1 \not\to I_1$ or $J_2 \not\to I_2$. Finally it is convenient to define the length $l(i_1, \ldots, i_k)$ of an index to be k.

2. Massey products. Let K be a cell complex and let $C^P(K)$ denote the cellular cochain groups defined over some coefficient ring. Assume that K is equipped with an associative cochain approximation to the cup product written

$$[\xi] \cup [\eta] = [\xi][\eta] = [\xi\eta], \quad \text{see [16]}.$$

Definition 2.1. Given elements u_i in H^1K, $i = 1, \ldots, n$ a *defining set* for the Massey product of length n, $\langle u_1, \ldots, u_m\rangle$ or $\langle u_I\rangle$, where $I = (1, \ldots, n)$, is an upper triangle array

$$(a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n-1} & * \\ & a_{22} & \ldots & a_{2n-1} & a_{2n} \\ & & \cdot & \cdot & \cdot \\ & & & & a_{nn} \end{bmatrix}$$

satisfying the following conditions:

1. a_{ij} lies in $C^1(K)$, $1 \leqq i \leqq j \leqq n$ and $(i, j) \neq (1, n)$.
2. a_{ii} is a cocycle representation of u_i, $i = 1, 2, \ldots, n$ and
3. if

$$\widetilde{a}_{ij} = \sum_{k=i}^{j-1} a_{ik}a_{k+1,j'}; \quad 1 \leqq i < j \leqq n$$

then

$$\widetilde{a}_{ij} = \delta a_{ij}, (i, j) \neq (1, n).$$

The element corresponding to the right-hand corner

$$\widetilde{a}_{1n} = \sum_{k=1}^{n-1} a_{lk}a_{k+1,n}$$

is then a cocycle representing some class in H^2K, written $u(a_{ij})$ and called the value of the defining set. The product $\langle u_1, \ldots, u_n\rangle$ is defined if there is some defining set for it. In this case $\langle u_1, \ldots, u_n\rangle$ is the subset of H^2K consisting of all values $u(a_{ij})$. The submatrices a_{ij}, $i' \leqq i \leqq j \leqq j'$ define Massey products of length $j' - i' + 1$ called *subproducts*. So that in order for $\langle u_1, \ldots, u_n\rangle$ to be defined it is necessary (but not sufficient) that each of these Massey subproducts, for $i' < j'$, contain zero. If they do contain zero and zero only then $\langle u_1, \ldots, u_n\rangle$ is defined and is said to be *strictly defined*. The indeterminacy In $\langle u_1, \ldots, u_n\rangle$ of a Massey product is the subset

$$\{a - b | a, b \in \langle u_1, \ldots, u_n \rangle \}.$$

A lemma will now be stated. Its proof can be found in [**10**].

LEMMA 2.2. *If $\langle u_1, \ldots, u_n \rangle$ is strictly defined then the indeterminacy*

$$\text{In } \langle u_1, \ldots, u_n \rangle = \{0\}$$

if and only if each

$$\langle u_1, \ldots, u_{k-1}, x_k, u_{k+2}, \ldots, u_n \rangle = \{0\},$$

where $1 \leqq k \leqq n - 1$ and x_k is any element of H^1K.

Massey products satisfy a naturality condition defined only up to inclusion because of indeterminacy. Let $f:L \to K$ be a cellular map and suppose that $u_1, \ldots, u_n$ are elements of H^1K.

LEMMA 2.3. *If $\langle u_1, \ldots, u_n \rangle$ is defined then $\langle f^*u_1, \ldots, f^*u_n \rangle$ is defined and*

$$f^* \langle u_1, \ldots, u_n \rangle \subset \langle f^*u_1, \ldots, f^*u_n \rangle.$$

3. Homology of complexes. The purpose of this section is to introduce the notation which will be used in the sequel concerning the cohomology of the group presentations.

Let $K = \{x_1, \ldots, x_n | R\}$, $L = \{y_1, \ldots, y_m | S\}$ be complexes. The cellular chain groups $C_1(K)$ and $C_1(L)$ are generated by the x_i and y_i respectively. Let x_i^* be the cochain defined by

$$x_i^*(x_j) = \delta_{ij}.$$

Then the x_i^* generate $C^1(K)$. Similarly the y_i^* generate the cochain group $C^1(L)$. The chain groups $C_2(K)$ and $C_2(L)$ are generated by $r_i \in R$ and $s_i \in S$. With a notation similar to that above $C^2(K)$ and $C^2(L)$ are generated by r_i^* and s_i^* respectively.

Let $\epsilon_i(r_j)$ denote the total exponent of x_i in r_j. Although x_i and y_i are always cycles, x_i^* is only a cocycle if

$$\epsilon_i(r_j) = 0 \quad \text{for all } r_j \in R.$$

Dually r_i^* is always a cocycle but r_i is only a cycle if

$$\epsilon_j(r_i) = 0 \quad \text{for all } j.$$

LEMMA 3.1. *Let $K = \{x_1, \ldots, x_n | R\}$, $L = \{y_1, \ldots, y_m | S\}$ be complexes. Assume that all $\epsilon_i(r_j)$ and all $\epsilon_i(s_j)$ vanish. Then with the notation above*

(a) *H^1K is freely generated by the $u_i = [x_i^*]$, $1, \ldots, n$ and $H^1(L)$ is freely generated by the $v_i = [y_i^*]$, $i = 1, \ldots, m$.*

(b) *$H^2(K)$ is freely generated by the $\rho_i = [r_i^*]$, $r_i \in R$ and $H^2(L)$ is freely generated by the $\sigma_i = [s_i^*]$, $s_i \in S$.*

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Let $\phi:L \rightarrow K$ be a cellular map and let $[\phi_i^{\ j}]$ be the $n \times m$ matrix

$$[\phi_i^{\ j}] = [\epsilon_i(\phi(y_j))].$$

Consider also the matrix $[\lambda_{ij}]$ which is given by the total exponent of r_i in $\phi(s_j)$.

LEMMA 3.2. *With the notation above the induced map*
(c) $\phi^*:H^1(K) \rightarrow H^1(L)$ *is given by*

$$\phi^*(u_i) = \sum_j \phi_i^{\ j} v_j.$$

(d) $\phi^*:H^2(K) \rightarrow H^2(L)$ *is given by*

$$\phi^*(\rho_i) = \sum_j \lambda_{ij}\sigma_j.$$

Proof. For (c) note that

$$(\phi^*(u_i), [y_j]) = (x_i^*, \phi(y_j)) = \epsilon_i(\phi(y_j)) = \phi_i^{\ j}.$$

For (d) suppose that

$$\phi(s_j) = \prod_{j_k} g_{j_k} r_{j_k}^{\ \eta_{j_k}} g_{j_k}^{-1}.$$

Then

$$\lambda_{ij} = \sum_{j_k = i} \eta_{j_k}$$

and so

$$(\phi^*(\rho_i), [s_j]) = (r_i^*, \phi(s_j)) = \lambda_{ij}.$$

4. Magnus expansions and Fox derivatives. In what follows F will always denote the free group on some variables $x_1, \ldots, x_n$. Its group ring will be denoted by $\mathbf{Z}[F]$. The augmentation homomorphism

$$\epsilon:\mathbf{Z}[F] \rightarrow \mathbf{Z}$$

is induced by $\epsilon(g) = 1$ if $g \in F$.

For each i, $1 \leqq i \leqq n$, the Fox or nonabelian derivative

$$\partial_i = \partial/\partial x_i:\mathbf{Z}[F] \rightarrow \mathbf{Z}[F]$$

is uniquely defined by the conditions:

$$\partial_i(x_j) = \delta_{ij} \quad \text{and} \quad \partial_i(uv) = \partial_i(u)\epsilon(v) + u\partial_i(v), \quad [\mathbf{3}].$$

Given any $f \in \mathbf{Z}[F]$ there is a Taylor expansion

$$f - \epsilon f = \sum_{i=1}^{n} \partial_i f(x_i - 1).$$

For any index $I = (i_1, \ldots, i_k)$ let

$$\partial_I : \mathbf{Z}[F] \rightarrow \mathbf{Z}[F]$$

denote the higher order derivative $\partial_{i_1} \circ \partial_{i_2} \circ \ldots \circ \partial_{i_k}$. The composite $\epsilon \circ \partial_I$ will be written ϵ_I. If $k = 0$ then $I = \phi$ and by convention ∂_ϕ is the identity and hence $\epsilon_\phi = \epsilon$. The integer $\epsilon_i(u)$ is equal to the total exponent of x_i in u considered earlier. The ∂_I are linear in the sense that

$$\partial_I(\lambda + \mu) = \partial_I(\lambda) + \partial_I(\mu) \quad \text{and}$$

$$\partial_I(\alpha\lambda) = \alpha\partial_I(\lambda) \text{ if } \alpha \epsilon \mathbf{Z}.$$

LEMMA 4.1. *For any index I and $\lambda, \mu \in \mathbf{Z}[F]$*

$$\epsilon_I(\lambda\mu) = \sum_{I_1 I_2 = I} \epsilon_{I_1}(\lambda)\epsilon_{I_2}(\mu)$$

where the summation is taken over all ordered pairs (I_1, I_2) such that $I_1 I_2 = I$, including (I, ϕ) and (ϕ, I).

