COMPOSITIO MATHEMATICA

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Compositio Math. **149** (2013), 417–429.

doi:10.1112/S0010437X1200070X

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY

https://doi.org/10.1112/S0010437X1200070X Published online by Cambridge University Press

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On admissible tensor products in p-adic Hodge theory

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Abstract

We prove that if W and W' are non-zero B-pairs whose tensor product is crystalline (or semi-stable or de Rham or Hodge–Tate), then there exists a character μ such that $W(\mu^{-1})$ and $W'(\mu)$ are crystalline (or semi-stable or de Rham or Hodge–Tate). We also prove that if W is a B-pair and if F is a Schur functor (for example Sym^n or Λ^n) such that $F(W)$ is crystalline (or semi-stable or de Rham or Hodge–Tate) and if the rank of W is sufficiently large, then there is a character μ such that $W(\mu^{-1})$ is crystalline (or semi-stable or de Rham or Hodge–Tate). In particular, these results apply to p-adic representations.

Introduction

Let K and E be finite extensions of $\mathbf{Q}_p$ and let $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$. Fontaine has defined the notions of crystalline, semi-stable and de Rham E-linear representations of G_K and proved that the corresponding categories are stable under sub-quotient, direct sum and tensor product. The goal of this note is to answer the following question: if V and V' are p-adic representations whose tensor product is crystalline (or semi-stable or de Rham or Hodge–Tate), then what can be said about V and V'?

Berger has defined the tensor category of $B_{|K}^{\otimes E}$-pairs, in which the objects are couples $W = (W_{\mathrm{e}}, W_{\mathrm{dR}}^+)$ such that W_{e} is a $\mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} E$-representation of G_K and W_{dR}^+ is a G_K-stable $\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} E$-lattice of $W_{\mathrm{dR}} = (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} E) \otimes_{(\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} E)} W_{\mathrm{e}}$. If $W = (W_{\mathrm{e}}, W_{\mathrm{dR}}^+)$ is a $B_{|K}^{\otimes E}$-pair, then the rank of W is defined to be $\mathrm{rank}_{(\mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} E)} W_{\mathrm{e}} = \mathrm{rank}_{(\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} E)} W_{\mathrm{dR}}^+$. If V is an E-linear representation of G_K, then $W(V) = ((\mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} E) \otimes_E V, (\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} E) \otimes_E V)$ is a $B_{|K}^{\otimes E}$-pair, and the functor $W(-)$ identifies the category of E-linear representations of G_K with a tensor subcategory of the category of $B_{|K}^{\otimes E}$-pairs. The notions of crystalline, semi-stable, de Rham, and Hodge–Tate objects may be extended in a natural way to objects in the category of $B_{|K}^{\otimes E}$-pairs in such a way that an E-linear representation V of G_K is crystalline (or semi-stable or de Rham or Hodge–Tate) if and only if the associated $B_{|K}^{\otimes E}$-pair $W(V)$ is. Using Fontaine's theory of $\mathbf{B}_{\mathrm{dR}}$-representations (see [Fon04]), we can show the following result.

Theorem 2.3.2. *Let W and W' be non-zero $B_{|K}^{\otimes E}$-pairs. If the $B_{|K}^{\otimes E}$-pair $W \otimes W'$ is Hodge–Tate, then there is a finite extension F/E and a character $\mu : G_K \to F^\times$ such that*

Received 16 January 2012, accepted in final form 27 July 2012, published online 14 February 2013.
2010 Mathematics Subject Classification 11F80 (primary).
Keywords: B-pair, crystalline representation, de Rham representation, Hodge–Tate representation, p-adic Galois representation, p-adic Hodge theory, Schur functor, semi-stable representation, tensor product.

the $B_{|K}^{\otimes F}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are Hodge–Tate. If, moreover, $W \otimes W'$ is de Rham, then so are $W(\mu^{-1})$ and $W'(\mu)$.

It is known that every de Rham $B_{|K}^{\otimes E}$-pair is potentially semi-stable, due to the results of [And02, Ber02, Ked04, Meb02]. The properties of $(\varphi, N, \mathrm{Gal}(L/K))$-modules allow us to understand the situation when W and W' are both potentially semi-stable.

Theorem 3.2.1. *Let W and W' be non-zero potentially semi-stable $B_{|K}^{\otimes E}$-pairs. If the $B_{|K}^{\otimes E}$-pair $W \otimes W'$ is semi-stable, then there is a finite extension F/E and a character $\mu : G_K \to F^\times$ such that the $B_{|K}^{\otimes F}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are semi-stable. If, moreover, $W \otimes W'$ is crystalline, then so are $W(\mu^{-1})$ and $W'(\mu)$.*

In particular, the above two theorems may be used to deduce analogous results for p-adic representations (see Corollaries 2.3.3 and 3.2.2).

The same methods used to prove Theorems 2.3.2 and 3.2.1 above may be used to understand the situation when the image of a B-pair by a Schur functor is crystalline (or semi-stable or de Rham or Hodge–Tate). An integer partition $u = (u_1, \ldots, u_r) \in \mathbf{N}_{>0}^r$ with $u_1 \geqslant \cdots \geqslant u_r$ of an integer n gives rise to the Schur functor $\mathrm{Schur}^u(-)$, which sends $B_{|K}^{\otimes E}$-pairs to $B_{|K}^{\otimes E}$-pairs. If $r = 1$ or if $u_1 = u_2 = \cdots = u_r$, then we put $r(u) = r + 1$ and we put $r(u) = r$ when this is not the case. In particular, if $u = (n)$, then $r(u) = 2$ and the associated Schur functor is $\mathrm{Sym}^n(-)$ and if $u = (1, \ldots, 1)$, then $r(u) = n + 1$ and the associated Schur functor is $\Lambda^n(-)$.

Theorem 2.4.2. *Let W be a $B_{|K}^{\otimes E}$-pair such that $\mathrm{rank}(W) \geqslant r(u)$. If the $B_{|K}^{\otimes E}$-pair $\mathrm{Schur}^u(W)$ is Hodge–Tate, then there is a finite extension F/E and a character $\mu : G_K \to F^\times$ such that the $B_{|K}^{\otimes F}$-pair $W(\mu^{-1})$ is Hodge–Tate. If, moreover, $\mathrm{Schur}^u(W)$ is de Rham, then $W(\mu^{-1})$ is de Rham.*

Theorem 3.3.2. *Let W be a potentially semi-stable $B_{|K}^{\otimes E}$-pair such that $\mathrm{rank}(W) \geqslant r(u)$. If the $B_{|K}^{\otimes E}$-pair $\mathrm{Schur}^u(W)$ is semi-stable, then there is a finite extension F/E and a character $\mu : G_K \to F^\times$ such that the $B_{|K}^{\otimes F}$-pair $W(\mu^{-1})$ is semi-stable. If, moreover, $\mathrm{Schur}^u(W)$ is crystalline, then so is $W(\mu^{-1})$.*

The above two theorems may be used to deduce analogous results for p-adic representations (see Corollaries 2.4.3 and 3.3.3).

In the discussion following Corollary 2.4.3, we show that the bounds on $\mathrm{rank}(W)$ in Theorems 2.4.2 and 3.3.2 are optimal.

It was shown by Skinner (see [Ski09, § 2.4.1]) that if V is a p-adic representation and if $\mathrm{Sym}^2(V)$ is crystalline, then Wintenberger's methods of [Win95, Win97] may be applied to show that there exists a quadratic character μ such that $V(\mu)$ is crystalline. It is likely that Wintenberger's methods can be used in the same fashion to give another proof of our Theorems 2.3.2, 3.2.1, 2.4.2, and 3.3.2.

1. Notation and generalities

1.1 Notation

Let $\overline{\mathbf{Q}}_p$ be an algebraic closure of $\mathbf{Q}_p$ and let $\mathbf{C}_p$ be a p-adic completion of $\overline{\mathbf{Q}}_p$. Let $\mathbf{Q}_p^{\mathrm{nr}}$ denote the maximal non-ramified extension of $\mathbf{Q}_p$ in $\overline{\mathbf{Q}}_p$. If $F/\mathbf{Q}_p$ is a finite extension, then we let F^{Gal}

denote the Galois closure of F in $\overline{\mathbf{Q}}_p$. Let $\mathbf{B}_{\mathrm{dR}}$, $\mathbf{B}_{\mathrm{dR}}^{+}$, $\mathbf{B}_{\mathrm{cris}}$, and $\mathbf{B}_{\mathrm{st}}$ denote Fontaine's rings as in [Fon94a] and let $\mathbf{B}_{\mathrm{e}} = \mathbf{B}_{\mathrm{cris}}^{\varphi=1}$. In this note, $E/\mathbf{Q}_p$ and $K/\mathbf{Q}_p$ denote finite extensions. If $\mathbf{B}$ is any of the above rings or any Galois sub-extension of $\overline{\mathbf{Q}}_p/K$, then $\mathbf{B}_E$ will denote the ring $\mathbf{B} \otimes_{\mathbf{Q}_p} E$ endowed with an action of $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ defined by $g(b \otimes e) = g(b) \otimes e$ for all $g \in G_K$. If W is a free $\mathbf{B}_E$-module of finite rank endowed with a semi-linear action of G_K, then we refer to W as a $\mathbf{B}_E$-representation of G_K.

