THE IDEAL LATTICE OF AN *MS*-ALGEBRA†

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Preliminaries. Recently we introduced the notion of an *MS*-algebra as a common abstraction of a de Morgan algebra and a Stone algebra [**2**]. Precisely, an *MS-algebra* is an algebra $\langle L; \wedge, \vee, ^\circ, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ such that $\langle L; \wedge, \vee, 0, 1\rangle$ is a distributive lattice with least element 0 and greatest element 1, and $x \to x^\circ$ is a unary operation such that $x \leqslant x^{\circ\circ}$, $(x \wedge y)^\circ = x^\circ \vee y^\circ$ and $1^\circ = 0$. It follows that $^\circ$ is a dual endomorphism of L and that $L^{\circ\circ} = \{x^{\circ\circ}; x \in L\}$ is a subalgebra of L that is called the *skeleton* of L and that belongs to **M**, the class of de Morgan algebras. Clearly, the class **MS** of *MS*-algebras is equational. All the subvarieties of **MS** were described in [**3**]. The lattice Λ (**MS**) of subvarieties of **MS** has 20 elements (see Fig. 1) and its non-trivial part (we exclude **T**, the class of one-element algebras) splits into the prime filter generated by **M**, that is $[\mathbf{M}, \mathbf{M}_1]$, the prime ideal generated by **S**, that is $[\mathbf{B}, \mathbf{S}]$, and the interval $[\mathbf{K}, \mathbf{K}_2 \vee \mathbf{K}_3]$.

In [**4**, Lemma 4] we observed that $\mathbf{V}_L$, the least subvariety of **MS** to which L belongs, is located in $[\mathbf{B}, \mathbf{S}]$, $[\mathbf{K}, \mathbf{K}_2 \vee \mathbf{K}_3]$ or $[\mathbf{M}, \mathbf{M}_1]$ if and only if $\mathbf{V}_{L^{\circ\circ}}$ equals **B**, **K** or **M** respectively.

In this paper, we investigate the ideal lattice $\mathcal{I}(L)$ of an *MS*-algebra L. We define on $\mathcal{I}(L)$ a unary operation $^\square$ in order to make it into an *MS*-algebra. We show that in case L is complete as a lattice, this can be done if and only if $L^{\circ\circ}$ is a dual Heyting lattice. We also prove that if L belongs to a class located in one of the intervals $[\mathbf{B}, \mathbf{S}]$, $[\mathbf{K}, \mathbf{K}_2 \vee \mathbf{K}_3]$ or $[\mathbf{M}, \mathbf{M}_1]$, then $\mathcal{I}(L)$ belongs to a class located in the same interval, usually "higher" in this interval, possibly at the top of this interval.

1. How to define a suitable unary operation $^\square$ on $\mathcal{I}(L)$. The ideal lattice of a bounded distributive lattice is a bounded distributive lattice. Hence it is natural to ask whether, given an *MS*-algebra L, the corresponding ideal lattice $\mathcal{I}(L)$ can be made into an *MS*-algebra. In what follows the unary operation on $\mathcal{I}(L)$ will be denoted by $^\square$. Of course for the embedding of L into $\mathcal{I}(L)$ to be interesting it has to satisfy the following condition:

$$(\forall x \in L)x^{\downarrow\square} = x^{\circ\downarrow},$$

in which $x^\downarrow$ means the principal ideal generated by x. In this manner $\langle L; ^\circ\rangle$ is (isomorphic to) a subalgebra of $\langle \mathcal{I}(L); ^\square\rangle$ and, when L satisfies the Ascending Chain Condition (that is, when all the ideals are principal), $\langle L; ^\circ\rangle$ and $\langle \mathcal{I}(L); ^\square\rangle$ are isomorphic *MS*-algebras.

