DIFFERENTIABLE POINTS OF ARCS IN CONFORMAL 3-SPACE

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Introduction. This paper is a generalization of the classification of the differentiable points in the conformal plane given in **(1)**. The main tools are the intersection and support properties of all the spheres through a differentiable point of an arc in conformal 3-space.

The discussion is also related to the classification **(2)** of the differentiable points of arcs in projective 4-space, since conformal 3-space can be represented on the surface of a 3-sphere in projective 4-space.

1. Pencils of spheres. In the following discussion, p, t, P, Q, and R will denote points of conformal space, while S and C will denote a sphere and a circle respectively. A sphere S decomposes the space into two open regions, its *interior* $\underset{\cdot}{S}$ and its *exterior* $\dot{S}$. If P is any fixed point which does not lie on S, the "interior" of S may be defined as the class of all points which do not lie on S and which are not separated from P by S. The exterior of S is then the class of points which are separated from P by S. The sphere through a proper circle C and a point $P \not\subset C$ will be denoted by $S(P; C)$. Much of the following discussion will depend on the use of pencils π of spheres and circles, determined by certain incidence and tangency conditions. A circle (point) which is common to all the spheres (circles) of a pencil is called a *fundamental circle* (*fundamental point*) of the pencil. In the pencil π of spheres through a fundamental circle C, there exists one and only one sphere $S(P; \pi)$ of π through any point P which does not lie on C. Similarly, in the pencil π of the spheres (circles) which touch a given sphere (circle) at a given point Q, there is one and only one sphere $S(P; \pi)$ [circle $C(P; \pi)$] of π which passes through any point $P \neq Q$. The *fundamental point* Q is regarded as a *point-sphere* (*point-circle*) belonging to π.

2. Convergence. A sequence of points $P_1, P_2, \ldots,$ is said to be *convergent* to P if to every sphere S with $P \subset \underset{\cdot}{S}$, there corresponds a positive integer $N = N(S)$, such that $P_\nu \subset \underset{\cdot}{S}$ if $\nu > N$. The convergence of spheres and circles to a point is defined in a similar fashion.

A sequence of spheres $S_1, S_2, \ldots,$ is said to be *convergent* to S if to every pair of points $P \subset \underset{\cdot}{S}$ and $Q \subset \dot{S}$, there corresponds a positive integer $N = N(P, Q)$, such that $P \subset \underset{\cdot}{S}_\nu$ and $Q \subset \dot{S}_\nu$ for every $\nu > N$.

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Finally a sequence of circles C_ν is said to be convergent to a circle C, if to every circle D which links with C, there corresponds a positive integer $N = N(D)$ such that C_ν links with D whenever ν is greater than N.

3. Arcs. An arc A is the continuous image of a real interval. If a sequence of points of this parameter interval converges to a point p, the corresponding sequence of image points is defined to be convergent to the image of p. The same small italics $p, t, \ldots$, will be used to denote both the points of the parameter interval and their images on A. The *end-(interior-) points* of A are the images of the end-(interior-) points of the parameter interval. A *neighbourhood* of p on A is the image of a neighbourhood of the parameter p on the parameter interval. If p is an interior point of A, this neighbourhood is decomposed by p into two (open) *one-sided neighbourhoods.* The images of distinct points of the parameter interval are considered to be different points of A, even though they may coincide in space. The notation $Q \neq P$ will indicate that the points Q and P do not coincide.

4. Differentiability. Let p be a fixed point of an arc A, and let t be a variable point of A. If P, Q, and p are mutually distinct points, the unique circle through these points will be denoted by $C(P, Q; \gamma_0)$. The symbol γ_0 itself will denote the family of all circles through p, including the point-circle p.

A is called *once-differentiable* at p if the following condition Γ_1 is satisfied:

Γ_1: If the parameter t is sufficiently close to, but different from, the parameter p, the circle $C(P, t; \gamma_0)$ is uniquely defined, and converges if t tends to p.

Thus the limit circle, which will be denoted by $C(P; \gamma_1)$, is independent of the way t converges to p. The family of all such circles, (i.e., the circles $C(P; \gamma_1)$ for all points $P \neq p$), together with the point circle p, will be denoted by γ_1.

A is called *twice-differentiable* at p if, in addition to the condition Γ_1, the following condition is also satisfied:

Γ_2: If the parameter t is sufficiently close to, but different from, the parameter p, the circle $C(t; \gamma_1)$ is uniquely defined, and converges if t tends to p.

The limit circle of the sequence $C(t; \gamma_1)$ will be denoted by $C(\gamma_2)$, the *osculating circle* of A at p, and, occasionally, also by the symbol γ_2 alone.

5. Structure of the families of circles through p. In this section, relations among the families of circles $\gamma_0, \gamma_1, \gamma_2$ are discussed.

THEOREM 1. *Suppose A satisfies condition Γ_1 at p. Then t does not coincide with p if the parameter t is sufficiently close to, but different from, the parameter p.*

Proof. Let P be any point different from p. By condition Γ_1, $C(P, t; \gamma_0)$ is uniquely defined when the parameter t is close to, but different from, the parameter p. Thus $t \neq p$.

THEOREM 2. *Suppose A satisfies condition Γ_1 at p. Then the angle at p between any two circles of γ_1 is* 0.