1.2 The category of $B_{|K}^{\otimes E}$-pairs

A $B_{|K}^{\otimes E}$-*pair* is a couple $W = (W_{\mathrm{e}}, W_{\mathrm{dR}}^{+})$ where W_{e} is a $\mathbf{B}_{\mathrm{e},E}$-representation of G_K and W_{dR}^{+} is a G_K-stable $\mathbf{B}_{\mathrm{dR},E}^{+}$-lattice of $W_{\mathrm{dR}} := (\mathbf{B}_{\mathrm{dR},E}) \otimes_{(\mathbf{B}_{\mathrm{e},E})} W_{\mathrm{e}}$. We define $\mathrm{rank}(W)$ to be the rank of W_{e} as a $\mathbf{B}_{\mathrm{e},E}$-module. If W and W' are $B_{|K}^{\otimes E}$-pairs, then $W \otimes W' = (W_{\mathrm{e}} \otimes_{\mathbf{B}_{\mathrm{e},E}} W'_{\mathrm{e}}, W_{\mathrm{dR}}^{+} \otimes_{\mathbf{B}_{\mathrm{dR},E}^{+}} W_{\mathrm{dR}}'^{+})$ is a $B_{|K}^{\otimes E}$-pair. If F/E and L/K are finite extensions and if W is a $B_{|K}^{\otimes E}$-pair, then $F \otimes_E W|_{G_L}$ is a $B_{|L}^{\otimes F}$-pair. If V is an E-linear representation of G_K, then we let $W(V)$ denote the $B_{|K}^{\otimes E}$-pair $((\mathbf{B}_{\mathrm{e},E}) \otimes_E V, (\mathbf{B}_{\mathrm{dR},E}^{+}) \otimes_E V)$. The properties of $B_{|K}^{\otimes E}$-pairs are developed in [Ber08, BC10, Nak09]. In this note, we consider only tensor products of non-zero $B_{|K}^{\otimes E}$-pairs.

1.3 Representations with coefficients in an extension

Let $F/\mathbf{Q}_p$ be a finite extension such that $K \supset F^{\mathrm{Gal}}$. If $\mathbf{B} \in \{\mathbf{C}_p, \mathbf{B}_{\mathrm{dR}}\}$ or if $\mathbf{B}$ is any Galois sub-extension of $\overline{\mathbf{Q}}_p/K$, then the map

$$\begin{aligned} \mathbf{B} \otimes_{\mathbf{Q}_p} F &\simeq \bigoplus_{h: F \to \overline{\mathbf{Q}}_p} \mathbf{B} \\ (b \otimes f) &\mapsto (b \cdot h(f))_h \end{aligned} \tag{1}$$

(where h runs over the embeddings of F into $\overline{\mathbf{Q}}_p$) is an isomorphism of $\mathbf{B}$-algebras which commutes with the action of G_K.

In particular, a $\mathbf{B}_F$-representation W of G_K decomposes into a direct sum $W = \bigoplus_{h: F \to \overline{\mathbf{Q}}_p} W_h$ as a $\mathbf{B}$-representation of G_K, where W_h is the sub-$\mathbf{B}$-representation of $\mathrm{rank}_{\mathbf{B}} W_h = \mathrm{rank}_{\mathbf{B}_F} W$ coming from the h-factor map $(b \otimes f) \mapsto b \cdot h(f) : \mathbf{B} \otimes_{\mathbf{Q}_p} F \to \mathbf{B}$ of the map (1) above. A $\mathbf{B}_{\mathrm{dR},F}$-representation W of G_K is de Rham if and only if the $\mathbf{B}_{\mathrm{dR}}$-representations W_h are de Rham for each embedding $h : F \to \overline{\mathbf{Q}}_p$ and a $\mathbf{C}_{p,F}$-representation W of G_K is Hodge–Tate if and only if the $\mathbf{C}_p$-representations W_h are Hodge–Tate for all embeddings $h : F \to \overline{\mathbf{Q}}_p$.

Lemma 1.3.1. *If W and W' are $\mathbf{B}_F$-representations of G_K and if $W = \bigoplus_h W_h$ and $W' = \bigoplus_h W'_h$ are their decompositions as described above, then the decomposition of the $\mathbf{B}_F$-representation $W \otimes_{\mathbf{B}_F} W'$ is given by $\bigoplus_{h: F \to \overline{\mathbf{Q}}_p} (W_h \otimes_{\mathbf{B}} W'_h)$.*

1.4 Schur functors applied to B-pairs

Let $n \geqslant 1$ be an integer and let $n = u_1 + \cdots + u_r$ be an integer partition such that $u_i \geqslant u_{i+1} \geqslant 1$ for all $i \in \{1, \ldots, r-1\}$, which we denote by $u = (u_1, \ldots, u_r)$. We represent u by its Young diagram Y_u, which is a diagram of n-many boxes arranged into left-justified rows such that the ith row from the top contains u_i-many boxes. We let v_j denote the length of the jth column from the left. Put $r(u) = r + 1$ if Y_u is a rectangle (i.e., if $u_1 = \cdots = u_r$) and put $r(u) = r$ if Y_u is not a rectangle.

https://doi.org/10.1112/S0010437X1200070X Published online by Cambridge University Press

If $d \geqslant 1$ is an integer, then a *tableau on Y_u with values in* $\{1, \ldots, d\}$ is a labeling of the boxes of Y_u with elements in $\{1, \ldots, d\}$ such that the labeling is weakly increasing from left to right and strongly increasing from top to bottom; we let $T = (t_{ij})$ denote a tableau with the integer $t_{ij} \in \{1, \ldots, d\}$ in the jth column of the ith row of Y_u. If $d \geqslant r$, then there is a tableau on Y_u which has i in each box of the ith row from the top; we refer to this tableau as the standard tableau, and we denote it by T_1. If $d \geqslant r(u)$, then there are tableaux $T_2, \ldots, T_d$ on Y_u with values in $\{1, \ldots, d\}$ such that for all $i \in \{1, \ldots, d-1\}$, there is an integer $j \in \{1, \ldots, d-1\}$ such that T_j and T_{j+1} have the same entries in all but one box, and in this box T_j contains i and T_{j+1} contains $i+1$.

Let R be a commutative ring with 1. The partition u gives rise to the Schur functor $\mathrm{Schur}^u(-)$, which sends R-modules to R-modules. If M is an R-module, then $\mathrm{Schur}^u(M)$ may be realized as a quotient of the R-module $\Lambda^{v_1}(M) \otimes_R \cdots \otimes_R \Lambda^{v_{u_1}}(M)$. If $\{m_1, \ldots, m_k\} \subset M$ and if $T = (t_{ij})$ is a tableau on Y_u with values in $\{1, \ldots, k\}$, then we let m_T denote the image of the element $(m_{t_{11}} \wedge \cdots \wedge m_{t_{v_1 1}}) \otimes \cdots \otimes (m_{t_{1u_1}} \wedge \cdots \wedge m_{t_{v_{u_1} u_1}})$ in $\mathrm{Schur}^u(M)$. If M is a free R-module of finite rank with basis $(e_1, \ldots, e_d)$, then $\mathrm{Schur}^u(M)$ is a free R-module with basis $(e_T)_T$, where T ranges over all tableaux on Y_u with values in $\{1, \ldots, d\}$.

For example, if M is an R-module, then the Schur module associated to the partition $u = (n)$ is $\mathrm{Sym}^n(M)$ and the Schur module associated to the partition $u = (1, \ldots, 1)$ is $\Lambda^n(M)$. The fundamental properties of tableaux and Schur modules are developed in [Ful97].

If $W = (W_{\mathrm{e}}, W^+_{\mathrm{dR}})$ is a $B^{\otimes E}_{|K}$-pair, then $\mathrm{Schur}^u(W) = (\mathrm{Schur}^u(W_{\mathrm{e}}), \mathrm{Schur}^u(W^+_{\mathrm{dR}}))$ is a $B^{\otimes E}_{|K}$-pair. If V is an E-linear representation of G_K, then we have an isomorphism of $B^{\otimes E}_{|K}$-pairs $\mathrm{Schur}^u(W(V)) \xrightarrow{\sim} W(\mathrm{Schur}^u(V))$.

Lemma 1.4.1. *Let $F/\mathbf{Q}_p$ be a finite extension such that $K \supset F^{\mathrm{Gal}}$ and let $\mathbf{B} \in \{\mathbf{C}_p, \mathbf{B}_{\mathrm{dR}}\}$. If W is a $\mathbf{B}_F$-representation of G_K and if $W = \bigoplus_{h: F \to \overline{\mathbf{Q}}_p} W_h$ is the decomposition of W as a $\mathbf{B}$-representation of G_K as in § 1.3, then the decomposition of the $\mathbf{B}_F$-representation* $\mathrm{Schur}^u(W)$ *as a $\mathbf{B}$-representation is given by* $\mathrm{Schur}^u(W) = \bigoplus_{h: F \to \overline{\mathbf{Q}}_p} \mathrm{Schur}^u(W_h)$.

2. Hodge–Tate tensor products and Schur B-pairs

2.1 Sen's theory of $\mathbf{C}_p$-representations

Let $\chi : G_K \to \mathbf{Z}_p^\times$ denote the cyclotomic character, $H_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K_\infty)$ its kernel, and $\Gamma_K = \mathrm{Gal}(K_\infty/K)$. In [Sen80], Sen associates to a $\mathbf{C}_{p,E}$-representation W of G_K a $K_{\infty,E}$-module $D_{\mathrm{sen}}(W)$, which is free of rank $d = \mathrm{rank}_{\mathbf{C}_{p,E}}(W)$ and is endowed with a K_∞-semi-linear E-linear action of Γ_K, together with a $K_{\infty,E}$-linear operator Θ_W which gives the action of $\mathrm{Lie}(\Gamma_K)$ on $D_{\mathrm{sen}}(W)$. The action of Γ_K commutes with Θ_W, and therefore the characteristic polynomial P_W of Θ_W has coefficients in $K^{\Gamma_K}_{\infty,E} = K \otimes_{\mathbf{Q}_p} E$.