Proof. This is obviously true if $l(I) = 0$. The proof follows routinely by induction on $l(I)$.

COROLLARY 4.2.

$$\epsilon_I(\lambda_1\lambda_2 \ldots \lambda_j) = \sum_{I_1 I_2 \ldots I_j = I} \epsilon_{I_1}(\lambda_1)\epsilon_{I_2}(\lambda_2) \ldots \epsilon_{I_j}(\lambda_j).$$

COROLLARY 4.3. *If $I \neq \phi$ and $g \in F$,*

$$\epsilon_I(g^{-1}) = \sum_{I_1 I_2 \ldots I_j = I} (-1)^j \epsilon_{I_1}(g)\epsilon_{I_2}(g) \ldots \epsilon_{I_j}(g),$$

where the sum is taken over all $I_1 \ldots I_j = I$ with $I_l \neq \phi$, $l = 1, \ldots, j$.

Proof. Consider the expansion of $0 = \epsilon_I(gg^{-1})$ and use induction on $l(I)$.

Definition. Let H be a subgroup of the group G and let $[G, H]$ denote the subgroup of G generated by all elements $ghg^{-1}h^{-1}$ where $g \in G$ and $h \in H$. The *lower central series* $G = G_1 \supseteq G_2 \supseteq \ldots$ is defined inductively by $G_1 = G$ and $G_{n+1} = [G, G_n]$.

THEOREM. 4.4. *Let $F = F(x_1, \ldots, x_n)$ be freely generated by $x_1, \ldots, x_n$. Then $g \in F_k$ if and only if $\epsilon_I(g) = 0$ for all $0 < l(I) < k$.*

The proof of 4.4 can be found in [**3**], and since the ϵ_I are also the terms in the Magnus expansion (see later) a proof can also be found in [**8**] and [**18**]. The mod$p$ version of 5.4 in [**14**] will be used in a later paper.

Definition. Let χ be the free associative power series ring in the noncommuting variables $X_1, \ldots, X_n$ and having coefficients in $\mathbf{Z}$. If $I = (i_1, \ldots, i_k)$, where $1 \leqq i_j \leqq n$, $j = 1, \ldots, k$ let X_I be the monomial

$$X_I = X_{i_1} \ldots X_{i_k}.$$

Thus the elements of χ can be expressed uniquely as formal sums

$$Q = \sum C_I X_I$$

where the summation is over the set of all indices and the C_I lie in $\mathbf{Z}$. The dimension of X_I is

$$\dim X_I = k = l(I).$$

In general the dimension of an arbitrary element $Q \in \chi$ is the minimal degree of the monomials occuring in the power series Q. Thus $\dim 0 = \infty$, $\dim k = 0$ if k is a non-zero element of $\mathbf{Z}$, and $\dim u = 0$ if u is a unit in χ. A result in [**7**] is that the elements $1 + X_1, \ldots, 1 + X_n$ generate a free group of rank n. Inverses are given by

$$(1 + X_i)^{-1} = 1 - X_i + X_i^2 - \ldots.$$

The Magnus expansion in the Magnus ring χ is the group homomorphism $M:F \rightarrow \chi$ given by

$$M(x_i) = 1 + X_i.$$

The coefficients in the expansion as a power series are just the ϵ_I. So

$$M(g) = \sum \epsilon_I(g) X_I.$$

This can be seen by repeated use of the Taylor expansion of g.

If $M(g) - 1$ has dimension k then the leading polynomial of $M(g)$ is defined to be the homogeneous polynomial of degree k in $M(g) - 1$. For example the leading polynomial of $M(xyx^{-1}y^{-1})$ is $XY - YX$. An application of 4.4 is that the leading polynomial of $M(g)$ has dimension k if and only if g lies in F_k but not F_{k+1}.

The proof of the following can be found in [**3**].

LEMMA 4.5. (a) *Suppose* $f \in F_i$, $g \in F_j$ *and* $l(I) \leqq \min(i, j)$. *Then*

$$\epsilon_I(fg) = \epsilon_I(f) + \epsilon_I(g).$$

(b) *Suppose* $l(I) = l(J') = i$, $l(J) = l(I') = j$, $IJ = I'J'$ *and* $f \in F_i$, $g \in F_j$. *Then*

$$\epsilon_{IJ}[f, g] = \epsilon_I(f)\epsilon_J(g) - \epsilon_{I'}(g)\epsilon_{J'}(f).$$

Definition. Let F be free on $x_1, \ldots, x_n$, G free on $y_1, \ldots, y_m$ and let $\phi:G \rightarrow F$ be a homomorphism. Let $\phi_i^{\,j}$ be the $n \times m$ matrix

$$[\phi_i^j] = [\epsilon_i(\phi(y_j))]$$

and let $\phi_{i_1,\ldots,i_k}^{j_1,\ldots,j_k}$ be the 'tensor' $\phi_{i_1}^{j_1}\phi_{i_2}^{j_2}\ldots\phi_{i_k}^{i_k}$.

The following important rule describes how the ϵ_I behave under homomorphisms.

THEOREM 4.6. (The chain rule) *With the notation above*
(a) *for* $g \in G$,

$$\epsilon_i(\phi(g)) = \sum_{j=1}^{m} \phi_i{}^j\epsilon_j(g);$$

(b) *for* $g \in G_k$,

$$\epsilon_{(i_1,\ldots,i_k)}(\phi(g)) = \sum_{(j_1,\ldots,j_k)} \phi_{i_1,\ldots,i_k}^{j_1,\ldots,j_k}\epsilon_{j_1,\ldots,j_k}(g).$$

Proof. (a) Note that both the mappings

$$g \to \epsilon_i(\phi(g)) \quad \text{and} \quad g \to \sum_{j=1}^{m} \phi_i^j\epsilon_j(g)$$

are group homomorphisms $G \to \mathbf{Z}$. Thus it is only necessary to consider the special case $g = y_l$ where the result is obvious ($\epsilon_j(y_l) = \delta_{jl}$).

(b) Once again by 4.5 (a) it is only necessary to consider the case $g = [g_1, g_2]$ where $g_1 \in F_{k_1}$, $g_2 \in F_{k_2}$ and $k_1 + k_2 = k$.

To shorten notation use the summation convention that repeated indices are summed over. I_1, J_1, K_1 will denote general indices of length k_1, and I_2, J_2, K_2 of length k_2.

$$\begin{aligned}\epsilon_I(\phi[g_1, g_2]) &= \epsilon_I[\phi(g_1), \phi(g_2)] \\ &= \epsilon_{I_1}(\phi(g_1))\epsilon_{I_2}(\phi(g_2)) - \epsilon_{I'_2}(\phi(g_2))\epsilon_{I'_1}(\phi(g_1)) \text{ by 4.5 (b)} \\ &= \phi_{I_1}^{J_2}\epsilon_{J_1}(g_1)\phi_{I_2}^{K_2}\epsilon_{K_2}(g_2) - \phi_{I'_2}^{J'_2}\epsilon_{J'_2}(g_2)\phi_{I'_1}^{K_1}\epsilon_{K_1}(g_1)\end{aligned}$$

(by induction on k)

$$\begin{aligned}&= \phi_I{}^J(\epsilon_{J_1}(g_1)\epsilon_{J_2}(g_2) - \epsilon_{J'_2}(g_2)\epsilon_{J'_1}(g_1)) \\ &= \phi_I{}^J\epsilon_J[g_1, g_2] \quad \text{by Lemma 4.5 (b).}\end{aligned}$$

5. Commutators with weight. First we recall the definition of a bracket arrangement [**8**].

Definition 5.1. A *bracket arrangement* consists of brackets and asterisks (which act as place holders) and comes assigned with a weight. The only bracket arrangement of weight 1 is $\beta = [*] = *$. To define bracket arrangements of higher weight we proceed inductively. Thus suppose we have defined them for all weights $< k$. Then the bracket arrangements of weight k are $\beta = [\beta_1, \beta_2]$ where β_1, β_2 are bracket arrangements of weights k_1, k_2 and $k = k_1 + k_2$. The weight of β is denoted by $\omega(\beta)$.