Proof. Let P, Q, R_1, R_2 be variable points, and let R_1 and R_2 converge to the same point R. Suppose there is a fixed sphere separating R from both P and Q. Then

$$\lim \angle\, [C(P, R_1, R_2),\quad C(Q, R_1, R_2)] = 0 \tag{1}$$

whether or not the circles themselves converge. In particular, the angle between $C(P; \gamma_1)$ and $C(Q; \gamma_1)$ is equal to 0.

COROLLARY 1. *If $C(P; \gamma_1)$ and $C(Q; \gamma_1)$ have another point in common, they are identical; thus there is one and only one circle of γ_1 through each point $P \neq p$.*

COROLLARY 2. *γ_1 consists of those circles C which meet a given circle of γ_1 at p at the angle* 0. *Thus the circles of γ_1 all touch at p.*

Proof. Let $P \subset C$, $P \neq p$. Suppose C meets some circle of γ_1 at angle 0 at p. Then C and $C(P; \gamma_1)$ also meet at angle 0 at p and have the point P in common. Hence they are identical.

COROLLARY 3. *If Γ_1 holds for a single point $P \neq p$, then it holds for all such points.*

Proof. If $Q \neq p$, by (1)

$$\lim \angle\, [C(Q, t, p),\quad C(P, t, p)] = 0.$$

Thus $C(Q, t; \gamma_0)$ converges to the unique circle through Q which touches $C(P; \gamma_1)$ at p.

THEOREM 3. *Suppose A satisfies the conditions Γ_1 and Γ_2 at p. Then*

$$\gamma_0 \supset \gamma_1 \supset \gamma_2. \tag{2}$$

Proof. It is clear that $\gamma_0 \supset \gamma_1$. If $C(\gamma_2) = p$, it belongs to γ_1 by definition. Suppose $C(\gamma_2) \neq p$. Then $C(\gamma_2)$, being the limit of a sequence of circles $C(t; \gamma_1)$ each of which touches a given circle $C(P; \gamma_1)$ of γ_1, must itself touch $C(P; \gamma_1)$ at p. Thus $C(\gamma_2) \in \gamma_1$ (cf. Theorem 2, Corollary 2.)

COROLLARY. *If $P \subset C(\gamma_2)$, $P \neq p$, then $C(\gamma_2) = C(P; \gamma_1)$.*

The conditions Γ_1 and Γ_2 are independent. For example, suppose a rectangular cartesian coordinate system is introduced and the arc

$$x = t,\ y = t^2,\ z = \begin{cases} (1 - \sqrt{1 - t^2 - t^4}) \sin t^{-1}, & 0 < |t| \leqslant \frac{1}{2} \\ 0 & , t = 0 \end{cases}$$

is considered. Γ_1 is satisfied at $t = 0$, but Γ_2 is not satisfied there.

6. Differentiable points of arcs. In addition to the conditions Γ_1 and Γ_2, three more conditions involving spheres are introduced. Suppose P, Q,

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and R are any three fixed points such that P, Q, R, and p do not lie on the same circle. It will be convenient to denote the unique sphere through p and the points P, Q, and R by the symbol $S(P, Q, R; \sigma_0)$. σ_0 will denote the family of all spheres through p, including the point-sphere p.

A is called *thrice-differentiable* at p if the following three conditions are satisfied:

Σ_1: The sphere $S(P, Q, t; \sigma_0)$ is uniquely defined when the parameter t is sufficiently close to, but different from, the parameter p, and converges, if t tends to p, to a limit sphere which will be denoted by $S(P, Q; \sigma_1)$.

Σ_2: The sphere $S(P, t; \sigma_1)$ is uniquely defined when the parameter t is sufficiently close to, but different from, the parameter p, and converges, if t tends to p, to a limit sphere which will be denoted by $S(P; \sigma_2)$.

Σ_3: The sphere $S(t; \sigma_2)$ is uniquely defined when the parameter t is sufficiently close to, but different from, the parameter p, and converges, if t tends to p, to a limit sphere which will be denoted by $S(\sigma_3)$.

The families of all the spheres $S(P, Q; \sigma_1)$, (i.e., the spheres $S(P, Q; \sigma_1)$ for all pairs of points P and Q such that $P \neq Q$ and P, $Q \neq p$), together with the point-sphere p, will be denoted by σ_1. Similarly, the family of all the spheres $S(P; \sigma_2)$ (including the point-sphere p if $C(\gamma_2) = p$) will be denoted by σ_2. The unique *osculating sphere* $S(\sigma_3)$ will occasionally be denoted by σ_3 alone.

The point p is called a *differentiable point* of A if A is thrice-differentiable at p.

7. Structure of the families of spheres through p. Although the conditions Γ_1 and Γ_2 are independent, not all the conditions Γ_1, Γ_2, Σ_1, Σ_2, and Σ_3 are independent. In addition, the families of spheres σ_0, σ_1, σ_2, and σ_3 are closely connected with the families of circles γ_0, γ_1, and γ_2.

THEOREM 4. *Suppose A satisfies condition Σ_1 at p. Let C be any fixed circle. Then $t \not\subset C$ if the parameter t is sufficiently close to, but different from, the parameter p.*

Proof. The assertion is clearly true if $p \not\subset C$. Suppose $p \subset C$, and let P, Q, p be mutually distinct points on C. By condition Σ_1, $S(P, Q, t; \sigma_0)$ is defined when t is sufficiently close to p. Thus $t \not\subset C(P, Q, p) = C$.