Suppose that E contains K^{Gal} for the remainder of this subsection. If $h : K \to E$ is an embedding, then we may associate to W the set of its h-weights $\mathrm{Wt}^h(W) := \{x \in \overline{\mathbf{Q}}_p | P^h_W(x) = 0\}$ of roots of P^h_W counted with multiplicity, where P^h_W is the polynomial of degree d with coefficients in E obtained by applying the map $(k, e) \mapsto h(k) \cdot e : K \otimes_{\mathbf{Q}_p} E \to E$ to the coefficients of P_W. For example, if $\mathbf{C}_{p,E}(i)$ denotes the $\mathbf{C}_{p,E}$-representation associated to the i-fold twist by the cyclotomic character ($i \in \mathbf{Z}$) and if $h : K \to E$ is an embedding, then the h-weight of $\mathbf{C}_{p,E}(i)$ is i.

Sen showed in [Sen80, 2.3] that a $\mathbf{C}_p$-representation W of G_K is Hodge–Tate if and only if it is semi-simple with integer Sen weights. In particular, a $\mathbf{C}_{p,E}$-representation W of G_K is

Hodge–Tate if and only if it is semi-simple as a $\mathbf{C}_p$-representation of G_K and for each embedding $h: E \to K$, the h-weights of W are in $\mathbf{Z}$.

If all Sen weights of a $\mathbf{C}_p$-representation W are in $\mathbf{Z}$, then [Fon04, Theorem 2.14] implies that W is a direct sum of indecomposable $\mathbf{C}_p$-representations of the form $\mathbf{C}_p[i;d] := \mathbf{C}_p(i) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(0;d)$ where $i \in \mathbf{Z}$ is a Sen weight of W and $\mathbf{Z}_p(0;d)$ is the $\mathbf{Z}_p$-module of polynomials in $\log t$ of degree less than or equal to d with coefficients in $\mathbf{Z}_p$. The $\mathbf{C}_p$-representation $\mathbf{C}_p[i;d]$ is simple if and only if $d = 0$.

The $K_{\infty,E}$-representation $D_{\mathrm{sen}}(W)$ and its operator Θ_W satisfy the following properties.

Proposition 2.1.1. *Let E and K be finite extensions of $\mathbf{Q}_p$ and let W and W' be $\mathbf{C}_{p,E}$-representations of G_K.*

(i) *If W' is a sub-representation of W, then $\Theta_W|_{W'} = \Theta_{W'}$ and $\Theta_{W/W'}$ is the canonical operator induced by Θ_W. In particular, if $0 \to W' \to W \to W'' \to 0$ is an exact sequence of $\mathbf{C}_{p,E}$-representations, then $P_{\Theta_W} = P_{\Theta_{W'}} P_{\Theta_{W''}}$. If $E \supset K^{\mathrm{Gal}}$, then $\mathrm{Wt}^h(W) = \mathrm{Wt}^h(W') \sqcup \mathrm{Wt}^h(W'')$ (counted with multiplicity).*

(ii) *If F/E is a finite extension, then $D_{\mathrm{sen}}(F \otimes_E W) = F \otimes_E D_{\mathrm{sen}}(W)$ and $\Theta_{F\otimes W}$ is the F-linearization of Θ_W. In particular, if $E \supset K^{\mathrm{Gal}}$, then the h-weights of W are the same as those of $F \otimes_E W$.*

(iii) *We have a natural isomorphism $D_{\mathrm{sen}}(W \otimes_{\mathbf{C}_{p,E}} W') = D_{\mathrm{sen}}(W) \otimes_{K_{\infty,E}} D_{\mathrm{sen}}(W')$ of $K_{\infty,E}$-representations of Γ_K and the Sen operator on $D_{\mathrm{sen}}(W \otimes_{\mathbf{C}_{p,E}} W')$ is $\Theta_W \otimes \mathrm{Id} + \mathrm{Id} \otimes \Theta_{W'}$. In particular, if $E \supset K^{\mathrm{Gal}}$, then for each embedding $h: K \to E$ the h-weights of $W \otimes_{\mathbf{C}_{p,E}} W'$ are the elements $s + s'$, where s is an h-weight of W and s' is an h-weight of W'.*

(iv) *If L/K is a finite Galois extension, then $D_{\mathrm{sen}}(W|_{G_L}) = L_\infty \otimes_{K_\infty} D_{\mathrm{sen}}(W)$ as an $L_{\infty,E}$-representation of Γ_L, and $\Theta_{W|_{G_L}}$ is the L_∞-linearization of Θ_W.*

Corollary 2.1.2. *Suppose $E \supset K^{\mathrm{Gal}}$ and let W be a $\mathbf{C}_{p,E}$-representation of G_K. If $h: K \to E$ is an embedding and if $a_{1,h}, \ldots, a_{d,h}$ denote the h-weights of W, then the h-weights of $\mathrm{Schur}^u(W)$ are the elements $a_T = \sum_{i,j} a_{t_{ij},h}$ for any tableau $T = (t_{ij})$ on the Young diagram of u with values in $\{1, \ldots, d\}$.*

Lemma 2.1.3. *Suppose $E \supset K^{\mathrm{Gal}}$, let $h_1, \ldots, h_r$ denote the embeddings of K into E, and let $\omega_1, \ldots, \omega_r$ be elements of E. There exists a finite Galois extension F/E and a character $\mu: G_K \to F^\times$ such that $\mathrm{Wt}^{h_i}(F(\mu)) = \{\omega_i\}$ for $i = 1, \ldots, r$.*

Proof. Let $\chi_K: G_K \to \mathcal{O}_K^\times$ be the character associated to a Lubin–Tate module over $\mathcal{O}_K$. The h-weight of $K(\chi_K)$ is 1 if h is the inclusion of K in E, and 0 otherwise [Col93, Theorem I.2.1].

If $\omega \in E$, then $\omega = p^{-n}\omega'$ for some $\omega' \in \mathcal{O}_E$, and some integer $n \geqslant 0$. Consider the topological factorization $\mathcal{O}_K^\times = [k_K^\times] \times (1 + \mathfrak{m}_K)$. Consider a topological factorization of the $\mathbf{Z}_p$-module $1 + \mathfrak{m}_K$ into $\mathbf{Z}/p^a\mathbf{Z} \times \mathbf{Z}_p^r$, where $a \geqslant 0$ and $r = [K : \mathbf{Q}_p]$. Let $\langle \chi_K \rangle$ denote the projection of χ_K onto the submodule $\mathbf{Z}_p^r$ in this factorization. If $y_1, \ldots, y_r$ are a $\mathbf{Z}_p$-basis of $\mathbf{Z}_p^r$, and if F/E is an extension containing $z_1, \ldots, z_r \in 1 + \mathfrak{m}_F$ such that $z_i^{p^n} = y_i$, then the map $\mu(y_1^{a_1} \cdot \cdots \cdot y_r^{a_r}) := z_1^{\omega' a_1} \cdot \cdots \cdot z_r^{\omega' a_r}$ composed with $\langle \chi_K \rangle$ is a character whose h-weight is $p^{-n}\omega' = \omega$ when $h = id$ and 0 otherwise. We denote this character by $\langle \chi_K \rangle^\omega$.

We may suppose that F is Galois over K. Given $\omega_1, \ldots, \omega_r \in E$, the product of characters $\prod \langle h_i^{-1}(\chi_K) \rangle^{\omega_i}$ has h_i-weight equal to ω_i for each $1 \leqslant i \leqslant r$, where $h_i^{-1}: F \to F$ is the inverse of an automorphism $h_i: F \to F$ extending $h_i: K \to E \subset F$. □

https://doi.org/10.1112/S0010437X1200070X Published online by Cambridge University Press

In particular, if $W=(W_{\mathrm{e}},W_{\mathrm{dR}}^+)$ is a $B_{|K}^{\otimes E}$-pair, then all of the above may be applied to the $\mathbf{C}_{p,E}$-representation $\overline{W}=W_{\mathrm{dR}}^+/tW_{\mathrm{dR}}^+$. We say that a $B_{|K}^{\otimes E}$-pair W is Hodge–Tate if the $\mathbf{C}_{p,E}$-representation $\overline{W}$ is Hodge–Tate. We let $\mathrm{Wt}(\overline{W})$ denote the set of all Sen weights associated to $\overline{W}$.

2.2 Fontaine's theory of $\mathbf{B}_{\mathrm{dR}}$-representations

Let W be a $\mathbf{B}_{\mathrm{dR}}$-representation of G_K and let $\mathcal{W}\subset W$ be a G_K-stable $\mathbf{B}_{\mathrm{dR}}^+$-lattice. The quotient $\overline{\mathcal{W}}:=\mathcal{W}/t\mathcal{W}$ is a $\mathbf{C}_p$-representation of G_K, and we may therefore associate to it the set $\mathrm{Wt}(\overline{\mathcal{W}})$ of its Sen weights, which is a set of elements of $\overline{\mathbf{Q}}_p$ of cardinal $\dim_{\mathbf{B}_{\mathrm{dR}}} W$ which is stable by the action of G_K. The following proposition shows that all lattices of W have the same Sen weights up to integers, so that the set of Sen weights modulo $\mathbf{Z}$ of a lattice $\mathcal{W}$ is an invariant of W.