For example the bracket arrangements of weights 2 and 3 are $[*, *]$, $[*, [*, *]]$, $[[*, *], *]$. To any β we associate a tree $T(\beta)$ with a root (distinguished vertex) as follows. If $\omega(\beta) = 1$ then $T(\beta)$ is a single vertex which is the root. Now assume we have defined these trees for all weights $< k$ and that $\beta = [\beta_1, \beta_2]$ has weight k. Then $T(\beta)$ is the tree in figure 1 and v is its root, where v_1 and v_2 are the roots of $T(\beta_1)$, $T(\beta_2)$. We always orient the trees so that left-right ordering is preserved and so that the new root is at the bottom. The weight of $T(\beta)$ is defined to be $\omega(\beta)$. In figure 2 the trees of weight $\leqq 3$ are exhibited.

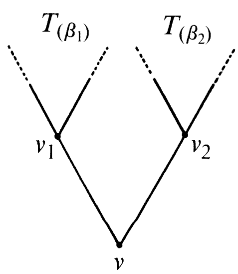


Figure 1

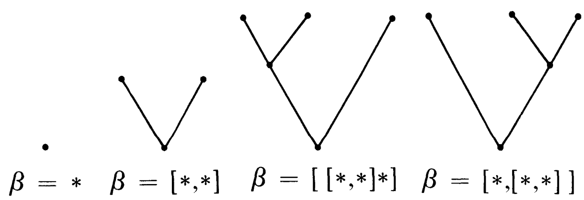


Figure 2

It is clear that there exists a one to one correspondence between bracket arrangements and trees of a certain type; namely those finite non-empty trees for which there is a left-right, up-down ordering as above and satisfying:

The highest vertices are each connected to the tree below by a single edge if the weight > 1.

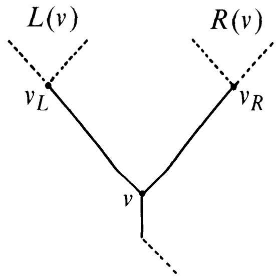
the root is connected to the tree above by 2 edges if the weight is > 1.

each in-between vertex is connected to the tree above by 2 edges and to the tree below by one edge.

Let the class of such trees be denoted $\mathscr{T}$. If v is a vertex in some $T \in \mathscr{T}$ we can pick out an upper tree $U(v)$, a left-hand tree $L(v)$ and a right-hand tree $R(v)$; see figure 3.

In order to have an independent set of commutators we work with the so called basic commutators. They have both weight and ordering.

Definition 5.2. The *basic commutators of weight* 1 are $x_1, \ldots, x_n$ and the ordering is $x_1 < \ldots < x_n$. Now assume we have defined the basic commutators together with their ordering for all weights $< k$. Then the



$$U(v) = L(v) \cup R(v)$$
$$v = L(v) \cap R(v)$$
$$v_L = \text{left hand vertex}$$
$$v_R = \text{right hand vertex}$$

Figure 3

basic commutators of weight k are the elements of the form $c = [c_1, c_2]$ where c_1, c_2 are basic commutators of weights k_1, k_2 and $k = k_1 + k_2$. Moreover we require $c_1 < c_2$, and if $c_2 = [c_3, c_4]$ we also require that $c_1 \geqq c_3$. The ordering is such that anything of weight k is greater than commutators of lower weight, whereas two basic commutators of weight k are ordered lexicographically, i.e., $[c_1, c_2] < [c_1', c_2']$ if and only if either $c_1 < c_1'$, or $c_1 = c_1'$ and $c_2 < c_2'$.

THEOREM 5.3. **[6]** and **[4]**. *Suppose* $w \in F$ *and* $k \geqq 1$. *Then there is a unique expression*

$$w = u\, c_m^{n_m} \dots c_1^{n_1},$$

where

(a) $u \in F_{k+1}$

(b) *the* c_i *are basic commutators of weights* $\leqq k$

(c) $c_1 < c_2 < \dots < c_m$.

Note that the tail of the sequence is of the form

$$x_n^{\epsilon_n} x_{n-1}^{\epsilon_{n-1}} \dots x_1^{\epsilon_1}$$

where ϵ_i is the total exponent of x_i in w.

Say that $c_m^{n_m} \dots c_1^{n_1}$ is *in collected form.*

Now suppose β and $I = (i_1, \dots, i_k)$ are given with $\omega(\beta) = k$. Let $\beta(I)$ denote the commutator element in F_k obtained by substitution of $x_{i_1}, \dots, x_{i_k}$ in consecutive locations.

Assume that $\beta(I) \in F_k - F_{k+1}$, which would be the case if $\beta(I)$ were basic. To such a pair we now associate a labelled tree $T(\beta, I)$ which will just be $T(\beta)$ with each vertex having a label from the free group F. The labelling is defined inductively as follows:

$$\text{if } \omega(\beta) = 1 \text{ and } I = i \text{ then } T(\beta, I) = \overset{x_i}{\cdot}$$

Assume the labelling has been accomplished for all trees of weight $< k$ and that $\beta = [\beta_1, \beta_2]$ has weight k. If $l(I) = k$ we break I up as $I = I_1 \cdot I_2$, where

$$l(I_1) = \omega(\beta_1) \quad \text{and} \quad l(I_2) = \omega(\beta_2).$$

Thus $T(\beta_1)$ and $T(\beta_2)$ are labelled and this gives the labelling for the corresponding sub-trees of $T(\beta)$. Finally the root of $T(\beta)$ is labelled with the commutator $[L_1, L_2]$, where L_1 and L_2 are the labels for the roots of $T(\beta_1)$, $T(\beta_2)$. Notice that $[L_1, L_2] = \beta(I)$ since by induction

$$L_1 = \beta_1(I_1), L_2 = \beta_2(I_2)$$

and therefore

$$[L_1, L_2] = [\beta_1(I_1), \beta_2(I_2)] = \beta(I).$$

The index I and the monomial $u(T) = X_I$ of a labelled tree T are defined by

$$u(T) = X_{i_1} \dots X_{i_r} = X_I.$$

Definition 5.4. The allowable operations on $T \in \mathscr{T}$ are generated by the following elementary moves: for some vertex $v \in T$ we interchange $L(v)$ with $R(v)$, keeping the left-right and up-down orderings within $L(v)$, $R(v)$ the same, while leaving v, $T - U(v)$ fixed (see figure 4). The sign of such a move is -1 and the sign of an allowable operation is the product of the signs of its elementary moves. If T has labels then so does every tree obtained from it by allowable operations.

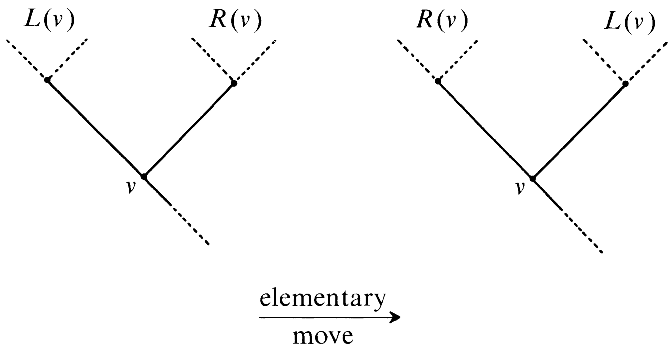


Figure 4

Given an element $f \in F$ we let $\mathscr{L}(f)$ denote the leading polynomial of $M(f)$. By (4.4) and the equation

$$M(f) = \sum_I \epsilon_I(f) X_I$$

it follows that

$$\mathscr{L}(f) = \sum_{l(I)=k} \epsilon_I(f)X_I \quad \text{if } f \in F_k - F_{k+1}.$$

LEMMA 5.5. *Suppose β is a bracket arrangement of weight k and $I = (i_1, \dots, i_k)$ is an index such that $\beta(I) \in F_k - F_{k+1}$. Then*

$$\mathscr{L}(\beta(I)) = \sum_I \zeta(T)u(T)$$

where the summation is over all labelled trees T obtained from $T(\beta, I)$ by allowable operations and $\zeta(T)$ is the sign of the allowable operation.

