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PRINCIPAL INDECOMPOSABLE MODULES FOR SOME THREE-DIMENSIONAL SPECIAL LINEAR GROUPS

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Let k be a finite field of characteristic 2, and let G be the three dimensional special linear group over k. The principal indecomposable modules of G over k are constructed from tensor products of the irreducible modules, and formulae for their dimensions are given.

Let G be a finite group and k a field, and let kG be the group algebra. The right regular module, also denoted by kG, has a finite number of isomorphism types of indecomposable direct summands $P_1, \ldots, P_r$. These are the *Principal Indecomposable Modules*: they are precisely the indecomposable projective kG-modules and the projective covers of the irreducible kG-modules.

The aim of this paper is to construct the principal indecomposables over a field of characteristic 2 for the 3-dimensional special linear groups $\mathrm{SL}\left(3, 2^n\right)$. This is a natural sequel to work of Alperin [2] on the principal indecomposables for $\mathrm{SL}\left(2, 2^n\right)$, and uses similar methods. The idea is to start with the Steinberg module S. Since S is projective, so is $S \otimes U$ for any irreducible U (indeed for any module U [5, Example 62.2]) and each principal indecomposable occurs as a direct summand of one of these [1]. Finding these direct summands is reduced to counting

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how many times S appears in the tensor product of two irreducibles. In Theorem 1 the principal indecomposables are displayed as tensor products or differences of tensor products of certain modules. The dimensions of the principal indecomposables can then be calculated and are given in Theorem 2.

This paper is based on the author's Oxford University D.Phil Thesis (1979).

1. Preliminaries

Assume k is a splitting field for G and let $V_1, \ldots, V_r$ be all the different isomorphism types of irreducible kG-modules. The principal indecomposables are in one-to-one correspondence $P_i \leftrightarrow V_i$ with the irreducibles and there is the decomposition

$$kG \simeq \sum_{i=1}^{r\oplus} (\dim V_i)P_i$$

(where for $n \in \mathbb{Z}$, nM denotes n copies of the module M). P_i is called the projective cover of V_i and has V_i both as its unique bottom factor (minimal submodule) and unique top factor (irreducible quotient) [5, §§54, 58]. We also denote the projective cover of an irreducible U by $P(U)$.

For two kG-modules V and W let (V, W) be the dimension of the k-vector space $\mathrm{Hom}_{kG}(V, W)$. By Schur's lemma

$$(P_i, V_j) = (V_i, V_j) = \delta_{ij} .$$

The next two easy lemmas give a method for finding the direct summands of a projective module.

LEMMA 1.1. *If k is a splitting field for G and $V_1, \ldots, V_r$ are the non-isomorphic irreducible kG-modules then for any projective kG-module P ,*

$$P \simeq \sum_{i=1}^{r\oplus} (P, V_i)P(V_i) .$$

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Proof. Since any projective module is a direct sum of principal indecomposables, this is immediate from the remarks above. //

Also by general properties of projective modules, for any module W, (P_i, W) is the number of composition factors of W isomorphic to V_i.

For a module M, M^* denotes the dual module $\mathrm{Hom}_k(M, 1)$ where 1 is the trivial module.

LEMMA 1.2. *For any three kG-modules, A, B and C,*

$$(A \otimes B, C) = (A, B^* \otimes C) .$$

Proof. We have the usual isomorphism of vector spaces

$$(*) \qquad i : \mathrm{Hom}_k(A \otimes B, C) \to \mathrm{Hom}_k\big(A, \mathrm{Hom}_k(B, C)\big) .$$

$\mathrm{Hom}_k(B, C)$ is made into a kG-module by defining

$$(fg)(b) = f\big(bg^{-1}\big)g \quad \text{for} \quad f \in \mathrm{Hom}_k(B, C) , \quad b \in B , \quad g \in G .$$

If both sides of (*) are made into kG-modules in the same way, then i is a kG-isomorphism. Since $\mathrm{Hom}_k(B, C) \simeq B^* \otimes C$ the result follows. //

2. The irreducible modules

Henceforth k is the field of 2^n elements and G is $\mathrm{SL}(3, k)$. Let V_0 be the standard 3-dimensional module for G; that is, the row vectors of length 3. If σ is the Galois automorphism of k, then σ acts on $g \in G$ by acting on its entries. V_i, the ith Galois conjugate of V_0, is defined as follows: V_i consists of the row vectors of length 3 and for $v \in V_i$, $g \in G$,

$$v \cdot g = v\big(g^{\sigma^i}\big)$$

taking the usual V_0 action on the right. This gives $n - 1$ new modules $V_1, \ldots, V_{n-1}$. Let $N = \{0, 1, \ldots, n-1\}$. Where necessary regard N as $\mathbb{Z}/n\mathbb{Z}$ so that for $i \in N$, $i + 1$ always makes sense. For a subset K of N we let $\overline{K}$ denote the complement of K in N. For $I \subseteq N$ we define

$$V_I = \prod_{i \in I}^{\otimes} V_i \ .$$

We also have the duals V_i and $V_I^* \simeq \prod_{i \in I}^{\otimes} V_i^*$.