Proposition 2.2.1. *Let W be a $\mathbf{B}_{\mathrm{dR}}$-representation of G_K. If $\mathcal{W}$ and $\mathcal{W}'$ are two G_K-stable $\mathbf{B}_{\mathrm{dR}}^+$-lattices of W, then each Sen weight of $\overline{\mathcal{W}'}$ may be written in the form $\alpha+i$ where α is a Sen weight of $\overline{\mathcal{W}}$ and $i\in\mathbf{Z}$.*

Proof. Let $c\geqslant 0$ be an integer such that the lattice $t^c\mathcal{W}'$ is contained in $\mathcal{W}$ and let $c'\geqslant 0$ be an integer such that the lattice $t^{c'}\mathcal{W}$ is contained in $t^c\mathcal{W}'$.

Consider the sequence of G_K-stable lattices,

$$t^c\mathcal{W}'=t^c\mathcal{W}'+t^{c'}\mathcal{W}\subset t^c\mathcal{W}'+t^{c'-1}\mathcal{W}\subset\cdots\subset t^c\mathcal{W}'+t\mathcal{W}\subset t^c\mathcal{W}'+\mathcal{W}=\mathcal{W},$$

and let $\mathcal{X}_k$ denote the lattice $t^c\mathcal{W}'+t^{c'-k}\mathcal{W}$ (for $0\leqslant k\leqslant c'$). We have G_K-equivariant inclusions $t\mathcal{X}_{k+1}\subset\mathcal{X}_k\subset\mathcal{X}_{k+1}$ for $k=0,1,\ldots,c'-1$; we therefore have exact sequences of $\mathbf{C}_p$-representations,

$$\mathcal{X}_{k+1}/t\mathcal{X}_{k+1}\to\mathcal{X}_{k+1}/\mathcal{X}_k\to 0\quad\text{and}\quad 0\to t\mathcal{X}_{k+1}/t\mathcal{X}_k\to\mathcal{X}_k/t\mathcal{X}_k\to\mathcal{X}_{k+1}/t\mathcal{X}_{k+1},$$

which, taken together with parts (i) and (iii) of Proposition 2.1.1, and since $x\mapsto tx$ induces an isomorphism of $(\mathcal{X}_{k+1}/\mathcal{X}_k)(1)$ onto $t\mathcal{X}_{k+1}/t\mathcal{X}_k$, implies that $\mathrm{Wt}(\overline{\mathcal{X}_k})\subset\mathrm{Wt}(\overline{\mathcal{X}_{k+1}})\cup(\mathrm{Wt}(\overline{\mathcal{X}_{k+1}})+1)$. By recurrence, the Sen weights of $\mathcal{X}_0=t^c\mathcal{W}'$ are all of the form $\alpha+i$, where α is a Sen weight of $\overline{\mathcal{X}_{c'}}=\overline{\mathcal{W}}$ and i is an integer. Again by part (iii) of Proposition 2.1.1, the Sen weights of $\mathcal{W}'$ are of the form $\alpha+i$ where α is a Sen weight of $\overline{\mathcal{W}}$. □

If W is a $\mathbf{B}_{\mathrm{dR}}$-representation of G_K and if $\mathcal{W}\subset W$ is a G_K-stable lattice, we call the image of the set $\mathrm{Wt}(\overline{\mathcal{W}})$ modulo $\mathbf{Z}$ the set of *de Rham weights* of W, and we denote this set by $\mathrm{Wt}_{\mathrm{dR}}(W)$. The set of de Rham weights of W is endowed with an action of G_K. Fontaine's theorem [Fon04, 3.19] shows that any $\mathbf{B}_{\mathrm{dR}}$-representation W decomposes along the set of G_K-orbits in $\mathrm{Wt}_{\mathrm{dR}}(W)$, and that W is de Rham if and only if it is semi-simple with de Rham weights in $\mathbf{Z}$.

If the de Rham weights of W are all in $\mathbf{Z}$, then Fontaine's theorem [Fon04, 3.19] implies that W is a direct sum of indecomposable objects of the form $\mathbf{B}_{\mathrm{dR}}[\{0\};d]:=\mathbf{B}_{\mathrm{dR}}\otimes_{\mathbf{Z}_p}\mathbf{Z}_p(0;d)$ where $\mathbf{Z}_p(0;d)$ is the $\mathbf{Z}_p$-module of polynomials in one variable $X=\log t$ of degree less than or equal to d with coefficients in $\mathbf{Z}_p$, such that $g(X)=X+\log(\chi(g))$ for all $g\in G_K$. The $\mathbf{B}_{\mathrm{dR}}$-representation $\mathbf{B}_{\mathrm{dR}}[\{0\};d]$ is simple if and only if $d=0$.

2.3 Hodge–Tate and de Rham tensor products of B-pairs

Let $W=(W_{\mathrm{e}},W_{\mathrm{dR}}^+)$ be a $B_{|K}^{\otimes E}$-pair. We say that W is *de Rham* if the $\mathbf{B}_{\mathrm{dR}}$-representation W_{dR} of G_K is de Rham. We say that W is *Hodge–Tate* if the $\mathbf{C}_{p,E}$-representation $\overline{W}=W_{\mathrm{dR}}^+/tW_{\mathrm{dR}}^+$ of G_K is Hodge–Tate.

Lemma 2.3.1. *If W and W' are $\mathbf{C}_p$-representations of G_K with Sen weights in $\mathbf{Z}$ such that $W \otimes_{\mathbf{C}_p} W'$ is Hodge–Tate, then W and W' are Hodge–Tate.*

If W and W' are $\mathbf{B}_{\mathrm{dR}}$-representations of G_K with de Rham weights in $\mathbf{Z}$ such that $W \otimes_{\mathbf{B}_{\mathrm{dR}}} W'$ is de Rham, then W and W' are de Rham.

Proof. Let W and W' be $\mathbf{B}_{\mathrm{dR}}$-representations of G_K with de Rham weights in $\mathbf{Z}$. By Fontaine's theorem [Fon04, 3.19], W and W' admit unique decompositions $W \simeq \bigoplus_{i=1}^{r} \mathbf{B}_{\mathrm{dR}}[\{0\}; d_i]^{e_i}$ and $W' \simeq \bigoplus_{j=1}^{r'} \mathbf{B}_{\mathrm{dR}}[\{0\}; d'_j]^{e'_j}$. The $\mathbf{B}_{\mathrm{dR}}$-representations W and W' are de Rham if and only if all of the d_i and d'_j are equal to zero. If $W \otimes_{\mathbf{B}_{\mathrm{dR}}} W'$ is de Rham, then $\mathbf{B}_{\mathrm{dR}}[\{0\}; d_i] \otimes_{\mathbf{B}_{\mathrm{dR}}} \mathbf{B}_{\mathrm{dR}}[\{0\}; d'_j]$ is de Rham for every $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant r'$. Suppose, for example, that W is not de Rham, so that we may assume $d_1 > 0$. Let $U = \mathbf{B}_{\mathrm{dR}}[\{0\}; d_1] \otimes_{\mathbf{B}_{\mathrm{dR}}} \mathbf{B}_{\mathrm{dR}}[\{0\}; d'_1]$, let $v_1 = 1 \otimes 1$, and let $(v_1, v_2, \ldots, v_f)$ be a K-basis of $D_{\mathrm{dR}}(U) = U^{G_K}$, where $f = (d_1 + 1)(d'_1 + 1)$. If U is de Rham, then the element $X \otimes 1 \in U$ may be written as a sum $X \otimes 1 = b_1(1 \otimes 1) + \sum_{i=2}^{f} b_i v_i$ with $b_i \in \mathbf{B}_{\mathrm{dR}}$ for all $1 \leqslant i \leqslant f$. Since $g(X \otimes 1) = X \otimes 1 + \log(\chi(g))(1 \otimes 1)$ for all $g \in G_K$, we have $g(b_1) - b_1 = \log(\chi(g))$ for all $g \in G_K$. If $b_1 \in \mathbf{B}_{\mathrm{dR}}^{+}$, then $g(\theta(b_1)) - \theta(b_1) = \log \chi(g)$ for all $g \in G_K$, which is impossible since $g \mapsto \log \chi(g)$ is a generator of the one-dimensional K-vector space $H^1(G_K, \mathbf{C}_p)$. If $b_1 \in t^h \mathbf{B}_{\mathrm{dR}}^{+} \backslash t^{h+1} \mathbf{B}_{\mathrm{dR}}^{+}$ for some $h < 0$, then $b_1 = t^h b'$ for a unique $b' \in \mathbf{B}_{\mathrm{dR}}^{+} \backslash t\mathbf{B}_{\mathrm{dR}}^{+}$ and $\chi(g)^h g(b') - b' \in t^{-h} \mathbf{B}_{\mathrm{dR}}^{+} \subset t\mathbf{B}_{\mathrm{dR}}^{+}$, so that reducing modulo t would imply that $\theta(b') \in \mathbf{C}_p(h)^{G_K} = \{0\}$, which is a contradiction. We therefore see that W and W' must be de Rham.