Proof. This is trivially true for weight 1. Thus suppose it has been proven for all weights $< k$ and that $\beta = [\beta_1, \beta_2]$ has weight $k = k_1 + k_2$, where $k_1 = \omega(\beta_1)$, $k_2 = \omega(\beta_1)$. Now

$$\beta(I) = [L_1, L_2] \in F_k - F_{k+1}$$

and so by (4.5)

$$\begin{aligned}\mathscr{L}(\beta(I)) &= \sum_{l(I)=k} \epsilon_I[L_1, L_2]X_I \\ &= \sum_{l(I)=k} \{\epsilon_{I_1}(L_1)\epsilon_{I_2}(L_2) - \epsilon_{J_1}(L_2)\epsilon_{J_2}(L_1)\}X_I\end{aligned}$$

where

$$I = I_1 \cdot I_2 = J_1 \cdot J_2,\ l(I_1) = l(J_2) = k_1,\ l(I_2) = l(J_1) = k_2.$$

Thus

$$\begin{aligned}\mathscr{L}(\beta(I)) &= \Sigma\epsilon_{I_1}(L_1)\epsilon_{I_2}(L_2)X_{I_1}X_{I_2} - \Sigma\epsilon_{J_1}(L_2)\epsilon_{J_2}(L_1)X_{J_1}X_{J_2} \\ &= \mathscr{L}(L_1)\mathscr{L}(L_2) - \mathscr{L}(L_2)\mathscr{L}(L_1).\end{aligned}$$

By induction we now have

$$\mathscr{L}(L_1)\mathscr{L}(L_2) = \sum_{T_1} \zeta(T_1)u(T_1) \sum_{T_2} \zeta(T_2)u(T_2) = \sum_T \zeta(T)u(T)$$

where the last summation is over all T arising from applying allowable operations within $T(\beta_1)$ and $T(\beta_2)$ with no flip about the root of $T(\beta)$. The term $-\mathscr{L}(L_2)\mathscr{L}(L_1)$ corresponds exactly to similar allowable operations after a flip about the root of $T(\beta)$.

Example 5.6. As an illustration of 5.5 consider

$$\beta = [*, [*, [*, *]]] \quad \text{and} \quad I = (2, 1, 1, 2).$$

There are 8 allowable operations (see figure 5). Thus we have

$$\mathscr{L}[x_2, [x_1, [x_1, x_2]]] = -2\,X_{2121} - X_{1122} + X_{2211} + 2X_{1212}.$$

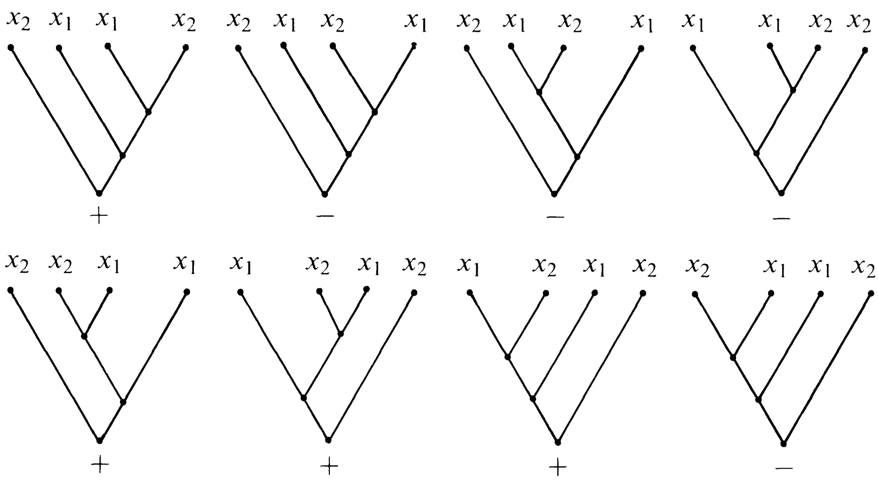


Figure 5

Finally, we have

LEMMA 5.7. *Suppose* $c = \beta(I)$. *Then* $\epsilon_J(c^{\pm 1}) = 0$ *if* $l(J) > 0$ *and* $J \not\to I$.

Proof. This is trivially true for weight 1. Thus assume it is true for all weights $< k$ and that $c = [c_1, c_2]$. By (4.2) we have

$$\epsilon_J(c) = \sum_{J_1J_2J_3J_4=J} \epsilon_{J_1}(c_1)\epsilon_{J_2}(c_2)\epsilon_{J_3}(c_1{}^{-1})\epsilon_{J_4}(c_2{}^{-1}).$$

Then the proof proceeds by a case by case examination, the cases being:

(a) none of the J_i are empty

(b) exactly one of the J_i is empty, etc.

For example, if $J_2 = J_4 = \emptyset$ and $J_1 \neq \emptyset \neq J_3$ we have the contribution

$$\sum_{J_1 \neq \phi \neq J_3} \epsilon_{J_1}(c_1)\epsilon_{J_3}(c_1{}^{-1}) = -\epsilon_J(c_1) - \epsilon_J(c_1{}^{-1}) \text{ by (4.1)}$$

$$J_2 = J_4 = \emptyset$$

and the fact that

$$0 = \epsilon_J(c_1 \cdot c_1{}^{-1}).$$

Likewise we have the contribution $-\epsilon_J(c_2) - \epsilon_J(c_2{}^{-1})$ coming from $J_1 = J_3 = \emptyset, J_2 \neq \emptyset = J_4$. If exactly 3 of $J_1, \ldots, J_4$ are $\emptyset$ the contribution is

$$\epsilon_J(c_1) + \epsilon_J(c_1{}^{-1}) + \epsilon_J(c_2) + \epsilon_J(c_2{}^{-1}).$$

Assuming that $J \not\to I$ all other cases give 0. A similar argument works for $\epsilon_J(c^{-1})$.

6. Basic complexes and hyperbolic defining sets.

Definition 6.1. A *basic complex of weight* $\omega(K) > 1$ is defined to be a one relator complex $K = \{x_1, \ldots, x_n | \beta(I)\}$ where $\beta = [\beta_1, \beta_2]$ is a bracket arrangement of weight $\omega(K)$ and $I = I_1 I_2$ is so that $\beta(I)$, $\beta_1(I_1)$ and $\beta_2(I_2)$ are basic commutators. Write

$$K = [K_1, K_2]$$

where

$$K_i = \{x_1, \ldots, x_n | \beta_i(I_i)\}.$$

6.2. *Geometric cohomology.* Before pursuing the main result a geometric picture of the various cochains which can be used for calculating the product structure of any 2-complex will be given. See also [**2**]. Consider as an example the complex

$$\{x_1, x_2 | x_1{}^2 x_2 x_1{}^{-1} x_2{}^{-1}\}$$

and some simplicial subdivision K. If α is a simplex than α^* will denote the 1-cochain with value 1 on α and value zero elsewhere. The orientation of a 1-cochain is pictured as an arrow $\rightarrow$ and the orientation of a 2-cochain as a curved arrow $\circlearrowleft$.

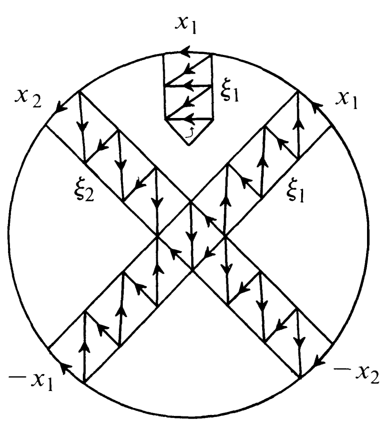


Figure 6a

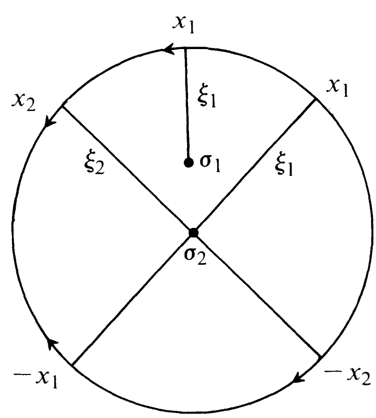


Figure 6b

Consider figure 6(a) which is a representation of part of K with some cochains. Then

$$\delta\xi_1 = \sigma_1^*, \quad \xi_1\xi_2 = \xi_1 \cup \xi_2 = \sigma_2{}^*$$

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(with appropriate orderings) and $\delta\xi_2 = 0$. These cochains may be pictured symbolically as in 6(b) which also admits the following interpretation. Embed K as the spine of some n-manifold M with boundary ∂M. Then

$$H^i(K) \leftarrow H^i(M)$$

is an isomorphism for all i. But by Lefschetz duality

$$H^i(M) = H_{n-i}(M, \partial M).$$

If $i = 1, 2$ elements of $H_{n-i}(M, \partial M)$ may be represented by properly embedded submanifolds meeting K transversely in geometric pictures of cocycles. If these submanifolds are in general position with respect to K then their intersection with any cell will be a codimension i submanifold with boundary in the boundary of the cell. Cochains which are not cocycles also admit a similar interpretation as submanifolds with non-proper boundary.

Alternatively the symbolic picture in 6(b) is the 'limit' as the size of the simplices tend to zero. For more details see [**1**], [**5**] or [**17**].

6.3. *Cocycles as line elements in the hyperbolic plane.* In this section a collection of geodesics and part geodesics is defined in the hyperbolic plane which will be the underlying space of a defining set for Massey products in basic complexes.

Represent the hyperbolic plane $\mathscr{H}$ as the open unit disc $|z| < 1$. As usual the lines of $\mathscr{H}$ are the arcs of circles in $\mathscr{H}$ orthogonal to the boundary circle $\partial\mathscr{H}$. Any one relator complex can be obtained by making identifications on $\partial\mathscr{H}$. This identification is specified by the relator written anticlockwise around the boundary.