Moreover, we have to take into account the fact that a Stone algebra is an *MS*-algebra in which the unary operation, generally denoted by *, is entirely determined by the lattice structure of $L: x^*$ is the largest element of L that is disjoint from x and is

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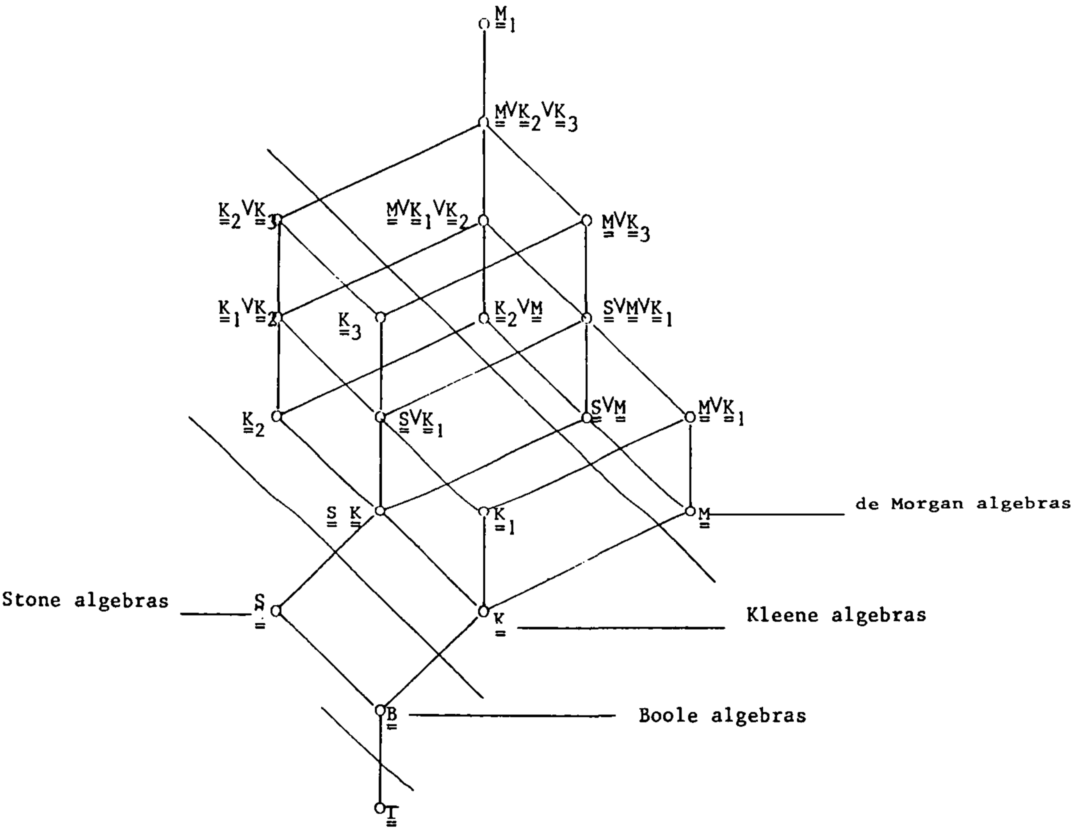

Figure 1.

called the *pseudocomplement* of x. The ideal lattice $\mathscr{I}(L)$ of a pseudocomplemented lattice is a pseudocomplemented lattice in which the pseudocomplement I^* of an ideal I is defined by $I^* = \bigcap_{i \in I} i^{*\downarrow}$. Finally, if the pseudocomplemented lattice is complete, then we have the equality $I^* = (\sup I)^{*\downarrow}$, which follows from the well-known 0-distributive property enjoyed by any pseudocomplemented lattice:

$$\text{if} \quad (\forall \alpha \in A)\, y \wedge x_\alpha = 0, \quad \text{then} \quad y \wedge \Big(\bigvee_{\alpha \in A} x_\alpha\Big) = 0.$$

The constraints of these observations suggest that we define on $\mathscr{I}(L)$, for $L \in \mathbf{MS}$, the unary operation $^\square$ by

$$I^\square = \bigcap_{a \in I} a^{\circ\downarrow} \tag{1}$$

or, equivalently, in case L is complete,

$$I^\square = (\sup I)^{\circ\downarrow}. \tag{2}$$

The equivalence of (1) and (2) will be proved in the next section.