The same arguments together with Fontaine's theorem [Fon04, 2.14] show that if W and W' are $\mathbf{C}_p$-representations of G_K with Sen weights in $\mathbf{Z}$ such that $W \otimes_{\mathbf{C}_p} W'$ is Hodge–Tate, then W and W' are Hodge–Tate. □

Theorem 2.3.2. *Let W and W' be non-zero $B_{|K}^{\otimes E}$-pairs. If the $B_{|K}^{\otimes E}$-pair $W \otimes W'$ is Hodge–Tate, then there is a finite extension F/E and a character $\mu : G_K \to F^{\times}$ such that the $B_{|K}^{\otimes F}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are Hodge–Tate. If, moreover, $W \otimes W'$ is de Rham, then so are $W(\mu^{-1})$ and $W'(\mu)$.*

Proof. Let W and W' be $B_{|K}^{\otimes E}$-pairs and suppose that the $B_{|K}^{\otimes E}$-pair $W \otimes W'$ is Hodge–Tate. By extending scalars if necessary, we may suppose that $E/\mathbf{Q}_p$ is finite Galois and contains K, so that the methods of § 2.1 apply.

Let $r = \mathrm{rank}(W)$ and let $r' = \mathrm{rank}(W')$. For each embedding $h : K \to E$, let $a_{1,h}, \ldots, a_{r,h}$ denote the h-weights of the $\mathbf{C}_{p,E}$-representation $\overline{W}$ and let $a'_{1,h}, \ldots, a'_{r',h}$ denote the h-weights of $\overline{W'}$. Part (iii) of Proposition 2.1.1 implies that if $h : K \to E$ is an embedding, then the h-weights of $\overline{W \otimes W'}$ are the elements $a_{i,h} + a'_{j,h}$ for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant r'$, which are integers since the $\mathbf{C}_{p,E}$-representation $\overline{W \otimes W'} = \overline{W} \otimes_{\mathbf{C}_{p,E}} \overline{W'}$ is Hodge–Tate. By Lemma 2.1.3, there is a finite Galois extension F/E and a character $\mu : G_K \to F^{\times}$ such that for all embeddings $h : K \to E \subset F$, the h-weight of the $\mathbf{C}_{p,F}$-representation $\overline{W(F(\mu))}$ is $a_{1,h}$.

We now show that the $B_{|K}^{\otimes F}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are Hodge–Tate. If $h : K \to E \subset F$ is an embedding, then parts (ii) and (iii) of Proposition 2.1.1 imply that the h-weights of $W(\mu^{-1})$ are the integers $a_{i,h} - a_{1,h}$ (for $1 \leqslant i \leqslant r$) and the h-weights of $W'(\mu)$ are the integers $a_{1,h} + a'_{j,h}$ for $1 \leqslant j \leqslant r'$. Since being Hodge–Tate is the same as being potentially Hodge–Tate, it suffices to show that the $B_{|F}^{\otimes F}$-pairs $W(\mu^{-1})|_{G_F}$ and $W'(\mu)|_{G_F}$ are Hodge–Tate. Let $\overline{W(\mu^{-1})} = \bigoplus_{h : F \to F} \overline{W(\mu^{-1})}_h$ and $\overline{W'(\mu)} = \bigoplus_{h : F \to F} \overline{W'(\mu)}_h$ be the decompositions of $\mathbf{C}_{p,F}$-representations of G_F as described in § 1.3. The $\mathbf{C}_p$-representations $\overline{W(\mu^{-1})}_h$ and $\overline{W'(\mu)}_h$

https://doi.org/10.1112/S0010437X1200070X Published online by Cambridge University Press

have weights in $\mathbf{Z}$ for every h. The isomorphism

$$\overline{W(\mu^{-1}) \otimes W'(\mu)} \simeq \bigoplus_{h:F\to F} \overline{W(\mu^{-1})}_h \otimes_{\mathbf{C}_p} \overline{W'(\mu)}_h$$

of $\mathbf{C}_p$-representations of G_F as in Lemma 1.3.1 implies that $\overline{W(\mu^{-1})}_h \otimes_{\mathbf{C}_p} \overline{W'(\mu)}_h$ is Hodge–Tate for each embedding $h: F \to F$. By Lemma 2.3.1, $\overline{W(\mu^{-1})}_h$ and $\overline{W'(\mu)}_h$ are Hodge–Tate for each embedding $h: F \to F$, and therefore $\overline{W(\mu^{-1})}$ and $\overline{W'(\mu)}$ are Hodge–Tate. Therefore, the $B_{|K}^{\otimes F}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are Hodge–Tate.

Suppose now that $E/\mathbf{Q}_p$ is a finite Galois extension and that W and W' are $B_{|K}^{\otimes E}$-pairs such that the $B_{|K}^{\otimes E}$-pair $W \otimes W'$ is de Rham. By the above, there is a finite Galois extension F/E and a character $\mu: G_K \to F^\times$ such that the $B_{|K}^{\otimes F}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are Hodge–Tate. We now show that $W(\mu^{-1})$ and $W'(\mu)$ are de Rham. It suffices to show that the restrictions of $W(\mu^{-1})$ and $W'(\mu)$ to G_F are de Rham. Let $W(\mu^{-1})_{\mathrm{dR}} = \bigoplus_{h:F\to F} W(\mu^{-1})_{\mathrm{dR},h}$ and $W'(\mu)_{\mathrm{dR}} = \bigoplus_{h:F\to F} W'(\mu)_{\mathrm{dR},h}$ be the decompositions of $\mathbf{B}_{\mathrm{dR}}$-representations of G_F as in § 1.3. For each embedding $h: F \to F$, the $\mathbf{B}_{\mathrm{dR}}$-representations $W(\mu^{-1})_{\mathrm{dR},h}$ and $W'(\mu)_{\mathrm{dR},h}$ have de Rham weights in $\mathbf{Z}$. By Lemma 1.3.1, the $\mathbf{B}_{\mathrm{dR}}$-representation $W(\mu^{-1})_{\mathrm{dR},h} \otimes_{\mathbf{B}_{\mathrm{dR}}} W'(\mu)_{\mathrm{dR},h}$ is de Rham for each embedding $h: F \to F$, and therefore so are $W(\mu^{-1})_{\mathrm{dR},h}$ and $W'(\mu)_{\mathrm{dR},h}$ by Lemma 2.3.1. Therefore, the $B_{|K}^{\otimes F}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are de Rham. □

Corollary 2.3.3. *Let $E/\mathbf{Q}_p$ and $K/\mathbf{Q}_p$ be finite extensions, and let V and V' be non-zero E-linear representations of G_K. If $V \otimes_E V'$ is Hodge–Tate, then there is a finite extension F/E and a character $\mu: G_K \to F^\times$ such that $V(\mu^{-1})$ and $V'(\mu)$ are Hodge–Tate. If, moreover, $V \otimes_E V'$ is de Rham, then so are $V(\mu^{-1})$ and $V'(\mu)$.*

2.4 Hodge–Tate and de Rham Schur B-pairs

In what follows, let $n \geqslant 1$ be an integer and let $u = (u_1, \ldots, u_r)$ denote an integer partition $n = u_1 + \cdots + u_r$ $(u_i \geqslant u_{i+1} \geqslant 1)$ of n. If $u_1 = \cdots = u_r$, put $r(u) = r + 1$. Otherwise, put $r(u) = r$.

Lemma 2.4.1. *If W is a $\mathbf{C}_p$-representation of G_K having Sen weights in $\mathbf{Z}$ such that $\dim_{\mathbf{C}_p}(W) \geqslant r(u)$ and $\mathrm{Schur}^u(W)$ is Hodge–Tate, then W is Hodge–Tate.*

If W is a $\mathbf{B}_{\mathrm{dR}}$-representation of G_K having de Rham weights in $\mathbf{Z}$ such that $\dim_{\mathbf{B}_{\mathrm{dR}}}(W) \geqslant r(u)$ and $\mathrm{Schur}^u(W)$ is de Rham, then W is de Rham.

Proof. Let W be a $\mathbf{B}_{\mathrm{dR}}$-representation of G_K having de Rham weights in $\mathbf{Z}$ such that $\dim_{\mathbf{B}_{\mathrm{dR}}}(W) \geqslant r(u)$. If W is not de Rham, then Fontaine's theorem [Fon04, 3.19] gives a decomposition $W = \mathbf{B}_{\mathrm{dR}}[\{0\}; d] \oplus W'$ for some $d > 0$, so that

$$\mathrm{Schur}^u(W) \simeq \bigoplus_{\lambda,\mu} (\mathrm{Schur}^\lambda(\mathbf{B}_{\mathrm{dR}}[\{0\}; d]) \otimes_{\mathbf{B}_{\mathrm{dR}}} \mathrm{Schur}^\mu(W'))^{\oplus c^u_{\lambda,\mu}}$$

as a $\mathbf{B}_{\mathrm{dR}}$-representation of G_K, where $c^u_{\lambda,\mu} \geqslant 0$ denotes the Littlewood–Richardson number. There are λ and μ such that $c^u_{\lambda,\mu}$ and $\mathrm{Schur}^\lambda(\mathbf{B}_{\mathrm{dR}}[\{0\}; d]) \otimes_{\mathbf{B}_{\mathrm{dR}}} \mathrm{Schur}^\mu(W')$ are non-zero, and such that $d + 1 \geqslant r(\lambda)$. This can be seen by using the fact that $c^u_{\lambda,\mu}$ is equal to the number of pairs of tableaux T of shape λ and U of shape μ such that the product tableau $T \cdot U$ is equal to the standard tableau T_1 on the Young diagram of u. Details on this combinatorial argument may be found in the author's forthcoming thesis.