Consider the torus $\{x, y|xyx^{-1}y^{-1}\}$. The generating cocycles u_1, u_2 can be represented by the x-axis as in figure 7.

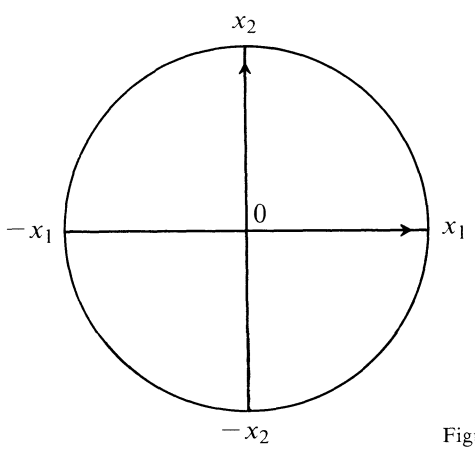

Figure 7

the orientation of the axes is towards positive from negative

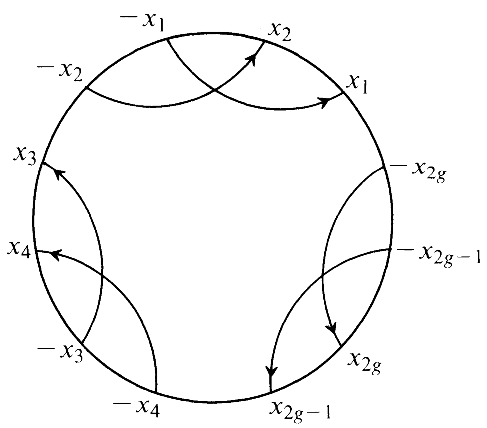

Figure 8

The cup product u_1u_2 is represented by the intersection of the axes at the origin 0 and has positive sign. For orientable surfaces of genus larger than 1 the cocycles may be represented by geodesics as in figure 8. The cup product structure can be read off from the meeting of the geodesics. This notion is behind the proof of the main theorem 7.4. Choose a point $0'$ on the x-axis in $\mathscr{H}$. This will be chosen sufficiently far along the positive end of the x-axis as circumstances dictate. Consider now the hyperbolic transformation ψ which translates 0 to $0'$ and then rotates about $0'$ clockwise through an angle $3\pi/4$. Let Ψ be the set of transformations

$$\Psi = \{\psi, i\psi, -\bar{\psi}, -i\bar{\psi}\}.$$

The effect of these four transformations on the x-axis is shown in figure 9.

Notice that $0'$ is chosen so that the four clusters of lines are disjoint.

Definition 6.4. Let $\beta = [\beta_1, \beta_2]$ be a bracket arrangement. A system of lines and points $H(\beta)$, called a *hyperbolic defining set* (h.d.s. for short), will now be constructed. If $\beta = *$ is the trivial bracket then $H(\beta)$ is defined to be the x-axis. Assume that $H(\beta_1)$ and $H(\beta_2)$ have been defined inductively. Let γ_x be the line joining $-\ 0'$ to $0'$ and let $\gamma_y = i\,\gamma_x$. There are four cases

if $\beta_1 = *$, $\beta_2 = *$ let

$$H(\beta) = H(\beta_1) \cup iH(\beta_2) = \cup \text{ axes}$$

if $\beta_1 = *$, $\beta_2 \neq *$ let

$$H(\beta) = H(\beta_1) \cup i\psi H(\beta_2) \cup -\ i\bar{\psi}H(\beta_2) \cup \gamma_y$$

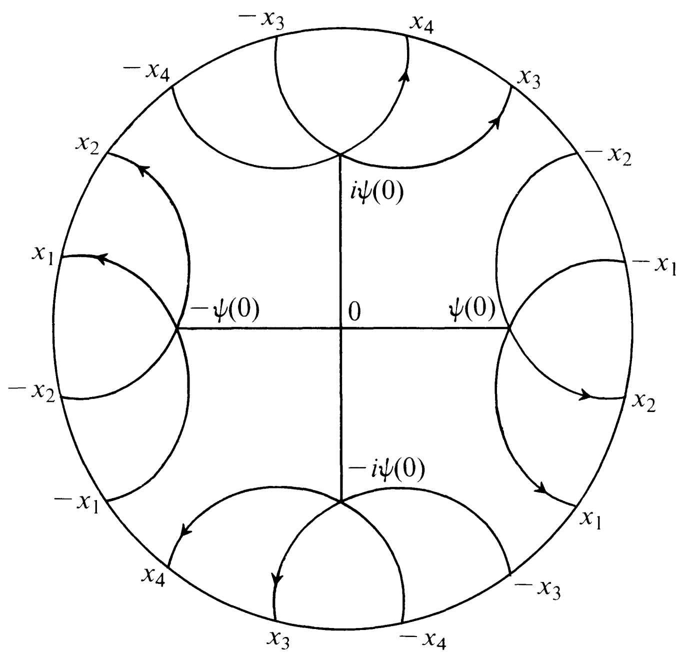

Figure 9

if $\beta_1 \neq \beta_2 = *$ let

$$H(\beta) = \psi H(\beta_1) \cup -\bar{\psi} H(\beta_1) \cup iH(\beta_2) \cup \gamma_x$$

if $\beta_1 \neq *$, $\beta_2 \neq *$ let

$$H(\beta) = \psi H(\beta_1) \cup -\bar{\psi} H(\beta_1) \cup i\psi H(\beta_2) \cup - i\bar{\psi} H(\beta_2) \cup \gamma_x \cup \gamma_y.$$

Examples of $H(\beta)$ are given by figure 9, if

$$\beta = [\,[*, *], [*, *]\,]$$

and by figure 10 if

$\beta = [*, [*, [*, *]\,]\,]$. Also Figure 7 gives $H(\beta)$ when $\beta = [*, *]$.

Notice that $H(\beta)$ is invariant under reflexions in the x-axis and in the y-axis and these symmetries are preserved under further commutation.

The line elements of $H = H(\beta)$ are the arcs

$$\alpha = \psi_1\psi_2 \ldots \psi_r(z)$$

where z is one of the axes or γ_x or γ_y and

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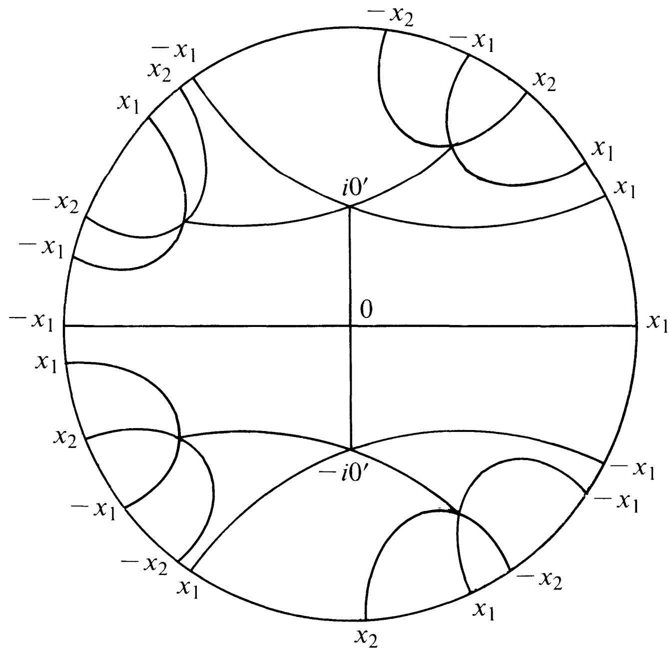

Figure 10

$$\psi_i \in \Psi, \quad i = 1, \dots, r.$$

If z is an axis then α is called a cocycle element. Notice that the line elements have orientations as the image of an axis or part axis. The points of H are the corresponding images of 0, $\hat{\alpha} = \psi_1 \dots \psi_r(0)$. If α is not a cocycle element then it has an initial end point $i(\alpha)$ and a final end point $f(\alpha)$, both points of H. Thus the line elements form a system of geodesics meeting in the points of H.

The cochain complex C^* and its products which will now be defined correspond naturally to a representation of the cohomology of K. Let $C^1(H)$ be the free abelian group on the line elements of H and $C^2(H)$ be likewise on the points of H. $C^{o(H)}$ is infinite cyclic. Coboundary operators are defined as follows: $\delta: C^{o(H)} \to C^1(H)$ is zero and $\delta: C^1(H) \to C^2(H)$ is given by

$$\delta(\alpha) = \begin{cases} 0 \text{ if } \alpha \text{ is a cocycle element} \\ f(\alpha) - i(\alpha) \text{ otherwise.} \end{cases}$$

It is easy to see that $C^1 = A^1 \oplus B^1$, where $B^1 = \ker \delta$ is freely generated by the cocycle elements.