2. When is $\langle \mathcal{I}(L); {}^{\square}\rangle$ an MS-algebra? One could hope that, for any *MS*-algebra L, $\mathcal{I}(L)$ endowed with the operation ${}^{\square}$ defined as in (1) be an *MS*-algebra. Clearly, for any $I \in \mathcal{I}(L)$, $I \leq I^{\square\square}$ and $L^{\square} = \{0\}$. Unfortunately, easy examples show that $(I \wedge J)^{\square} = I^{\square} \vee J^{\square}$ does not hold generally. We shall show that the latter equality is conditioned by a certain distributivity property of $L^{\circ\circ}$.

We already noticed that $L^{\circ\circ}$ is a subalgebra of L. But, even if L is complete, $L^{\circ\circ}$ is not a complete sublattice of L: the inequalities

$$\Big(\bigwedge_{\alpha\in A} x_\alpha\Big)^{\circ} \geq \bigvee_{\alpha\in A} x_\alpha^{\circ},$$

$$\Big(\bigwedge_{\alpha\in A} x_\alpha\Big)^{\circ\circ} \leq \bigwedge_{\alpha\in A} x_\alpha^{\circ\circ},$$

$$\Big(\bigvee_{\alpha\in A} x_\alpha\Big)^{\circ\circ} \geq \bigvee_{\alpha\in A} x_\alpha^{\circ\circ}$$

cannot be generally replaced by equalities. Nevertheless, the dual of the first formula gives rise to an equality.

Lemma 1. *Let L be a complete MS-algebra. Then for every family $(x_\alpha)_{\alpha\in A}$ of elements of L,*

$$\Big(\bigvee_{\alpha\in A} x_\alpha\Big)^{\circ} = \bigwedge_{\alpha\in A} x_\alpha^{\circ}.$$

Proof. Clearly, $\Big(\bigvee_{\alpha\in A} x_\alpha\Big)^{\circ} \leq \bigwedge_{\alpha\in A} x_\alpha^{\circ}$. Now $\bigwedge_{\alpha\in A} x_\alpha^{\circ} \leq x_\beta^{\circ}$ for every $\beta \in A$ so $x_\beta \leq x_\beta^{\circ\circ} \leq \Big(\bigwedge_{\alpha\in A} x_\alpha^{\circ}\Big)^{\circ}$ and hence $\bigvee_{\beta\in A} x_\beta \leq \Big(\bigwedge_{\alpha\in A} x_\alpha^{\circ}\Big)^{\circ}$. Consequently $\bigwedge_{\alpha\in A} x_\alpha^{\circ} \leq \Big(\bigwedge_{\alpha\in A} x_\alpha^{\circ}\Big)^{\circ\circ} \leq \Big(\bigvee_{\beta\in A} x_\beta\Big)^{\circ}$ and the proof is complete. □

Lemma 2. *Let L be a complete MS-algebra. Then for every ideal I of L*

$$I^{\square} = \bigcap_{a\in I} a^{\circ\downarrow} = (\sup I)^{\circ\downarrow}.$$

Proof. $x \in \bigcap_{a\in I} a^{\circ\downarrow} \Leftrightarrow x \leq a^{\circ}\ (\forall a \in I)$

$$\Leftrightarrow x \leq \bigwedge_{a\in I} a^{\circ} = \Big(\bigvee_{a\in I} a\Big)^{\circ} = (\sup I)^{\circ}$$

$$\Leftrightarrow x \in (\sup I)^{\circ\downarrow}.\quad \square$$

Before stating our Main Theorem, we recall that in a complete distributive lattice L, the following conditions are equivalent [**1**]:

(D_1) $x \vee \Big(\bigwedge_{\beta\in B} y_\beta\Big) = \bigwedge_{\beta\in B} (x \vee y_\beta)$,

(D_2) $\Big(\bigwedge_{\alpha\in A} x_\alpha\Big) \vee \Big(\bigwedge_{\beta\in B} y_\beta\Big) = \bigwedge_{\substack{\alpha\in A\\ \beta\in B}} (x_\alpha \vee y_\beta)$,

(D_3) L is dually relatively pseudocomplemented, i.e. L is a dual Heyting algebra.