The following example shows that Γ_1 does not imply Σ_1 in general (cf., however, Theorem 5). Consider the arc

$$x = t, y = \begin{cases} t^2 \cos t^{-1}, & 0 < |t| \leqslant 1 \\ 0 & , t = 0 \end{cases}, \quad z = \begin{cases} t^2 \sin t^{-1}, & 0 < |t| \leqslant 1 \\ 0 & , t = 0 \end{cases},$$

in the neighbourhood of $t = 0$. If $P = \infty$, $Q = (1, 0, 0)$, and $p = (0, 0, 0)$, the sphere $S(P, Q, t; \sigma_0)$ does not converge, while, e.g., $C(P, t; \gamma_0)$ converges to the x-axis and by Theorem 2, Corollary 3, Γ_1 is satisfied.

THEOREM 5. *If A satisfies Σ_1 at p, then Γ_1 holds there and*

$$C(Q;\gamma_1) = \bigcap_{P\neq Q,p} S(P, Q;\sigma_1). \tag{3}$$

Conversely, let A satisfy Γ_1 at p. Then Σ_1 holds at p for all pairs P, Q such that $P \not\subset C(Q;\gamma_1)$, and $S(P, Q;\sigma_1) = S[P; C(Q;\gamma_1)]$.

Proof. Suppose Σ_1 holds at p. Theorem 4 implies that $t \neq p$ if t is close to p. Let $Q \neq p$. Since $C(Q, t;\gamma_0) \subset S(P, Q, t;\sigma_0)$, any limit circle of $C(Q, t;\gamma_0)$ lies on $S(P, Q;\sigma_1)$ for every choice of $P \neq Q$, p. Thus if $P_1 \not\subset S(P_2, Q;\sigma_1)$, any limit circle of $C(Q, t;\gamma_0)$ lies on $S(P_1, Q;\sigma_1) \cap S(P_2, Q;\sigma_1)$ and is therefore uniquely determined. Hence $C(Q, t;\gamma_0)$ converges and we have

$$C(Q;\gamma_1) = \bigcap_{P\neq Q,p} S(P, Q;\sigma_1).$$

Conversely, suppose that Γ_1 holds. If $P \not\subset C(Q;\gamma_1)$, then $P \not\subset C(Q, t;\gamma_0)$ when t is sufficiently close to p and

$$S[P; C(Q;\gamma_1)] = \lim_{t\to p} S[P; C(Q, t;\gamma_0)] = \lim_{t\to p} S(P, Q, t;\sigma_0). \tag{4}$$

Thus for all pairs of points P and Q such that $P \not\subset C(Q;\gamma_1)$, $S(P, Q, t;\sigma_0)$ converges, Σ_1 is satisfied, and $S(P,Q;\sigma_1)$ is the sphere through P and $C(Q;\gamma_1)$.

COROLLARY. *There is only one sphere of σ_1 which contains two points not on the same circle of γ_1.*

Remark. Condition Γ_1 is still satisfied when Σ_1 is replaced by a weaker assumption:

Suppose $S'_1 = S(P_1, Q_1, t;\sigma_0) \to S_1$, $S'_2 = S(P_2, Q_2, t;\sigma_0) \to S_2$, and suppose further that $S_1 \cap S_2 = C \neq p$. Then Γ_1 holds at p. For, let $S'_1 \cap S'_2 = C'$. Then $C' \to C$ and $C' \supset p$ and t. As in equation (1), $\lim \angle\, [C(P_1, t;\gamma_0), C'] = 0$. Thus $C(P_1, t;\gamma_0)$ converges to the unique circle through P_1 which touches C at p. By Theorem 2, Corollary 3, Γ_1 holds at p.

If, however, $S_1 \cap S_2 = p$, Γ_1 need not hold; e.g., take $P_1 = \infty$, $Q_1 = (1,0,0)$, $P_2 = (0, 0, 2)$, $Q_2 = (1, 0, 1)$, $p = (0, 0, 0)$, and let A be the arc,

$$x = \begin{cases} t \sin t^{-1}, & 0 < |t| \leqslant 1 \\ 0 & , t = 0 \end{cases}, \quad y = t, \ z = t^2.$$

S'_1 converges to the xy-plane, S'_2 converges to the sphere $x^2 + y^2 + z^2 - 2z = 0$, but Γ_1 does not hold.

THEOREM 6. *Suppose Σ_1 holds at p. Choose $C \in \gamma_1$, $C \neq p$. Then σ_1 is the set of all spheres which touch C at p.*

Proof. Suppose a sphere $S(P, Q;\sigma_1)$ of σ_1 meets C in a point $R \neq p$. If $R \subset C(Q;\gamma_1)$, by Theorem 5 and Theorem 2, Corollary 1,

$$S(P, Q;\sigma_1) \supset C(Q;\gamma_1) = C$$

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while if $R \not\subset C(Q; \gamma_1)$,

$$\begin{aligned} S(P, Q; \sigma_1) &= S[R; C(Q; \gamma_1)] = S(R, Q; \sigma_1) \\ &= S[Q; C(R; \gamma_1)] = S(Q; C) \supset C. \end{aligned}$$

Conversely, suppose a sphere S touches C at p. If $S \supset C$, then $S \in \sigma_1$ (Theorem 5). If $S \cap C = p$, choose a point $Q \subset S$, $Q \neq p$. Let $C_0 = S(Q; C) \cap S$. Then C_0 touches C at p. By Theorem 2, Corollary 2, $C_0 \in \gamma_1$. Since $S \supset C_0$ and $C_0 \in \gamma_1$, it follows from Theorem 5 that $S \in \sigma_1$.