Every line element $\alpha = \psi_1 \ldots \psi_r(z)$, assuming $\omega(\beta) > 1$, has a perpendicular adjoint α^* passing through $\hat{\alpha}$. If $r > 0$ the mate of α is

$$\bar{\alpha} = \psi_1 \ldots \psi_{r-1}\phi_r(z),$$

where $\phi_r = -\bar{\psi}_r$ if z lies in the x-axis and $\phi_r = \bar{\psi}_r$ otherwise. If $r = 0$ the mate of z is z itself. Every point $P = \hat{\alpha}$ has a mate given by $\bar{P} = \hat{\bar{\alpha}}$. Notice that $\bar{0} = 0$ for the origin. Intersection induces a product $C^1 \otimes C^1 \to C^2$ given by

$$\alpha \cdot \alpha^* = \alpha^* \cdot \alpha = \pm\hat{\alpha} \quad \text{and} \quad \alpha_1 \cdot \alpha_2 = 0$$

otherwise, see figure 11.

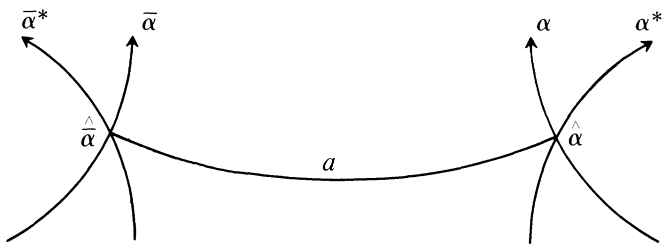

Figure 11

The sign is chosen so that

$$\gamma_x \cdot \gamma_y = \bar{0} = 0.$$

Note that $\bar{\alpha}^* \cdot \bar{\alpha} = \overline{\alpha \cdot \alpha^*}$ and so in general $\overline{c_1 \cdot c_2} = \bar{c}_2 \cdot \bar{c}_1$.

A chain $c = \sum \lambda_i \alpha_i$ in C^1 is said to be *equivariant* if

$$\bar{c} = \sum \lambda_i \bar{\alpha}_i = c.$$

Likewise a chain $d = \sum \mu_i p_i$ in C^2 is said to be *equivariant* if

$$\bar{d} = \sum \mu_i \bar{p}_i = -d.$$

Notice that if d in C^2 is equivariant there is a unique equivariant element a in A^1 such that $\delta a = d$ (see figure 11).

An index $I = (i_1, i_2, \ldots, i_n)$ is called *distinct* if all the elements i_k are distinct, $k = 1, \ldots, n$.

LEMMA 6.5. *If $K = \{x_1, \ldots, x_n | \beta(I)\}$ is a basic complex with weight > 1 where I is distinct and if $l(J) \leqq \omega(\beta)$, then there is a defining set for the Massey product $\langle u_J \rangle$ with corresponding cocycle $\epsilon_J(\beta(I))0^*$.*

Proof. The complex K may be thought of in the usual way as a bouquet of circles $x_1, \ldots, x_n$ with a 2-cell $\bar{\mathscr{H}} = \mathscr{H} \cup \partial\mathscr{H}$ attached by the word $\beta(I)$. Consider the h.d.s. $H(\beta)$ and a cocycle element α. The end points of α can be identified with the centre point of a 1-cell x_i in a unique way. The orientation of α is determined as in figure 12.

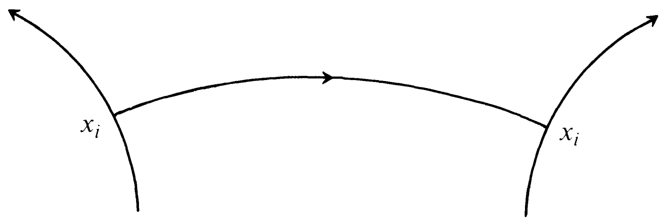

Figure 12

If $\alpha_1, \ldots, \alpha_k$ are associated with x_i then $\alpha_1 + \ldots + \alpha_k$ is a geometric cocycle representing the class u_i of $H^1(K)$.

A defining set (a_{ij}) for the Massey product $\langle u_{j_1}, \ldots, u_{j_k} \rangle$ using the product structure of $C^*(H(\beta))$ will now be constructed as follows: Let a_{ij} represent u_{j_i} as above. Note that a_{ii} is equivariant. The products $u_l \cdot u_{l+1}$ can now be identified with the intersection of the appropriate arcs and the product defined for the h.d.s. At this stage we are not using the hypothesis that I is disjoint.

Assume now that a_{ij} are constructed for all $0 \leqq j - i < r$ and are equivariant. This means that the products $a_{ij} \cdot a_{j+1,i+r}$ are also equivariant and hence there is a unique equivariant element $a_{i,i+r}$ in A^1 such that

$$\delta a_{i,i+1} = \sum_{j=1}^{i+r-1} a_{ij} \cdot a_{j+1,i+r}.$$

This completes the inductive step. We must now check that the geometric cochain corresponds appropriately. This is where the hypothesis of the distinct I is used.

The only non-zero product $u.v$ of a geometric cochain with support an h.d.s. can occur in the two cases illustrated by figure 13.

(a) (b)

Figure 13

Because I is distinct type (a) is the only case which can occur and this corresponds to our built in product in the h.d.s.

The value of the defining set is

$$\sum_{i=1}^{k-1} a_{1i}a_{i+1,k}.$$

Each evaluation $(a_{1i}, a_{i+1,k}, [K])$ is zero except when $i = k_1$ or k_2 and in these cases writing $c = \beta(I) = [c_1, c_2]$

$$(a_{1k_1}a_{k_1+1,k}, [K]) = \epsilon_{J_1}(c_1)\epsilon_{J_2}(c_2)$$

$$(a_{1k_2}a_{k_2+1,k}, [K]) = -\epsilon_{J_2'}(c_1)\epsilon_{J_1'}(c_2)$$

by induction on k. Hence the value of the defining set on $[K]$ is $\epsilon_J(c)$ by 4.5(b). As a geometric cocycle it consists of $\epsilon_J(c)$ times the central point 0.

THEOREM 6.6. *If $K = \{x_1, \ldots, x_n | \beta(I)\}$ is a basic complex of weight $\omega(\beta) > 1$ and J is an index with $l(J) \leqq \omega(\beta)$ then the Massey product $\langle u_J \rangle$ is defined and only has one element, given by the rule*

$$(\langle u_J \rangle, [K]) = \epsilon_J(\beta(I)).$$

In particular if $l(J) < \omega(\beta)$ the Massey product $\langle u_J \rangle$ vanishes.

Proof. If $I = (i_1, \ldots, i_k)$ let $I' = (1, 2, \ldots, k)$, so I' is distinct. If

$$K' = \{x_1, \ldots, x_n | \beta(I')\}$$

let $\phi:K' \to K$ be defined by the rule

$$l \to i_l,\ l = 1, \ldots, k.$$

Then ϕ_* is the identity on H_2. Assume the truth of (6.6) by induction for all J' with $l(J') < l(J)$. This means that $\langle u_j \rangle$ is both strictly and uniquely defined.

Then

$$(\langle u_{i_1}, \ldots, u_{i_k} \rangle, [K])$$
$$= (\langle u_{i_1}, \ldots, u_{i_k} \rangle, \phi_*[K'])$$
$$= (\phi^*\langle u_{i_1}, \ldots, u_{i_k} \rangle, [K])$$
$$= (\langle \phi^*u_{i_1}, \ldots, \phi^*u_{i_k} \rangle, [K'])$$

(by naturality and the fact that $\langle u_{i_1}, \ldots, u_{i_k} \rangle$ is uniquely defined)

$$= (\langle \Sigma\phi_{i_1}^{j_1}u_{j_1}, \ldots, \Sigma\phi_{i_k}^{j_k}u_{j_k} \rangle, [K]) \quad \text{by Lemma 3.2}$$
$$= (\Sigma\phi_{i_1 \ldots i_k}^{j_1 \ldots j_k}\langle u_{j_1}, \ldots, u_{j_k} \rangle, [K']) \quad \text{by linearity}$$
$$= \Sigma\phi_{i_1 \ldots i_k}^{j_1 \ldots j_k}\epsilon_{j_1 \ldots i_k} \quad \text{by 6.5 since } I' \text{ is distinct}$$
$$= \epsilon_{j_1 \ldots i_k}(\beta(I)) \quad \text{by 4.6(b).}$$

The next result considers the case when the length of the Massey product is greater than $\omega(K)$.