The $\mathbf{B}_{\mathrm{dR}}$-representations $\mathrm{Schur}^\lambda(\mathbf{B}_{\mathrm{dR}}[\{0\}; d])$ and $\mathrm{Schur}^\mu(W')$ have de Rham weights in $\mathbf{Z}$ by Lemma 2.1.1. If $\mathrm{Schur}^u(W)$ is de Rham, then so is $\mathrm{Schur}^\lambda(\mathbf{B}_{\mathrm{dR}}[\{0\}; d]) \otimes_{\mathbf{B}_{\mathrm{dR}}} \mathrm{Schur}^\mu(W')$

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and Lemma 2.3.1 implies that $\mathrm{Schur}^\lambda(\mathbf{B}_{\mathrm{dR}}[\{0\};d])$ is de Rham. Let $(1, X, X^2, \ldots, X^d)$ denote the standard $\mathbf{B}_{\mathrm{dR}}$-basis of $\mathbf{B}_{\mathrm{dR}}[\{0\};d]$. If T_1 is the standard tableau defined in § 1.4, then the element $e_{T_1} \in \mathrm{Schur}^\lambda(\mathbf{B}_{\mathrm{dR}}[\{0\};d])$ is such that $g(e_{T_1}) = e_{T_1}$ for all $g \in G_K$. Let T' be the tableau with values in $\{1, \ldots, d+1\}$ which is obtained from T_1 by adding 1 to the value in the bottom-most cell of the right-most column of Y_λ; this tableau T' exists since $d+1 \geqslant r(\lambda)$. A calculation shows that $g(e_{T'}) = e_{T'} + \nu \log \chi(g) e_{T_1}$, where ν is the length of the right-most column of Y_λ. If $\mathrm{Schur}^\lambda(\mathbf{B}_{\mathrm{dR}}[\{0\};d])$ is de Rham, then it admits a basis $(e_{T_1}, e_2, \ldots, e_f)$ of elements such that, for all $i = 2, \ldots, f$, $g(e_i) = e_i$ for all $g \in G_K$. If $b_1, \ldots, b_f \in \mathbf{B}_{\mathrm{dR}}$ are elements such that $e_{T'} = b_1 e_T + \sum_{i \geqslant 2} b_i e_i$, then $g(b_1) - b_1 = \nu \log \chi(g)$ for all $g \in G_K$, which is impossible. Therefore, W and W' must be de Rham.

One can prove the claim for $\mathbf{C}_p$-representations by using Fontaine's theorem [Fon04, 2.14] and applying the same arguments. □

THEOREM 2.4.2. *Let W be a $B_{|K}^{\otimes E}$-pair such that* $\mathrm{rank}(W) \geqslant r(u)$. *If the $B_{|K}^{\otimes E}$-pair* $\mathrm{Schur}^u(W)$ *is Hodge–Tate, then there is a finite extension F/E and a character $\mu : G_K \to F^\times$ such that the $B_{|K}^{\otimes F}$-pair $W(\mu^{-1})$ is Hodge–Tate. If, moreover,* $\mathrm{Schur}^u(W)$ *is de Rham, then $W(\mu^{-1})$ is de Rham.*

Proof. Let W be a $B_{|K}^{\otimes E}$-pair such that $d = \mathrm{rank}(W) \geqslant r(u)$ and suppose that $\mathrm{Schur}^u(W)$ is Hodge–Tate. By extending scalars if necessary, we may suppose that $E/\mathbf{Q}_p$ is finite Galois and contains K.

If $h : K \to E$ is an embedding, then let $a_{1,h}, \ldots, a_{d,h}$ denote the h-weights of $\overline{W}$. By Corollary 2.1.2, the h-weights of the $\mathbf{C}_{p,E}$-representation $\overline{\mathrm{Schur}^u(W)} = \mathrm{Schur}^u(\overline{W})$ are the elements of the form $a_{T,h} = \sum a_{t_{ij},h}$ for any tableau $T = (t_{ij})$ with values in $\{1, \ldots, d\}$ on the Young diagram of u. Since $\mathrm{Schur}^u(W)$ is Hodge–Tate, the elements $a_{T,h}$ are in $\mathbf{Z}$. Since $d = \mathrm{rank}(W) \geqslant r(u)$, considering the tableaux $T_1, \ldots, T_d$ in § 1.4 allows us to conclude that $a_{i,h} - a_{1,h} \in \mathbf{Z}$ for all $1 \leqslant i \leqslant d$. By Lemma 2.1.3, there is a finite Galois extension F/E and a character $\mu : G_K \to F^\times$ such that the $B_{|K}^{\otimes F}$-pair $W(F(\mu))$ has $a_{1,h}$ as its h-weight for each embedding $h : K \to E \subset F$.

We now show that the $B_{|K}^{\otimes F}$-pair $W(\mu^{-1})$ is Hodge–Tate. It suffices to show that the restriction of $W(\mu^{-1})$ to G_F are Hodge–Tate. Let $\overline{W(\mu^{-1})} = \bigoplus_{h:F\to F} \overline{W(\mu^{-1})}_h$ be the decomposition as a $\mathbf{C}_p$-representation of G_F as described in § 1.3. The $\mathbf{C}_p$-representation $\overline{W(\mu^{-1})}_h$ has Sen weights in $\mathbf{Z}$ for each embedding $h : F \to F$. By Lemma 1.4.1, the $\mathbf{C}_p$-representation $\mathrm{Schur}^u(\overline{W(\mu^{-1})}_h)$ of G_F is Hodge–Tate for each embedding $h : F \to F$. Since $\dim_{\mathbf{C}_p} \overline{W(\mu^{-1})}_h = \mathrm{rank}(W) \geqslant r(u)$, Lemma 2.4.1 implies that $\overline{W(\mu^{-1})}_h$ is Hodge–Tate for each embedding $h : F \to F$. The $B_{|K}^{\otimes F}$-pair $W(\mu^{-1})$ is therefore Hodge–Tate.

Suppose now that W is a $B_{|K}^{\otimes E}$-pair such that $\mathrm{rank}(W) \geqslant r(u)$ and $\mathrm{Schur}^u(W)$ is de Rham. There is a finite Galois extension F/E and a character $\mu : G_K \to F^\times$ such that the $B_{|K}^{\otimes E}$-pair $W(\mu^{-1})$ is Hodge–Tate. We now show that $W(\mu^{-1})$ is de Rham. Let $W(\mu^{-1})_{\mathrm{dR}} \simeq \bigoplus_{h:F\to F} W(\mu^{-1})_{\mathrm{dR},h}$ be the decomposition as a $\mathbf{B}_{\mathrm{dR}}$-representation of G_F as described in § 1.3. The $\mathbf{B}_{\mathrm{dR}}$-representation $W(\mu^{-1})_{\mathrm{dR},h}$ has de Rham weights in $\mathbf{Z}$ for each embedding $h : F \to F$. By Lemma 1.4.1, $\mathrm{Schur}^u(W(\mu^{-1})_{\mathrm{dR},h})$ is a de Rham $\mathbf{B}_{\mathrm{dR}}$-representation of G_F for each embedding $h : F \to F$ and therefore $W(\mu^{-1})_{\mathrm{dR},h}$ is de Rham for each embedding h since $\dim_{\mathbf{B}_{\mathrm{dR}}} W(\mu^{-1})_{\mathrm{dR},h} = \mathrm{rank}(W) \geqslant r(u)$. Therefore, the $B_{|K}^{\otimes F}$-pair $W(\mu^{-1})$ is de Rham. □

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Corollary 2.4.3. *Let $n \geqslant 1$ be an integer, let u be a partition of n, and let V be an E-linear representation of G_K such that $\dim_E(V) \geqslant r(u)$. If $\mathrm{Schur}^u(V)$ is Hodge–Tate, then there is a finite extension F/E and a character $\mu : G_K \to F^\times$ such that $V(\mu^{-1})$ is Hodge–Tate. If, moreover, $\mathrm{Schur}^u(V)$ is de Rham, then V is de Rham.*

We now show that the bound on $\mathrm{rank}(W)$ in Theorem 2.4.2 is optimal. If W is a $B^{\otimes E}_{|K}$-pair such that $\mathrm{rank}(W) < r(u)$, then $\mathrm{Schur}^u(W)$ is of rank 1 if $u_1 = \cdots = u_r$ and $\mathrm{Schur}^u(W) = 0$ otherwise. In the former case, $\mathrm{rank}(W) = r$ and $\mathrm{Schur}^u(W) = \bigotimes_{i=1}^r \det(W)$. Let V denote a two-dimensional $\mathbf{Q}_p$-vector space endowed with an action of $G_{\mathbf{Q}_p}$ such that $g \in G_{\mathbf{Q}_p}$ acts on a basis $\mathcal{E} = (e_1, e_2)$ by the matrix

$$\begin{pmatrix} 1 & \log_p(\chi(g)) \\ 0 & 1 \end{pmatrix}$$

so that V is not Hodge–Tate since $\mathbf{C}_p \otimes_{\mathbf{Q}_p} V = \mathbf{C}_p[\{0\}; 1]$, but $G_{\mathbf{Q}_p}$ acts trivially on $\Lambda^2 V$. There is no character $\mu : G_{\mathbf{Q}_p} \to E^\times$ such that $V(\mu^{-1})$ is Hodge–Tate; such a character would necessarily have weights in $\mathbf{Z}$, and Lemma 2.4.1 would imply that V itself is Hodge–Tate.