THEOREM 7. *If A satisfies Σ_1 and Σ_2 at p, then Γ_1 and Γ_2 will also hold at p, and equations* (3) *and*

$$C(\gamma_2) = \bigcap_{P \neq p} S(P; \sigma_2) \tag{5}$$

will be satisfied there. Conversely, let A satisfy Γ_1 and Γ_2 at p and let $C(\gamma_2) \neq p$. If $P \not\subset C(\gamma_2)$, then Σ_2 will hold at p for P, and $S(P; \sigma_2)$ will be the sphere through P and $C(\gamma_2)$.

Proof. Suppose Σ_1 and Σ_2 hold at p. In view of Theorem 5, we have only to show that Σ_2 implies Γ_2, and that (5) holds. If t is close to p, (3) implies that $C(t; \gamma_1) \subset S(P, t; \sigma_1)$ for every point $P \neq p$. Hence any limit circle of $C(t; \gamma_1)$ lies on $S(P; \sigma_2)$. Thus if $P_1 \not\subset S(P_2; \sigma_2)$ this limit circle lies on $S(P_1; \sigma_2) \cap S(P_2; \sigma_2)$ and is therefore uniquely determined. Hence $C(t; \gamma_1)$ converges, and

$$C(\gamma_2) = \bigcap_{P \neq p} S(P; \sigma_2).$$

Thus Σ_2 implies Γ_2 and (5) holds.

Conversely, suppose that Γ_1 and Γ_2 hold and $C(\gamma_2) \neq p$. If $P \not\subset C(\gamma_2)$, then $P \not\subset C(t; \gamma_1)$ when t is sufficiently close to p, and by Theorem 5,

$$S[P; C(\gamma_2)] = \lim_{t \to p} S[P; C(t; \gamma_1)] = \lim_{t \to p} S(P, t; \sigma_1).$$

Hence $S(P, t; \sigma_1)$ exists and converges. Thus $S(P; \sigma_2) = S[P; C(\gamma_2)]$.

COROLLARY 1. *If A satisfies Σ_1(Σ_1 and Σ_2) at p, then A is once-(twice-) differentiable there.*

In particular, this implies

COROLLARY 2. *If p is a differentiable point of A, then Γ_1 and Γ_2 hold there.*

COROLLARY 3. $S(\sigma_3) \supset C(\gamma_2)$.

Proof. By (5),

$$S(t; \sigma_2) \supset \bigcap_{P \neq p} S(P; \sigma_2) = C(\gamma_2).$$

Hence $S(\sigma_3) \supset C(\gamma_2)$.

This implies

COROLLARY 4. *If* $S(\sigma_3) = p$, *then* $C(\gamma_2) = p$.

COROLLARY 5. *If* $C(\gamma_2) \neq p$, σ_2 *consists of the spheres through* $C(\gamma_2)$.

The conditions Γ_1 and Γ_2 by themselves do not imply Σ_1 in general, whether or not $C(\gamma_2) = p$. Consider, for example, the arc

$$x = t,\ y = \begin{cases} t^3 \sin t^{-1}, & 0 < |t| \leqslant 1 \\ 0 & ,\ t = 0 \end{cases},\ z = \begin{cases} t^3 \cos t^{-1}, & 0 < |t| \leqslant 1 \\ 0 & ,\ t = 0 \end{cases}$$

which satisfies Γ_1 and Γ_2 at $t = 0$, $C(\gamma_2)$ being the x-axis. When $P = \infty$, $Q = (1, 0, 0)$, the sphere $S(P, Q, t; \sigma_0)$ is a plane through the x-axis, and this plane does not converge when $t \to 0$. Thus Σ_1 is not satisfied.

Condition Σ_1 is a very strong one for it implies not only Γ_1, but, as the following theorems show, Σ_2 and Γ_2 as well, and even Σ_3 in the case $C(\gamma_2) \neq p$.

THEOREM 8. *Suppose A satisfies* Σ_1 *at* p. *Then A also satisfies* Σ_2 *at* p.

Proof. Let P be any point $\neq p$. Theorem 4 implies that t does not lie on $C(P; \gamma_1)$ if t is close to p. Hence by Theorem 5, $S(P, t; \sigma_1) = S[t; C(P; \gamma_1)]$. Let $Q \subset C(P; \gamma_1)$, $Q \neq P, p$. Then $C(P; \gamma_1) = C(P, Q; \gamma_0)$. Thus $S(P, t; \sigma_1) = S[t; C(P, Q; \gamma_0)] = S(P, Q, t; \sigma_0)$, and Σ_1 now implies that

$$\lim_{t \to p} S(P, t; \sigma_1) = S(P, Q; \sigma_1). \tag{6}$$

Since $S(P; \sigma_2)$ exists for each point $P \neq p$, Σ_2 is satisfied.

COROLLARY 1. *If A satisfies* Σ_1 *at* p, *it also satisfies* Γ_2 *there.*

Proof. By Theorem 7, condition Σ_2 implies Γ_2.

COROLLARY 2. *If A satisfies* Σ_1 *at* p, p *is a differentiable point of A if and only if* $S(t; \sigma_2)$ *converges when t tends to* p.

Relation (6) implies

COROLLARY 3. $S(P; \sigma_2) \in \sigma_1$.