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Main Theorem. *Let L be a complete MS-algebra. For every ideal I of L define*

$$I^{\square} = (\sup I)^{\circ\downarrow}.$$

Then $\langle \mathcal{I}(L), {}^{\square}\rangle$ is an MS-algebra if and only if $L^{\circ\circ}$ is a dual Heyting algebra.

Proof. We have $L^{\square} = 1^{\circ\downarrow} = \{0\}$ and, for every ideal I,

$$I^{\square\square} = (\sup I^{\square})^{\circ\downarrow} = (\sup I)^{\circ\circ\downarrow} \supseteq (\sup I)^{\downarrow} \supseteq I.$$

Now on the one hand, for ideals I, J of L, we have, using Lemma 1,

$$\begin{aligned} I^{\square} \vee J^{\square} &= (\sup I)^{\circ\downarrow} \vee (\sup J)^{\circ\downarrow} \\ &= [(\sup I)^{\circ} \vee (\sup J)^{\circ}]^{\downarrow} \\ &= \left[\left(\bigvee_{i\in I} i\right)^{\circ} \vee \left(\bigvee_{j\in J} j\right)^{\circ}\right]^{\downarrow} \\ &= \left[\bigwedge_{i\in I} i^{\circ} \vee \bigwedge_{j\in J} j^{\circ}\right]^{\downarrow} \end{aligned}$$

and, on the other, again by Lemma 1,

$$\begin{aligned} (I \cap J)^{\square} = [\sup(I\cap J)]^{\circ\downarrow} &= \left[\bigvee_{\substack{i\in I\\ j\in J}} (i \wedge j)\right]^{\circ\downarrow} \\ &= \left[\bigwedge_{\substack{i\in I\\ j\in J}} (i^{\circ} \vee j^{\circ})\right]^{\downarrow}. \end{aligned}$$

It follows that $(I\cap J)^{\square} = I^{\square} \vee J^{\square}$, and that $\langle \mathcal{I}(L), {}^{\square}\rangle$ is an MS-algebra, if and only if $L^{\circ\circ}$ satisfies (D_2). □

Corollary 1. *Let I be an ideal of the complete MS-algebra L. Then $I = I^{\square\square}$ if and only if $I = x^{\circ\circ\downarrow}$ for some $x \in L$. If I is principal, then $I = I^{\square\square}$ if and only if $x \in I \Rightarrow x^{\circ\circ} \in I$.*

Proof. $I = I^{\square\square} \Leftrightarrow I = (\sup I)^{\circ\circ\downarrow}$.

If I is principal, then $I = (\sup I)^{\downarrow}$ so $I = I^{\square\square} \Leftrightarrow \sup I \in L^{\circ\circ} \Leftrightarrow (x \in I \Rightarrow x^{\circ\circ} \in I)$. □

Corollary 2. *Let L be a complete MS-algebra. Then $[\mathcal{I}(L)]^{\square\square} \cong L^{\circ\circ}$, $\mathcal{I}(L) \in \mathbf{S} \Leftrightarrow L \in \mathbf{S}$ and $\mathcal{I}(L) \in \mathbf{K}_2 \vee \mathbf{K}_3 \Leftrightarrow L \in \mathbf{K}_2 \vee \mathbf{K}_3$.*

Proof. The isomorphism of $[\mathcal{I}(L)]^{\square\square}$ and $L^{\circ\circ}$ follows directly from Corollary 1. Since $\mathcal{I}(L)$ and L have isomorphic skeletons, any identity involving only elements of $L^{\circ\circ}$ is satisfied or not by L and $\mathcal{I}(L)$ simultaneously. The varieties $\mathbf{S}$ and $\mathbf{K}_2 \vee \mathbf{K}_3$ are characterised [see **3**] by the axioms

$$(\forall x \in L) x^{\circ\circ} \wedge x^{\circ} = 0$$

and

$$(\forall x, y \in L)(x^{\circ\circ} \wedge x^{\circ}) \vee (y^{\circ\circ} \vee y^{\circ}) = y^{\circ\circ} \vee y^{\circ}$$