3. Semi-stable tensor products and Schur B-pairs

3.1 Semi-stable B-pairs

Let $W = (W_{\mathrm{e}}, W^+_{\mathrm{dR}})$ be a $B^{\otimes E}_{|K}$-pair. We say that W is *crystalline* if the $\mathbf{B}_{\mathrm{cris}}$-representation $(\mathbf{B}_{\mathrm{cris},E}) \otimes_{\mathbf{B}_{\mathrm{e},E}} W_{\mathrm{e}}$ of G_K is trivial. Similarly, we say that W is *semi-stable* if the $\mathbf{B}_{\mathrm{st}}$-representation $(\mathbf{B}_{\mathrm{st},E}) \otimes_{\mathbf{B}_{\mathrm{e},E}} W_{\mathrm{e}}$ of G_K is trivial. We say that W is *potentially crystalline* (or *potentially semi-stable*) if there is a finite extension L/K such that the $B^{\otimes E}_{|L}$-pair $W|_{G_L}$ is crystalline (or semi-stable). Note that if V is an E-linear representation of G_K, then V is crystalline (or semi-stable) if and only if the $B^{\otimes E}_{|K}$-pair $W(V)$ is crystalline (or semi-stable).

Let L/K be a finite Galois extension and let $L_0 = L \cap \mathbf{Q}_p^{\mathrm{nr}}$. If W is a $B^{\otimes E}_{|K}$-pair which is semi-stable when restricted to G_L, then $D_{\mathrm{st},L}(W) = (\mathbf{B}_{\mathrm{st},E} \otimes_{\mathbf{B}_{\mathrm{e},E}} W_{\mathrm{e}})^{G_L}$ is a free $L_{0,E}$-module such that $\mathrm{rank}_{L_{0,E}}(D_{\mathrm{st},L}(W)) = \mathrm{rank}(W)$, and it is endowed with an injective additive self-map φ that is E-linear and semi-linear for the absolute Frobenius automorphism σ on L_0, an $L_{0,E}$-linear nilpotent endomorphism N such that $N\varphi = p\varphi N$, and an E-linear and L_0-semi-linear action of $\mathrm{Gal}(L/K)$ which commutes with φ and N. The following follows from [Fon94b, 4.2.6, 5.1.5].

Proposition 3.1.1. *Let W be a potentially semi-stable $B^{\otimes E}_{|K}$-pair, semi-stable when restricted to G_L where L/K is finite and Galois. The $B^{\otimes E}_{|K}$-pair W is semi-stable if and only if the inertia group $I_{L/K}$ acts trivially on $D_{\mathrm{st},L}(W)$, and W is crystalline if and only if it is semi-stable and $N = 0$ on $D_{\mathrm{st},L}(W)$.*

3.2 Semi-stable tensor products

Theorem 3.2.1. *Let W and W' be non-zero potentially semi-stable $B^{\otimes E}_{|K}$-pairs. If the $B^{\otimes E}_{|K}$-pair $W \otimes W'$ is semi-stable, then there is a finite extension F/E and a character $\mu : G_K \to F^\times$ such that the $B^{\otimes F}_{|K}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are semi-stable. If, moreover, $W \otimes W'$ is crystalline, then so are $W(\mu^{-1})$ and $W'(\mu)$.*

Proof. Let L/K be a finite Galois extension such that W and W' are semi-stable as $B_{|L}^{\otimes E}$-pairs. By [Fon94b, 5.1.7], we have an isomorphism of E-$(\varphi, N, \mathrm{Gal}(L/K))$-modules:

$$D_{\mathrm{st},L}(W \otimes W') \xleftarrow{\sim} D_{\mathrm{st},L}(W) \otimes_{L_{0,E}} D_{\mathrm{st},L}(W').$$

Let $\mathcal{E} \subset D_{\mathrm{st},L}(W)$ and $\mathcal{E}' \subset D_{\mathrm{st},L}(W')$ be $L_{0,E}$-bases, so that the set $\mathcal{E} \otimes \mathcal{E}'$ of elementary tensors is a basis of $D_{\mathrm{st},L}(W \otimes W')$. For all $g \in G_K$, let $U_g = \mathrm{Mat}(g|\mathcal{E}) \in \mathrm{GL}_d(L_{0,E})$ and let $U'_g = \mathrm{Mat}(g|\mathcal{E}') \in \mathrm{GL}_{d'}(L_{0,E})$. By Proposition 3.1.1, $I_{L/K}$ acts trivially on $D_{\mathrm{st},L}(W \otimes W')$, and we have $\mathrm{Mat}(g|\mathcal{E} \otimes \mathcal{E}') = U_g \otimes U'_g = \mathrm{Id}$ for all $g \in I_{L/K}$, so that $U_g = \eta_g \,\mathrm{Id}$ and $U'_g = \eta_g^{-1}\,\mathrm{Id}$ with $\eta_g \in (L_{0,E})^\times$. The relation $\varphi g = g\varphi$ on $D_{\mathrm{st},L}(W)$ translates to the matrix relation $\mathrm{Mat}(\varphi|\mathcal{E}) \cdot \sigma(U_g) = U_g \cdot g(\mathrm{Mat}(\varphi|\mathcal{E}))$ for all $g \in \mathrm{Gal}(L/K)$, so that for all $g \in I_{L/K}$, we have $\eta_g \in (L_{0,E})^{\sigma=1} = E$ and therefore $\eta_g \in E^\times$.

We now show that there is a finite extension F/E such that the character $\eta : I_{L/K} \to E^\times$ can be extended to a character $\mu : \mathrm{Gal}(L/K) \to F^\times$. Let $\omega \in \mathrm{Gal}(L/K)$ be such that its residual image generates the cyclic group $\mathrm{Gal}(k_L/k_K)$. If $g \in \mathrm{Gal}(L/K)$, then we can write $g = g'\omega^i$ for a unique $g' \in I_{L/K}$ and unique $0 \leqslant i \leqslant f-1$, where $f = [k_L : k_K]$. Let $\xi \in \overline{\mathbf{Q}}_p$ be an fth root of $\eta(\omega^f)$. Since $\eta(\omega g' \omega^{-1}) = \eta(g')$ for all $g' \in I_{L/K}$, putting $F = E(\xi)$ and $\mu(g) := \eta(g')\xi^i$ defines a homomorphism $\mu : G_K \to F^\times$.

The $B_{|K}^{\otimes F}$-pairs $W(\mu^{-1})$ and $W'(\mu)$ are semi-stable, by Proposition 3.1.1. If, moreover, $W \otimes W'$ is crystalline, then the $B_{|K}^{\otimes F}$-pair $W(\mu^{-1}) \otimes W'(\mu)$ is crystalline as well and by the isomorphism of F-$(\varphi, N, \mathrm{Gal}(L/K))$-modules recalled above, we have

$$D_{\mathrm{st},L}(W(\mu^{-1}) \otimes W'(\mu)) \xleftarrow{\sim} D_{\mathrm{st},L}(W(\mu^{-1})) \otimes_{L_{0,F}} D_{\mathrm{st},L}(W'(\mu)).$$

The monodromy operator $N \otimes \mathrm{Id} + \mathrm{Id} \otimes N'$ is zero, and therefore the matrices of N and N' are scalar multiples of the identity. Since N and N' are nilpotent, these scalars are necessarily zero since $L_{0,F}$ is reduced, and thus $W(\mu^{-1})$ and $W'(\mu)$ are crystalline by Proposition 3.1.1. □

COROLLARY 3.2.2. *Let V and V' be non-zero potentially semi-stable E-linear representations of G_K. If $V \otimes_E V'$ is semi-stable, then there is a finite extension F/E and a character $\mu : G_K \to F^\times$ such that the F-linear representations $V(\mu^{-1})$ and $V'(\mu)$ are semi-stable. If, moreover, $V \otimes_E V'$ is crystalline, then so are $V(\mu^{-1})$ and $V'(\mu)$.*

3.3 Semi-stable Schur B-pairs

In this subsection, $n \geqslant 1$ is an integer and $u = (u_1, \ldots, u_r)$ denotes an integer partition $n = u_1 + \cdots + u_r$ such that $u_i \geqslant u_{i+1} \geqslant 1$ for all $i \in \{1, \ldots, r-1\}$.