THEOREM 9. *Suppose A satisfies* Σ_1 (*and hence* Σ_2, Γ_1, *and* Γ_2) *at* p, *and suppose* $C(\gamma_2) \neq p$. *Then A also satisfies* Σ_3 *at* p.

Proof. If t is close to but different from p, $S(t; \sigma_2)$ is defined. By Theorem 4, $t \not\subset C(\gamma_2)$, and by Theorem 7, $S(t; \sigma_2) = S[t; C(\gamma_2)]$. Let $P \subset C(\gamma_2)$, $P \neq p$. Then by the corollary of Theorem 3, $C(\gamma_2) = C(P; \gamma_1)$ and hence $S(t; \sigma_2) = S[t; C(P, \gamma_1)] = S(P, t; \sigma_1)$. Σ_2 now implies that

$$\lim_{t \to p} S(t; \sigma_2) = \lim_{t \to p} S(P, t; \sigma_1) = S(P; \sigma_2) \tag{7}$$

Thus $S(t; \sigma_2)$ converges and Σ_3 holds.

COROLLARY. *If A satisfies condition* Σ_1 *at* p *and if* $C(\gamma_2) \neq p$, *then* p *is a differentiable point of A.*

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The following example shows that p need not be a differentiable point of A when Σ_1 is satisfied and $C(\gamma_2) = p$. Consider the arc defined by

$$x = t^2, y = t^3, z = \begin{cases} t^4 \sin t^{-1}, & 0 < |t| \leqslant 1 \\ 0 & , t = 0 \end{cases}.$$

It can readily be verified that A satisfies Σ_1 at $t = 0$, and that the spheres of σ_2 touch the xy-plane at the origin. Thus $C(\gamma_2)$ is a point circle. However, as t tends to 0, $S(t; \sigma_2)$ oscillates and $x^2 + y^2 + z^2 \pm z = 0$ are two accumulation spheres of the sequence $S(t; \sigma_2)$. Thus Σ_3 does not hold at $t = 0$.

THEOREM 10. *Let Σ_1 hold at p, and let $C(\gamma_2) = p$. Then σ_2 is the set of spheres which touch a given sphere of σ_2 at p.*

Proof. Let P and Q be variable points and let C be a variable circle converging to a fixed point. Suppose there is a fixed sphere which separates this point from P and Q. Then

$$\lim \angle\ [S(P; C), S(Q; C)] = 0$$

whether or not the spheres $S(P; C)$ and $S(Q; C)$ themselves converge. In particular, let P and Q be fixed points $\neq p$ and let $C = C(t; \gamma_1) \rightarrow p$, as $t \rightarrow p$, $t \subset A, t \neq p$. Then

$$(8) \qquad \angle\ [S(P; \sigma_2), S(Q; \sigma_2)] = \lim_{t \to p} \angle\ [S(P, t; \sigma_1), S(Q, t; \sigma_1)] = 0.$$

Hence any two spheres of σ_2 touch at p.

Conversely, let S be a sphere which touches $S(P; \sigma_2)$. Choose a point $Q \subset S$, $Q \neq p$. Then $S(Q; \sigma_2)$ also touches $S(P; \sigma_2)$ at p and $S(Q; \sigma_2) = S$. Thus $S \in \sigma_2$.

COROLLARY 1. *σ_2 is the family of spheres, the intersection of any two of which is* $C(\gamma_2)$ (cf. Theorem 7, Corollary 5).

COROLLARY 2. *There is one and only one sphere of σ_2 through each point $\not\subset C(\gamma_2)$; that is, if $Q \subset S(P; \sigma_2)$, $Q \not\subset C(\gamma_2)$, then $S(P; \sigma_2) = S(Q; \sigma_2)$.*

THEOREM 11. *If p is a differentiable point of A, then*

$$(9) \qquad \sigma_0 \supset \sigma_1 \supset \sigma_2 \supset \sigma_3.$$

Proof. Evidently $\sigma_0 \supset \sigma_1$. Theorem 8, Corollary 3 shows that $\sigma_1 \supset \sigma_2$. This can also be seen as follows:

Let $P \neq p$. By Theorem 6, any sphere $S(P; \sigma_2)$ of σ_2 is the limit of a sequence of spheres $S(P, t; \sigma_1)$ each of which touches a proper circle $C \in \gamma_1$ at p. Thus $S(P; \sigma_2)$ also touches C at p, and $S(P; \sigma_2) \in \sigma_1$.

Let $C(\gamma_2) \neq p$. By Theorem 7, Corollary 5, σ_2 is the set of all the spheres through $C(\gamma_2)$. Hence $S(\sigma_3)$, being the limit of a sequence of spheres through $C(\gamma_2)$, is itself a sphere through $C(\gamma_2)$, and thus a sphere of σ_2. Relation (7) also implies that $\sigma_2 \supset \sigma_3$ when $C(\gamma_2) \neq p$.

Suppose $C(\gamma_2) = p$. By Theorem 10, σ_2 is the set of all the spheres which touch a given sphere $\neq p$ of σ_2 at p. Hence $S(\sigma_3)$, being the limit of a sequence of such tangent spheres, is itself a sphere of σ_2.

This section can be summarized by the following remark:

Let p be a differentiable point of an arc A. Let $P \neq p$. In addition, if $S(\sigma_3) \neq p$, let $P \subset S(\sigma_3)$.