THEOREM 6.7. *Suppose $K = \{x_1, \dots, x_n | \beta(I)\}$ is a basic complex and J is an index such that $J \not\twoheadrightarrow I$. Then $\langle u_J \rangle$ is defined and contains the value zero.*

Proof. Using the notation of the previous theorem start to construct a defining set (a_{ij}) for $\langle u_J \rangle$ with underlying set an h.d.s. for β. By hypothesis and the above result there will be no obstruction when $l(J) \leqq \omega(K)$. Assume that a_{ij} is defined for $j - i < r$. A product $a_{ij} \cdot a_{j+1,i+r}$ defines subindices J_1, J_2 of J such that J_1J_2 is a sub-index of J, see figure 14. Then by induction one of $\langle u_{J_1} \rangle$ in K_1 or $\langle u_{J_2} \rangle$ in K_2 contains zero which is defined using an h.d.s. So the set (a_{ij}) may be defined for $j - i \geqq \omega(K)$ by putting $a_{ij} = 0$.

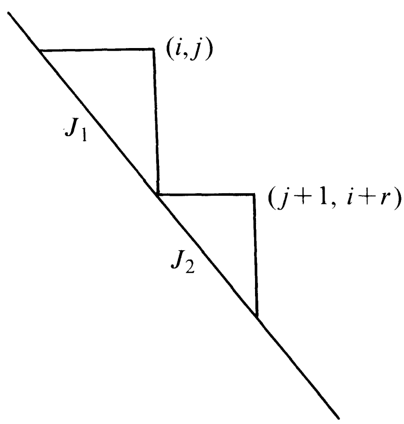

Figure 14

It is worthwhile pointing out that if $K = \{x_1, \dots, x_n | \beta(I)\}$ is a basic complex weight $k > 1$ then the Massey products $\langle u_J \rangle$, where $l(J) = k$, are given by the coefficients in the leading polynomial $\mathscr{L}(\beta(I))$. More precisely

$$\mathscr{L}(\beta(I)) = \sum_{l(J)=k} \epsilon_J(\beta(I))X_J = \sum_{l(J)=k} (\langle u_J \rangle, [K])X_J.$$

Thus all Massey products of length k can be computed by applying allowable operations to the labelled tree $T(\beta, I)$ (see 5.5). As an example consider the basic complex

$$K = \{x_1, x_2 | [x_2, [x_1, [x_1, x_2]]]\}.$$

Then from 5.6 we can read off the following Massey products

$$(\langle u_2, u_1, u_2, u_1\rangle, [K]) = -2, \quad (\langle u_1, u_1, u_2, u_2\rangle, [K]) = -1$$

$$(\langle u_2, u_2, u_1, u_1\rangle, [K]) = 1, \quad (\langle u_1, u_2, u_1, u_2\rangle, [K]) = 2,$$

and all other $(\langle u_J\rangle, [K]) = 0$ if $l(J) = 4$.

Notice that these products do not satisfy symmetry with respect to allowable operations. As an example consider

$$(\langle u_2, u_1, u_1, u_2\rangle, [K]) = 0;$$

whereas

$$(\langle u_2, u_1, u_2, u_1\rangle, [K]) = -2.$$

On the other hand the following symmetry result generalizes the skew symmetry of the cup product

$$u_1 \cup u_2 + u_2 \cup u_1 = 0.$$

Let I and J be sequences $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_s)$. Then a *shuffle* of I and J is a sequence

$$L = (k_1, \dots, k_{r+s})$$

with indices

$$1 \leqq \alpha_1 < \alpha_2 < \dots < \alpha_r \leqq r + s, \ 1 \leqq \beta_1 < \beta_2 < \dots < \beta_s \leqq r + s$$

such that

$$k_{\alpha_n} = i_n, \ n = 1, \dots, r \quad \text{and} \quad k_{\beta_m} = j_m, \ m = 1, \dots, s.$$

For example 2113 arrives as a shuffle of 21 and 13 in two ways. We write $L = I|J$ to denote a shuffle of I and J.

THEOREM 6.8. *Let* $K = \{x_1, \dots, x_n|\beta(IJ)\}$ *be a basic complex where* $l(I) > 0$, $l(J) > 0$. *Then*

$$\sum_L \langle u_L\rangle = 0$$

where the sum is taken over all $L = I|J$ *obtained by shuffling* I *and* J.

Proof. Since

$$(\langle u_L\rangle, [K]) = \epsilon_L(\beta(IJ))$$

the result follows from the shuffle identities of [**4**].

So for example writing $(1, 1, 2) = (1) \cdot (1, 2)$ we see

$$2\langle u_1, u_1, u_2\rangle + \langle u_1, u_2, u_1\rangle = 0.$$

Definition 6.9. In view of the above results the following definition may be made. Let $K = \{x_1, \dots, x_n | \beta(I)\}$ be a basic complex of weight ω and let J be an index. Then the *minimal Massey product* $\langle \langle u_J \rangle \rangle$ is defined in any of the following cases

1. $l(J) \leqq \omega$
2. $J \nrightarrow I$.

In these cases $\langle \langle u_J \rangle \rangle$ is the element of $\langle u_J \rangle$ given by the defining set constructed above. Its value on the top dimensional cycle is given by

$$(\langle \langle u_J \rangle \rangle, [K]) = \epsilon_J(\beta(I)).$$

It is not immediately clear that the definition is independent of the complex K chosen to represent the homotopy type. This will follow after the discussion on naturality in Section 8.

Since the $u_1, \dots, u_n$ form a basis for H^1 the general minimal Massey product $\langle \langle v_J \rangle \rangle$ where

$$v_j = \Sigma \lambda_j^{\,i} u_i$$

may be defined by linearity

$$\langle \langle v_J \rangle \rangle = \langle \langle \Sigma \lambda_J^{\,I} u_I \rangle \rangle = \Sigma \lambda_J^{\,I} \langle \langle u_I \rangle \rangle,$$

assuming all the $\langle \langle u_I \rangle \rangle$ are defined.

7. Massey products in general two-dimensional complexes. Let

$$r = \beta_1(I_1) \dots \beta_k(I_k)$$

be a word in $F(x_1, \dots, x_n)$ where $\beta_i(I_i)$, $i = 1, \dots, k$ are basic commutators. Then write the complex

$$K = \{x_1, \dots, x_n | r\}$$

formally as $K = K_1 \dots K_k$ where

$$K_i = \{x_1, \dots, x_n | \beta_i(I_i)\}, \ i = 1, \dots, k.$$

The Massey products of K can be calculated from those of K_i as follows:

Suppose the 2-cell of K is obtained by making identifications on the boundary of $\mathscr{H}$. Divide $\mathscr{H}$ into k sectors using k radii from the origin to the k^{th} roots of unity. Suppose that $\langle \langle u_J \rangle \rangle$ is defined in each K_i with underlying space an h.d.s. Then an element of the Massey product $\langle u_J \rangle$ can be defined in K by means of the h.d.s. lying in the k sectors in turn. The value of this element on $[K]$ is given by the formula $\sum_{i=1}^{k} \phi_i$ where ϕ_i

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is the value of the corresponding Massey product in K_i for $i = 1, \ldots, k$. If these conditions are satisfied then the minimal product $\langle \langle u_J \rangle \rangle$ is defined to be the above sum.

If all the $\beta_i(I_i) \in F_l$ we can compute $\langle \langle u_J \rangle \rangle$ in the terms of Fox derivatives. Assuming $l(J) \leqq l$ it follows from 6.5 that

$$(\langle \langle u_J \rangle \rangle, [K]) = \sum_{i=1}^{k} \phi_i = \sum_{i=1}^{k} (\langle \langle u_J \rangle \rangle, [K_i])$$

$$= \sum_{i=1}^{k} \epsilon_J(\beta_i(I_i)) = \epsilon_J(\beta_1(I_1) \ldots \beta_k(I_k)) = \epsilon_J(r).$$

THEOREM 7.1. *Let $K = \{x_1, \ldots, x_n | r\}$ where $r \in F_\omega$, $\omega > 2$. Then all Massey products of length less than ω are defined and vanish.*

Proof. The relator r can be written $r = c_1 \ldots c_\nu$ where each c_i is a commutator of weight greater than $\omega - 1$. Then it is easy to construct basic commutators $\beta_i(I_i)$ in new variables $y_1, \ldots, y_m$, $i = 1, \ldots, \nu$, of weights $\geqq \omega$, a complex

$$L = \{y_1, \ldots, y_m | \beta_1(I_1) \ldots \beta_\nu(I_\nu) = s\}$$

where $\beta_1(I_1) \ldots \beta_\nu(I_\nu)$ is in collected form, and a map $f:L \rightarrow K$ which is an isomorphism on H^2. Now for L all Massey products of length less than ω vanish by the arguments given above and (6.6). The vanishing of the Massey products in K follow by naturality.