respectively; hence the two logical equivalences mentioned in the statement. □

3. Going from L to $\mathscr{I}(L)$. Let $\langle L, {}^\circ\rangle$ be an *MS*-algebra and let us suppose that $\langle \mathscr{I}(L), {}^\square\rangle$ is also an *MS*-algebra. Of course, any identity which is satisfied by $\langle \mathscr{I}(L), {}^\square\rangle$ holds in $\langle L, {}^\circ\rangle$ too, but the converse is false. It means that $\mathbf{V}_L \leq \mathbf{V}_{\mathscr{I}(L)}$ and when one goes from L to $\mathscr{I}(L)$ the jump inside Λ (**MS**) can be important. Nevertheless Corollary 2 fixes some limits to this jump: if $\mathbf{V}_L$ belongs to $[\mathbf{B}, \mathbf{S}]$, $[\mathbf{K}, \mathbf{K}_2 \vee \mathbf{K}_3]$ or $[\mathbf{M}, \mathbf{M}_1]$, then $\mathbf{V}_{\mathscr{I}(L)}$ belongs to the same interval. The following examples will show that inside two of these three intervals the jump can be maximal, that is from bottom to top.

EXAMPLE 1. *If* $\mathbf{V}_L = \mathbf{B}$ *then* $\mathbf{V}_{\mathscr{I}(L)}$ *can be* $\mathbf{S}$.

Let L be a complete Boolean algebra with a non-principal ideal I (for instance, take, in the subset lattice of an infinite set, the ideal formed by the finite subsets). By Corollary 2, we have that $[\mathscr{I}(L)]^{\square\square} \cong L^{\circ\circ} = L$. Since $I < I^{\square\square}$ and $[\mathscr{I}(L)]^{\square\square}$ is the greatest Boolean subalgebra of $\mathscr{I}(L)$, it follows that I has no complement and $\mathbf{V}_{\mathscr{I}(L)} = \mathbf{S}$.

EXAMPLE 2. *If* $\mathbf{V}_L = \mathbf{M}$ *then* $\mathbf{V}_{\mathscr{I}(L)}$ *can be* $\mathbf{M}_1$.

Let $L = \mathbb{N}_\infty \times \mathbb{N}_\infty^d$ where $\mathbb{N}_\infty$ is the set of natural numbers completed by ∞ with the usual order, and $\mathbb{N}_\infty^d$ is the order dual of $\mathbb{N}_\infty$. Define on L the operation $^\circ$ by $(x, y)^\circ = (y, x)$. Then $\langle L; {}^\circ\rangle$ belongs properly to $\mathbf{M}$ and the fixed points are (a, a) for all $a \in \mathbb{N}_\infty$. Observe also that L is a dual Heyting algebra since any bounded chain is so and the class of dual Heyting algebras is equational. It then follows that, since $L^{\circ\circ} = L$, we have $\langle \mathscr{I}(L); \square\rangle$ is an *MS*-algebra. The ideal lattice $\mathscr{I}(L)$ has infinitely many non-principal ideals $I_k = \{(x, y) : x < \infty,\ y \geq k\}$, $k \in \mathbb{N}_\infty$. Its Hasse diagram is shown in Fig. 2.

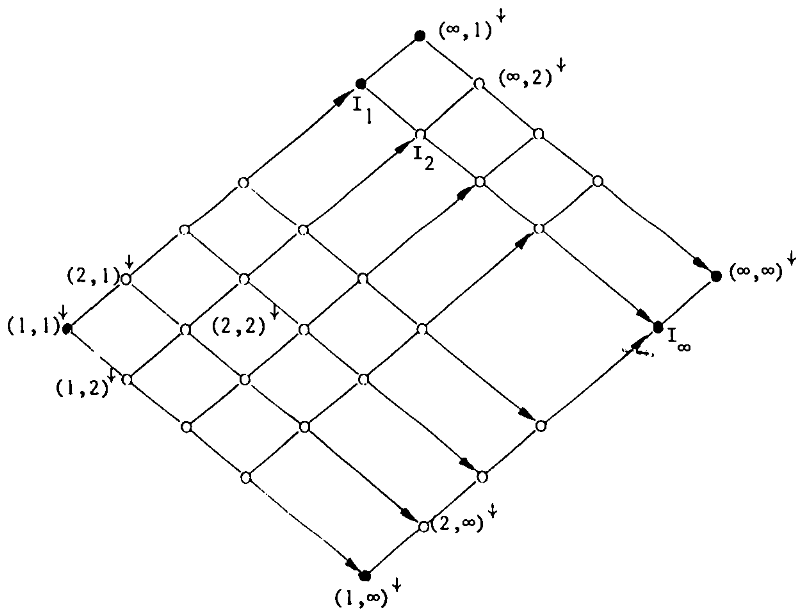

Figure 2.