Let

$$C = \begin{cases} C(\gamma_2) \text{ if } C(\gamma_2) \neq p \\ C(P;\gamma_1) \text{ if } C(\gamma_2) = p \end{cases}, \quad S = \begin{cases} S(\sigma_3) \text{ if } S(\sigma_3) \neq p \\ S(P;\sigma_2) \text{ if } S(\sigma_3) = p \end{cases}.$$

Then $C \subset S$, and the structures of γ_1, σ_1, and σ_2 are completely determined by C and S.

8. Support and intersection. Let p be an interior point of an arc A. Then we call p a *point of support* (*intersection*) with respect to a sphere S, if a sufficiently small neighbourhood of p is decomposed by p into two one-sided neighbourhoods which lie in the same region (in different regions) bounded by S. S is then called a *supporting* (*intersecting*) sphere of A at p. Thus S supports A at p if $p \not\subset S$. By definition, the point-sphere p always supports A at p.

It is possible for a sphere to have points $\neq p$ in common with every neighbourhood of p on A (cf., e.g., equation II, §10). In this case S neither supports nor intersects A at p.

9. Intersection and support properties of the families $\sigma_0 - \sigma_1$, $\sigma_1 - \sigma_2$, and $\sigma_2 - \sigma_3$. Throughout the remainder of the paper, the point p is assumed to be a differentiable interior point of A.

THEOREM 12. *Every sphere $\neq S(\sigma_3)$ either supports or intersects A at p.*

Proof. If a sphere S neither supports nor intersects A at p, then $p \subset S$ and there exists a sequence of points $t \to p$, $t \subset A \cap S$, $t \neq p$. We may assume that conditions Σ_1, Σ_2, and Σ_3 hold for this sequence. Choose points P and Q on S such that P, Q, and p are mutually distinct. Then condition Σ_1 implies $S = S(P, Q, t; \sigma_0)$ for each t, and hence $S = S(P, Q; \sigma_1)$.

By Theorem 5, $S = S(P, Q; \sigma_1) \supset C(P; \gamma_1)$. By Theorem 4, $t \not\subset C(P; \gamma_1)$ and again by Theorem 5, $S = S[t; C(P; \gamma_1)] = S(P, t; \sigma_1)$. Condition Σ_2 now implies that $S = S(P; \sigma_2)$.

Finally, by Theorem 7, $S \supset C(\gamma_2)$, and by Theorem 4, $t \not\subset C(\gamma_2)$. If $C(\gamma_2) \neq p$, Theorem 7 implies that $S = S[t; C(\gamma_2)] = S(t; \sigma_2)$, while if $C(\gamma_2) = p$, Theorem 10 implies that $S = S(t; \sigma_2)$. Applying the condition Σ_3, we are led to the conclusion $S = S(\sigma_3)$.

THEOREM 13. *If $S(\sigma_3) = p$, then the spheres of $\sigma_2 - \sigma_3$ all intersect A at p, or they all support.*

Proof. Let S' and S'' be two distinct spheres of $\sigma_2 - \sigma_3$. Since $S(\sigma_3) = p$, Theorem 7, Corollary 4 implies that $C(\gamma_2) = p$, and Theorem 10 implies that S' and S'' touch at p. Thus we may assume that $S'' \subset (p \cup \dot{S}')$ and $S' \subset (p \cup \underset{\cdot}{S}'')$. Suppose now, for example, that S' supports A at p while S'' intersects. Then $A \cap \dot{S}''$ is not void and hence we may assume $A \subset (p \cup \dot{S}')$. Let $t \to p$ in $A \cap \underset{\cdot}{S}''$; thus $t \subset \underset{\cdot}{S}'' \cap \dot{S}'$. Hence $S(t; \sigma_2) \subset (\underset{\cdot}{S}'' \cap \dot{S}') \cup p$. Consequently $S(t; \sigma_2)$ cannot converge to $S(\sigma_3) = p$ as $t \to p$. Thus S' and S'' must both support or both intersect A at p.

THEOREM 14. *If $S(\sigma_3) \neq p$ and $C(\gamma_2) = p$, then every sphere of $\sigma_2 - \sigma_3$ supports A at p.*

Proof. Suppose $C(\gamma_2) = p$, so that the spheres of σ_2 all touch at p (Theorem 10). Let $S \in \sigma_2$, $S \neq S(\sigma_3)$, $S \neq p$. If a sequence of points t exists such that $t \subset A \cap \dot{S}$, $t \to p$, then each $S(t; \sigma_2)$ lies in the closure of $\dot{S}$. Hence $S(\sigma_3)$ will lie in the same domain, and therefore even in $p \cup \dot{S}$. Similarly, the existence of a sequence $t' \subset \underset{\cdot}{S} \cap A$, $t' \to p$, implies that $S(\sigma_3) \subset p \cup \underset{\cdot}{S}$. Thus if S intersects A at p, $S(\sigma_3) \subset (p \cup \dot{S}) \cap (p \cup \underset{\cdot}{S}) = p$; in other words, $S(\sigma_3) = p$.