LEMMA 3.3.1. *Let L/K be a finite Galois extension and let D be an E-$(\varphi, N, \mathrm{Gal}(L/K))$-module such that $\mathrm{rank}(D) \geqslant r(u)$. If $I_{L/K}$ acts trivially on $\mathrm{Schur}^u(D)$, then $I_{L/K}$ acts on D via a character $\eta : I_{L/K} \to E^\times$. If $N = 0$ on $\mathrm{Schur}^u(D)$, then $N = 0$ on D.*

Proof. By extending scalars if necessary, we may suppose that $E \supset L$. We have an isomorphism of rings, $L_{0,E} \xrightarrow{\sim} \bigoplus_{h : L_0 \to \overline{\mathbf{Q}}_p} E$ on which $I_{L/K}$ acts trivially on both sides. We therefore see that D decomposes as an E-linear representation of $I_{L/K}$ into $D \simeq \bigoplus_h D_h$ where D_h is the E-linear representation of $I_{L/K}$ coming from the h-factor map $(\lambda, e) \mapsto h(\lambda)e : L_{0,E} \to E$. The corresponding decomposition of $\mathrm{Schur}^u(D)$ is given by $\mathrm{Schur}^u(D) \simeq \bigoplus_h \mathrm{Schur}^u(D_h)$, and by assumption $I_{L/K}$ acts trivially on each E-linear representation $\mathrm{Schur}^u(D_h)$. Let $I_{L/K}$ act $\overline{\mathbf{Q}}_p$-linearly on $\overline{D}_h = \overline{\mathbf{Q}}_p \otimes_E D_h$. Let $g \in I_{L/K}$. Since $I_{L/K}$ is finite, there is a $\overline{\mathbf{Q}}_p$-basis $\mathcal{E}_h^g = (e_{1,h}^g, \ldots, e_{d,h}^g)$ of $\overline{D}_h$ and elements $\lambda_{1,h}^g, \ldots, \lambda_{d,h}^g \in \overline{\mathbf{Q}}_p$ such that $g(e_{i,h}^g) = \lambda_{i,h}^g e_{i,h}^g$ for

https://doi.org/10.1112/S0010437X1200070X Published online by Cambridge University Press

all $i\in\{1,\dots,d\}$. Consider the $\overline{\mathbf{Q}}_p$-basis of $\mathrm{Schur}^u(\overline{D}_h)$ consisting of elements $e^g_{T,h}$, where T ranges over all tableaux on Y_u with values in $\{1,\dots,d\}$. One has $g(e^g_{T,h})=\lambda^g_{T,h}e^g_{T,h}$, where $\lambda^g_{T,h}=\prod_{i=1}^d(\lambda^g_{i,h})^{m_T(i)}$ and $m_T(i)$ denotes the number of times that i appears in the tableau T. Since $\dim_{\overline{\mathbf{Q}}_p}\overline{D}_h=\mathrm{rank}(D)\geqslant r(u)$, one sees that $\lambda^g_{1,h}=\lambda^g_{2,h}=\cdots=\lambda^g_{d,h}=\lambda^g_h$ by considering the tableaux $T_1,\dots,T_d$ as in § 1.4, and therefore $g(z)=\lambda^g_h z$ for all $z\in\overline{D}_h$. Note that we necessarily have $\lambda^g_h\in E$. We therefore see that for each embedding $h:L_0\to E$, $I_{L/K}$ acts on $\overline{D}_h$ by a character $\eta_h:I_{L/K}\to E^\times$, which translates to saying that $I_{L/K}$ acts on $\overline{D}$ by a character $\eta:I_{L/K}\to(L_{0,E})^\times$. Since $\varphi g=g\varphi$ for all $g\in I_{L/K}$ and $(L_{0,E})^{\sigma=1}=E$, we see that $\eta:I_{L/K}\to E^\times$.

Moreover, since N is an $L_{0,E}$-linear map, the factors in the decomposition $D\simeq\bigoplus_h D_h$ are N-stable. We let N again denote the E-linear nilpotent map induced on D_h. Since $N=0$ on $\mathrm{Schur}^u(\overline{D})=\bigoplus_h\mathrm{Schur}^u(\overline{D}_h)$, we see that $N=0$ on $\mathrm{Schur}^u(\overline{D}_h)$ for each embedding $h:L_0\to\overline{\mathbf{Q}}_p$. Let $(e'_{1,h},\dots,e'_{d,h})$ denote a Jordan canonical basis for N on $\overline{D}_h$. Suppose that $N\neq 0$, so that we may suppose $N(e'_{2,h})=e'_{1,h}$. If T is the tableau on Y_u in which i appears in all boxes of the ith row, except in the right-most column where $i+1$ appears, then a calculation shows that $N(e_{T,h})=e_{T',h}$, where T' is another tableau, therefore contradicting the fact that $N=0$ on $\overline{D}_h$. We therefore see that $N=0$ on each $\overline{D}_h$, so that $N=0$ on $\overline{D}$ and thus $N=0$ on D. □

Theorem 3.3.2. *Let W be a potentially semi-stable $B^{\otimes E}_{|K}$-pair such that $\mathrm{rank}(W)\geqslant r(u)$. If the $B^{\otimes E}_{|K}$-pair $\mathrm{Schur}^u(W)$ is semi-stable, then there is a finite extension F/E and a character $\mu:G_K\to F^\times$ such that the $B^{\otimes F}_{|K}$-pair $W(\mu^{-1})$ is semi-stable. If, moreover, $\mathrm{Schur}^u(W)$ is crystalline, then so is $W(\mu^{-1})$.*

Proof. Let L/K be a finite Galois extension such that W is semi-stable as a $B^{\otimes E}_{|L}$-pair, so that [Fon94b, 5.1.7] implies that we have an isomorphism of E-$(\varphi,N,\mathrm{Gal}(L/K))$-modules

$$\mathrm{Schur}^u(D_{\mathrm{st},L}(W))\xrightarrow{\sim}D_{\mathrm{st},L}(\mathrm{Schur}^u(W)).$$

If $\mathrm{Schur}^u(W)$ is semi-stable, then Proposition 3.1.1 implies that $I_{L/K}$ acts trivially on $\mathrm{Schur}^u(D_{\mathrm{st},L}(W))$. Lemma 3.3.1 implies that $I_{L/K}$ acts on $D_{\mathrm{st},L}(W)$ via a character $\eta:I_{L/K}\to E^\times$. By the same reasoning as in the proof Theorem 3.2.1, there is a finite extension F/E and a character $\mu:\mathrm{Gal}(L/K)\to F^\times$ such that $\mu|_{I_{L/K}}=\eta$. By Proposition 3.1.1, $W(\mu^{-1})$ is semi-stable.

If $\mathrm{Schur}^u(W)$ is crystalline, then $N=0$ on $\mathrm{Schur}^u(D_{\mathrm{st},L}(W))$. Lemma 3.3.1 implies that $N=0$ on $D_{\mathrm{st},L}(W)$, which implies the same for $D_{\mathrm{st},L}(W(\mu^{-1}))$, so that $W(\mu^{-1})$ is crystalline. □

Theorem 3.3.2 implies the following.

Corollary 3.3.3. *Let V be a potentially semi-stable E-linear representation of G_K such that $\dim_E V\geqslant r(u)$. If the E-linear representation $\mathrm{Schur}^u(V)$ of G_K is semi-stable, then there is a finite extension F/E and a character $\mu:G_K\to F^\times$ such that the F-linear representation $V(\mu^{-1})$ of G_K is semi-stable. If, moreover, $\mathrm{Schur}^u(V)$ is crystalline, then so is $V(\mu^{-1})$.*

Acknowledgements

This note is part of my PhD under the supervision of Laurent Berger. We are grateful to Kevin Buzzard, Frank Calegari, Pierre Colmez, Brian Conrad, Tong Liu and Liang Xiao for useful correspondence and to Jean-Marc Fontaine and Jean-Pierre Wintenberger for pointing out the Tannakian argument. Laurent Berger first heard about this question from Barry Mazur.

References

And02 Y. André, *Représentations galoisiennes et opérateurs de Bessel p-adiques*, Ann. Inst. Fourier (Grenoble) **52** (2002), 779–808.

Ber02 L. Berger, *Représentations p-adiques et équations différentielles*, Invent. Math. **148** (2002), 219–284.

Ber08 L. Berger, *Construction de (φ, Γ)-modules: représentations p-adiques et B-paires*, Algebra Number Theory **2** (2008), 91–120.

BC10 L. Berger and G. Chenevier, *Représentations potentiellement triangulines de dimension 2*, J. Théor. Nombres Bordeaux **22** (2010), 557–574.

Col93 P. Colmez, *Périodes des variétés abéliennes à multiplication complexe*, Ann. of Math. (2) **138** (1993), 625–683.

Fon94a J.-M. Fontaine, *Le corps des périodes p-adiques. Avec un appendice par Pierre Colmez: Les nombres algébriques sont denses dans B_{dR}^{+}*, Astérisque **223** (1994), 59–111.

Fon94b J.-M. Fontaine, *Représentations p-adiques semi-stables*, Astérisque **223** (1994), 113–184.

Fon04 J.-M. Fontaine, *Arithmétique des représentations galoisiennes p-adiques, Cohomologies p-adiques et applications arithmétiques (III)*, Astérisque **295** (2004), 1–115.

Ful97 W. Fulton, *Young tableaux. With applications to representation theory and geometry*, London Mathematical Society Student Texts, vol. 35 (Cambridge University Press, Cambridge, 1997).

Ked04 K. Kedlaya, *A p-adic local monodromy theorem*, Ann. of Math. (2) **160** (2004), 93–184.

Meb02 Z. Mebkhout, *Analogue p-adique du théorème de Turrittin et le théorème de la monodromie p-adique*, Invent. math. **148** (2002), 319–351.

Nak09 K. Nakamura, *Classification of two-dimensional split trianguline representations of p-adic fields*, Compos. Math. **145** (2009), 865–914.

Sen80 S. Sen, *Continuous cohomology and p-adic Galois representations*, Invent. Math. **62** (1980), 89–116.

Ski09 C. Skinner, *A note on the p-adic Galois representations attached to Hilbert modular forms*, Doc. Math. **14** (2009), 241–258.

Win95 J.-P. Wintenberger, *Relèvement selon une isogénie de systèmes ℓ-adiques de représentations galoisiennes associés aux motifs*, Invent. math. **120** (1995), 215–240.

Win97 J.-P. Wintenberger, *Propriétés du groupe tannakien des structures de Hodge p-adiques et torseur entre cohomologies cristalline et étale*, Ann. Inst. Fourier (Grenoble) **47** (1997), 1289–1334.

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