Clearly $(1, \infty)^{\downarrow\Box} = (\infty, 1)^{\downarrow}$, $(\infty, 1)^{\downarrow\Box} = (1, \infty)^{\downarrow}$, $(1, 1)^{\downarrow\Box} = (1, 1)^{\downarrow}$, $(\infty, \infty)^{\downarrow\Box} = (\infty, \infty)^{\downarrow} = I_{\infty}^{\Box}$ and $I_1^{\Box} = (1, \infty)^{\downarrow}$. Hence $\{(1, \infty)^{\downarrow}, (1, 1)^{\downarrow}, (\infty, \infty)^{\downarrow}, (\infty, 1)^{\downarrow}, I_1, I_{\infty}\}$ is a subalgebra of $\langle \mathcal{I}(L); {}^{\Box}\rangle$ isomorphic to M_1 [**2**, page 307]. Since $\mathbf{M} \vee \mathbf{K}_2 \vee \mathbf{K}_3$ is the largest subvariety of **MS** not containing M_1, we have that $\mathbf{V}_{\mathcal{I}(L)} = \mathbf{M}_1$.

EXAMPLE 3. *If* $\mathbf{V}_L = \mathbf{K}$ *then* $\mathbf{V}_{\mathcal{I}(L)}$ *can be* $\mathbf{K}_1 \vee \mathbf{K}_2$.

Let $L = \{-\infty\} \oplus \mathbb{Z} \oplus \{a\} \oplus \mathbb{Z} \oplus \{+\infty\}$ and define the operation $^\circ$ "symmetrically" with respect to the element a, which is consequently the only fixed point. Clearly $\langle L; {}^\circ\rangle$ belongs to **K** and $\langle \mathcal{I}(L); {}^{\Box}\rangle$ is an *MS*-algebra. Their Hasse diagrams are depicted in Fig. 3.

It is easy to verify that $\mathbf{V}_{\mathcal{I}(L)} = \mathbf{K}_1 \vee \mathbf{K}_2$.

COMMENT. We did not succeed in finding an *MS*-algebra L such that $\mathbf{V}_L = \mathbf{K}$ and $\mathbf{V}_{\mathcal{I}(L)} = \mathbf{K}_2 \vee \mathbf{K}_3$. One can easily see that the question is in fact the following: if the axiom $x \wedge x^\circ \leqslant y \vee y^\circ$ is satisfied in L, is it also satisfied in $\mathcal{I}(L)$? We leave the question open.

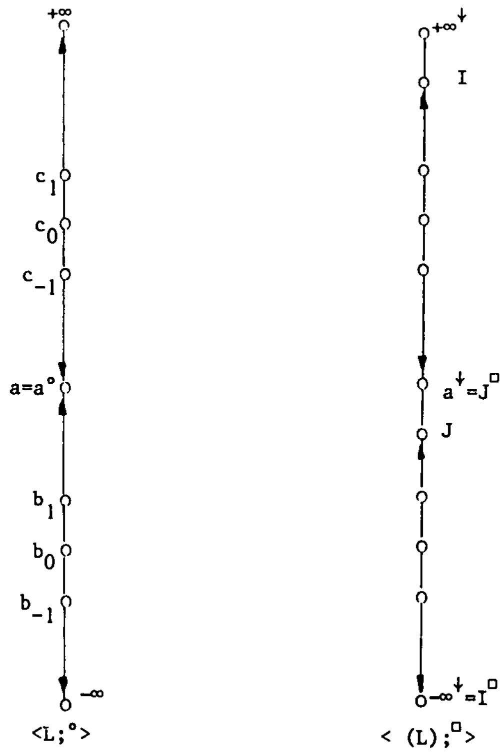

Figure 3.

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