THEOREM 15. *All the spheres of $\sigma_0 - \sigma_1(\sigma_1 - \sigma_2;\ \sigma_2 - \sigma_3)$ support A at p, or they all intersect.*

Proof. Let S' and S'' be two distinct spheres of $\sigma_0 - \sigma_1$ $(\sigma_1 - \sigma_2; \sigma_2 - \sigma_3)$. Suppose, for the moment, that the intersection $S' \cap S''$ is a proper circle $C_0 = C(P, Q; \gamma_0)$ $[C_1 = C(P; \gamma_1);\ C_2 = C(\gamma_2)]$ (cf. equations (3) and (5)). Suppose, for example, that S' intersects while S'' supports A at p. With no loss in generality, we may assume that $A \subset \dot{S}'' \cup p$. Thus $A \cap \underset{\cdot}{S}'$ and $A \cap \dot{S}'$ are not void. If $t \subset A \cap \underset{\cdot}{S}'$, by Theorems 4, 5, and 7, $S(P, Q, t; \sigma_0) = S(t; C_0)$ $[S(P, t; \sigma_1) = S(t; C_1); S(t; \sigma_2) = S(t; C_2)]$ lies in the closure of

$$(\underset{\cdot}{S}' \cap \dot{S}'') \cup (\dot{S}' \cap \underset{\cdot}{S}'').$$

Letting $t \to p$ on A, we conclude that $S(P, Q; \sigma_1)$ $[S(P; \sigma_2); S(\sigma_3)]$ lies in the same closed domain. By letting t' converge to p through $\dot{S}' \cap A$, we obtain symmetrically that $S(P, Q; \sigma_1)$ $[S(P; \sigma_2);\ S(\sigma_3)]$ also lies in the closure of

$$(\underset{\cdot}{S}' \cap \underset{\cdot}{S}'') \cup (\dot{S}' \cap \dot{S}'').$$

Hence $S(P, Q; \sigma_1)$ $[S(P; \sigma_2); S(\sigma_3)]$ lies in the intersection $S' \cup S''$ of these two domains, i.e., $S(P, Q; \sigma_1)$ $[S(P; \sigma_2); S(\sigma_3)]$ is either S' or S'', contrary to our assumptions. Thus S' and S'' both support or they both intersect in this case.

Suppose now that $S' \cap S'' = p$. In view of Theorems 13 and 14 there remain to be considered only the cases where S' and S'' belong to the family $\sigma_0 - \sigma_1$, or both belong to $\sigma_1 - \sigma_2$.

By Theorem 6, any sphere S through p which does not touch a circle C of γ_1 belongs to $\sigma_0 - \sigma_1$; by Theorem 6, Theorem 7, Corollary 5, and Theorem 10, any sphere S which touches a circle C of γ_1, but does not contain $C(\gamma_2)$ in case $C(\gamma_2) \neq p$, and does not touch a sphere of σ_2 in case $C(\gamma_2) = p$, belongs to $\sigma_1 - \sigma_2$. Hence there exists a sphere S of $\sigma_0 - \sigma_1$ $(\sigma_1 - \sigma_2)$ which intersects S' and S'' respectively in a proper circle. From the

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above, S and S', and also S and S'', both support or both intersect A at p. Thus S' and S'' both support or both intersect A at p.

THEOREM 16. *If $C(\gamma_2) \neq p$, every sphere of $\sigma_1 - \sigma_2$ supports A at p.*

Proof. Suppose $S \in \sigma_1 - \sigma_2$ intersects A at p. Let $t \to p$, $t \subset A \cap \underset{\cdot}{S}$, $t \neq p$. By Theorem 6, $C(t; \gamma_1)$ touches S at p and hence $C(t; \gamma_1) \subset \underset{\cdot}{S} \cup p$. Since $C(t; \gamma_1) \to C(\gamma_2)$ it follows that $C(\gamma_2) \subset \underset{\cdot}{S} \cup S$. If t' converges to p through $A \cap \dot{S}$, it follows symmetrically that $C(\gamma_2) \subset \dot{S} \cup S$. Thus $C(\gamma_2) \subset S$. Since $S \notin \sigma_2$, however, Theorem 7, Corollary 5 implies that $C(\gamma_2) = p$.

10. Characteristics and a classification of the differentiable points. The *characteristic* $(a_0, a_1, a_2, a_3; i)$ of a differentiable point p of an arc A is defined as follows:

$a_0, a_1, a_2 = 1$ or 2; $a_3 = 1, 2,$ or ∞; $i = 1, 2,$ or 3.

a_0 is even or odd according as the spheres of $\sigma_0 - \sigma_1$ support or intersect.

$a_0 + a_1$ is even or odd according as the spheres of $\sigma_1 - \sigma_2$ support or intersect.

$a_0 + a_1 + a_2$ is even or odd according as the spheres of $\sigma_2 - \sigma_3$ support or intersect.

$a_0 + a_1 + a_2 + a_3$ is even if $S(\sigma_3)$ supports and odd if $S(\sigma_3)$ intersects, while $a_3 = \infty$ if $S(\sigma_3)$ neither supports nor intersects.

$i = 1$ if $C(\gamma_2) \neq p$; $i = 2$ if $C(\gamma_2) = p$ and $S(\sigma_3) \neq p$; and $i = 3$ if $S(\sigma_3) = p$.

Theorems 16, 14, and the convention that $S(\sigma_3)$ supports when it is the point-sphere, lead to the restriction on the characteristic $(a_0, a_1, a_2, a_3; i)$ that $a_0 + \ldots + a_i$ is even. As a result, there are just 32 types of differentiable points; 12 when $i = 1$, 12 when $i = 2$, and 8 when $i = 3$.