Definition 7.2. Call a complex $K = \{x_1, \ldots, x_n | r\}$ *in collected form* if $r = c_1 \ldots c_k$ is a product of basic commutators in collected form. In this case write

$$r = r_p \ldots r_1$$

where either

$$r_i = 1 \quad \text{or} \quad r_i \in F^i - F^{i+1}$$

and put

$$K = K_p \ldots K_1$$

where

$$K_i = \{x_1, \ldots, x_n | r_i\}, \ i = 1, \ldots, p.$$

Now any element $r \in F$ can be written as $\omega c_1 \ldots c_k$ where $\omega \in F_{m+1}$, $c_1 \ldots c_k$ is in collected form, each c_i has weight $\leqq m$, and m is arbitrary. By the above results the complex $\{x, \ldots, x_n | r\}$ has the same Massey products up to size m as $\{x_1, \ldots, x_n | c_1 \ldots c_k\}$. In other words it may be assumed that any complex is in collected form in calculating its Massey products.

Definition 7.3. Let $K = K_p \dots K_2$ be in collected form as above where $r \in F_2$. If I is an index of length q the *minimal Massey product* $\langle \langle u_I \rangle \rangle$ is defined if $\langle \langle u_I \rangle \rangle$ is defined on K_i, $i = 2, \dots, q$ and

$$(\langle \langle u_I \rangle \rangle, [K_i]) = 0 \quad \text{for } i = 2, \dots, q - 1.$$

THEOREM 7.4. *With the notation above if the minimal Massey product is defined its value is given by*

$$(\langle \langle u_I \rangle \rangle, [K]) = \epsilon_I(r).$$

7.5. *Calculation of the Massey product when K has more than one 2-cell.* Let

$$K = \{x_1, \dots, x_n | r_1, \dots, r_m\}$$

and consider the $n \times m$ matrix

$$A = [\epsilon_i(r_j)], \quad i = 1, \dots, n; j = 1, \dots, m.$$

The move $x_i \to x_i x_j$ performs a row sum operation on A. The move $r_i \to r_i r_j$ performs a column sum operation. These moves do not change the homotopy type of K. By a sequence of such operations combined with permutations K may be replaced by $\{y_1, \dots, y_n | s_1, \dots, s_m\}$ in which A now takes the form

$$A = \begin{array}{cc} & \begin{array}{cc} m_1 & m - m_1 \end{array} \\ \begin{array}{c} n_1 \\ n - n_1 \end{array} & \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & D \end{array}\right] \end{array}$$

where D is a diagonal matrix with non-zero diagonal entries.

Therefore H^1 is generated by $u_1, \dots, u_{n_1}$ and the free part of H^2 by $s_1, \dots, s_{m_1}$. The value of any Massey product on the 2-cell corresponding to s_i follows as before.

8. Multilinearity, naturality and symmetry of Massey products. In general Massey products are far from linear. Consider Example 1 given in the next section. Here $\langle u_1 + u_6, u_2 + u_5, u_3 + u_8, u_4 + u_7 \rangle$ is defined whereas products such as $\langle u_1, u_2, u_3, u_4 \rangle$ are not defined. However statements about linearity can be made provided all Massey products are defined and are minimal. Let

$$K = \{x_1, \dots, x_n | r\}$$

and let

$$\xi_i = \sum_{j=1}^{n} \lambda_{ij} u_j, \quad 1 \leqq i \leqq k.$$

Let A_i be the set of integers j such $\lambda_{ij} \neq 0$. Suppose

$$\langle \langle u_{\nu_1}, u_{\nu_2}, \ldots, u_{\nu_k} \rangle \rangle$$

is defined where $(\nu_1, \nu_2, \ldots, \nu_k)$ is any index with $\nu_i \in A_i$. Then the Massey product $\langle \xi_1, \ldots, \xi_k \rangle$ is defined and we put

$$\langle \langle \xi_1, \ldots, \xi_k \rangle \rangle = \sum \lambda_{1\nu_1} \lambda_{2\nu_2} \ldots \lambda_{k\nu_k} \langle \langle u_{\nu_1}, u_{\nu_2}, \ldots, u_{\nu_k} \rangle \rangle.$$

This definition will make sense if it can be shown that the minimal product is natural.

Let $f:L \to K$ be a cellular map and assume that

$$f_*:H_2(L) \to H_2(K)$$

is an isomorphism. Let

$$\langle \langle \alpha_1, \ldots, \alpha_n \rangle \rangle \quad \text{and} \quad \langle \langle f^*\alpha_1, \ldots, f^*\alpha_n \rangle \rangle$$

be defined minimal Massey products.

THEOREM 8.1. *With the notation above*

$$f^* \langle \langle \alpha_1, \ldots, \alpha_n \rangle \rangle = \langle \langle f^*\alpha_1, \ldots, f^*\alpha_n \rangle \rangle.$$

Proof. Since the value of a minimal Massey product on the 2-cell is $\epsilon_J(r)$, the result will follow from a naturality argument similar to that given in 6.6.

The following theorem shows that minimal Massey products satisfy symmetry properties akin to those enjoyed by the cup product. For example,

$$2 \langle \langle u_1, u_2, u_2, u_3 \rangle \rangle + \langle \langle u_2, u_1, u_2, u_3 \rangle \rangle + \langle \langle u_1, u_2, u_3, u_2 \rangle \rangle$$
$$+ \langle \langle u_2, u_3, u_1, u_2 \rangle \rangle + \langle \langle u_2, u_1, u_3, u_2 \rangle \rangle = 0.$$

THEOREM 8.2. *Let $\langle \langle u_{IJ} \rangle \rangle$ be a minimal Massey product defined on the two dimensional complex K, $l(I) > 0$, $l(J) > 0$. Then*

$$\sum \langle \langle u_L \rangle \rangle = 0$$

where the sum is taken over all shuffles L of I and J.

Proof. Since the result is true for basic commutators, 6.8, it is true in general by addition.

9. Some examples. Finally some examples of Massey product behavior are given below.

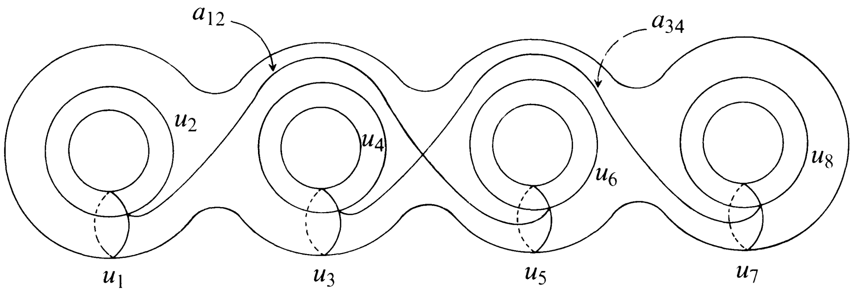
1. Let

$$K = \{x_1, \ldots, \ldots, x_8 | [x_1, x_2][x_3, x_4][x_5, x_6][x_7, x_8]\}$$

be the closed orientable surface of genus 4. Then $u_1, \ldots, u_8$ is a standard symplectic basis for H^1. The Massey product

$$\langle u_1 + u_6, u_2 + u_5, u_3 + u_8, u_4 + u_7 \rangle$$

exists and has indeterminacy the whole of H^2. A defining set for the generator of H^2 is given in figure 15.



$$a_{23} = a_{13} = a_{24} = 0$$

Figure 15.

2. Let

$$K = \{x_1, x_2, x_3, x_4 | [x_1, [x_2, x_3]][x_1, x_4]\}.$$

Then $\langle u_1, u_2, u_3 \rangle = H^2$ and $\langle \langle u_1, u_2, u_3 \rangle \rangle$ generates H^2.

3. The following example is due to O'Neill [**12**]. Let K be the complex

$$\{x_1, \ldots, x_5 | [x_1, [x_2, x_3]][x_1, x_5], [x_2, [x_3, x_4]][x_4, x_5]\}$$

with two 2-cells. Then $\langle u_1, u_2, u_3 \rangle$ and $\langle u_2, u_3, u_4 \rangle$ both contain zero, (due to indeterminacy). But $\langle u_1, u_2, u_3, u_4 \rangle$ is undefined. However note that both $\langle \langle u_1, u_2, u_3 \rangle \rangle$ and $\langle \langle u_2, u_3, u_4 \rangle \rangle$ are non-zero.

4. Consider

$$K = \{x_1, \ldots, x_5 | [x_1, [x_2, [x_3, x_4]]][x_1, [x_2, x_3]][x_1, x_5]\}.$$

Here $\langle u_1, u_2, u_3, u_4 \rangle$ is defined but not strictly defined. Note that $\langle \langle u_1, u_2, u_3 \rangle \rangle$ is non-zero.

5. Let T be the torus

$$T = \{x_1, x_2 | [x_1, x_2]\}$$

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then the Massey product $\langle u_1, u_1, u_1, u_1\rangle$ is defined and has indeterminacy. The minimal product $\langle\, \langle u_1, u_1, u_1, u_1\rangle\, \rangle$ is zero.

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