Examples of each of the 32 types are given by the curves

$$\text{(I)} \qquad x = t^m, \quad y = t^n, \quad z = t^r,$$

for the cases $a_3 = 1$ or 2, and

$$\text{(II)} \qquad x = t^m, y = t^n, z = \begin{cases} t^r \sin t^{-1}, & \text{if } 0 < |t| \leqslant 1 \\ 0 & , t = 0 \end{cases},$$

for the cases $a_3 = \infty$, all relative to the point $t = 0$. The indices, m, n, and r are positive integers and $m < n < r$. The different types are determined by the parities of the indices m, n, and r and the relative magnitudes of m, n, r, and $2m$. In each of these examples the circles of γ_1 and the spheres of σ_1 touch the x-axis at the origin. In the case $i = 1$, σ_2 is the family of planes through the x-axis; while in each of the cases $i = 2$ or 3, σ_2 is the family of spheres which touch the xy-plane at the origin (cf. Remark at the end of §7).

The first of the following tables lists examples of all the types of differentiable points together with their characteristics, while the second table summarizes the properties of these types. Congruences are mod 2.

TABLE I

$(a_0, a_1, a_2, a_3; i)$

Equation	$i = 1$	$m < 2m < n < r$			$i = 2$	$m < n < 2m < r$			$i = 3$	$m < n < r < 2m$		
I	$(1, 1, 1, 1; 1)$	$m \equiv 1$	$n \equiv 1$	$r \equiv 0$	$(1, 1, 2, 1; 2)$	$m \equiv 1$	$n \equiv 0$	$r \equiv 1$	$(1, 1, 1, 1; 3)$	$m \equiv 1$	$n \equiv 0$	$r \equiv 1$
	$(1, 1, 1, 2; 1)$			$r \equiv 1$	$(1, 1, 2, 2; 2)$			$r \equiv 0$	$(1, 1, 2, 2; 3)$			$r \equiv 0$
	$(1, 1, 2, 1; 1)$		$n \equiv 0$	$r \equiv 1$	$(1, 2, 1, 1; 2)$		$n \equiv 1$	$r \equiv 1$	$(1, 2, 2, 1; 3)$		$n \equiv 1$	$r \equiv 1$
	$(1, 1, 2, 2; 1)$			$r \equiv 0$	$(1, 2, 1, 2; 2)$			$r \equiv 0$	$(1, 2, 1, 2; 3)$			$r \equiv 0$
	$(2, 2, 1, 1; 1)$	$m \equiv 0$	$n \equiv 1$	$r \equiv 0$	$(2, 1, 1, 1; 2)$	$m \equiv 0$	$n \equiv 1$	$r \equiv 1$	$(2, 1, 1, 2; 3)$	$m \equiv 0$	$n \equiv 1$	$r \equiv 0$
	$(2, 2, 1, 2; 1)$			$r \equiv 1$	$(2, 1, 1, 2; 2)$			$r \equiv 0$	$(2, 1, 2, 1; 3)$			$r \equiv 1$
	$(2, 2, 2, 1; 1)$		$n \equiv 0$	$r \equiv 1$	$(2, 2, 2, 1; 2)$		$n \equiv 0$	$r \equiv 1$	$(2, 2, 1, 1; 3)$		$n \equiv 0$	$r \equiv 1$
	$(2, 2, 2, 2; 1)$			$r \equiv 0$	$(2, 2, 2, 2; 2)$			$r \equiv 0$	$(2, 2, 2, 2; 3)$			$r \equiv 0$
II	$(1, 1, 1, \infty; 1)$	$m \equiv 1$	$n \equiv 1$		$(1, 1, 2, \infty; 2)$	$m \equiv 1$	$n \equiv 0$					
	$(1, 1, 2, \infty; 1)$		$n \equiv 0$		$(1, 2, 1, \infty; 2)$		$n \equiv 1$					
	$(2, 2, 1, \infty; 1)$	$m \equiv 0$	$n \equiv 1$		$(2, 1, 1, \infty; 2)$	$m \equiv 0$	$n \equiv 1$					
	$(2, 2, 2, \infty; 1)$		$n \equiv 0$		$(2, 2, 2, \infty; 2)$		$n \equiv 0$					

TABLE II

i	$C(\gamma_2)$	$S(\sigma_3)$	Characteristic $(a_0, a_1, a_2, a_3; i)$	Restrictions		Examples: (I) or (II)				No. of types
								$C(\gamma_2)$	$S(\sigma_3)$	
1	$\neq p$	$\neq p$	$(a_0, a_1, a_2, a_3; 1)$ $a_3 = 1$ or 2 $(a_0, a_1, a_2, \infty; 1)$	$\sigma_1 - \sigma_2$ supports	$a_0 + a_1 \equiv 0$	$m < 2m < n < r$	I II	x-axis	xy-plane	8 4
2	$= p$	$\neq p$	$(a_0, a_1, a_2, a_3; 2)$ $a_3 = 1$ or 2 $(a_0, a_1, a_2, \infty; 2)$	$\sigma_2 - \sigma_3$ supports	$a_0 + a_1 + a_2 \equiv 0$	$m < n < 2m < r$	I II	$x = y = z = 0$	xy-plane	8 4
3	$= p$	$= p$	$(a_0, a_1, a_2, a_3; 3)$ $a_3 = 1$ or 2	σ_3 supports	$a_0 + a_1 + a_2 + a_3 \equiv 0$	$m < n < r < 2m$	I	$x = y = z = 0$	$x = y = z = 0$	8

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