Since V_0^* is afforded by the column vectors c of length 3 on which $g \in G$ acts by $c \to g^{-1}c$, $V_0^* \otimes V_0$ is afforded by the 3×3 matrices X on which g acts by $X \to g^{-1}Xg$, and so $V_0^* \otimes V_0 \simeq T \oplus W_0$ where T consists of the scalar matrices and W_0 those of trace zero. T is the trivial module while W_0 is easily seen to be irreducible of dimension 8 and self-dual. Let W_i be the ith Galois conjugate of W_0 and let $W_I = \prod_{i \in I}^{\otimes} W_i$. By a theorem of Brauer [3, 83.5], $SL\left(3,\ 2^n\right)$ has 4^n absolutely irreducible modules in characteristic 2 . 1, V_0, V_0^* and W_0 are the 4 irreducibles for $SL(3,\ 2)$ and so by Steinberg's tensor product theorem [4] the modules $M_0 \otimes M_1 \otimes \ldots \otimes M_{n-1}$ where $M_i \simeq 1$, V_i, V_i^* or W_i are the absolute irreducibles for $SL\left(3,\ 2^n\right)$. We call M_i the ith tensor factor of the irreducible. Note that since all the absolute irreducibles can be written over k , k is a splitting field so Lemma 1.1 applies.

We shall also use a more convenient notation. For an ordered triple $(I,\ J,\ K)$ of pairwise disjoint subsets of N , let $V(I,\ J,\ K) = V_I \otimes V_J^* \otimes W_K$. Then the irreducibles are indexed by all such triples. Let $P(I,\ J,\ K)$ be the projective cover of $V(I,\ J,\ K)$. The module W_N of dimension 8^n is the Steinberg module [5] and henceforth we denote it by S . It is the reduction modulo 2 of an ordinary irreducible and so by a theorem of Brauer and Nesbitt [3, 86.3] it is a principal indecomposable and is projective.

We now obtain some information about the tensor product of two irreducibles.

LEMMA 2.1. *$V_0 \otimes V_0$ has a unique composition series*

$$V_0 \otimes V_0 \supset X \supset Y \supset \{0\}$$

where $(V_0 \otimes V_0)/X \simeq V_0^* \simeq Y$ *and* $X/Y \simeq V_1$.

Proof. $V_0 \otimes V_0$ is afforded by the 3×3 matrices M over k , and $g \in G$ acts by $M \to g^t M g$, where g^t is the transpose of g . X consists of the symmetric matrices and Y those with zeros on the leading diagonal.

LEMMA 2.2. *Assume* $n \geq 2$. *$V_0 \otimes W_0$ is indecomposable with composition factors (counting multiplicities):*

$$V_0,\ V_1^*,\ V_0,\ V_0^* \otimes V_1,\ V_1^*,\ V_0 .$$

V_0 is the unique top and the unique bottom factor. $V_0 \otimes W_0$ has a submodule and a quotient isomorphic to $V_0^* \otimes V_0^*$.

Proof. $V_0 \otimes V_0$ has composition factors $V_0^*,\ V_1,\ V_0^*$ so $V_0^* \otimes V_0^*$ has composition factors $V_0,\ V_1^*,\ V_0$, and $V_0^* \otimes V_0 \otimes V_0$ has composition factors

$$V_0,\ V_1^*,\ V_0,\ V_0^* \otimes V_1,\ V_0,\ V_1^*,\ V_0 .$$

But $V_0^* \otimes V_0 \otimes V_0 \simeq V_0 \oplus (V_0 \otimes W_0)$ so the composition factors are as claimed. To find the top factor it is easily checked using Lemma 1.2, that $(V_0 \otimes W_0,\ U)$ is zero for each composition factor U , except that it is 1 for $U = V_0$. Similarly check $(U,\ V_0 \otimes W_0)$ to find the bottom factor.

Now $V_0 \otimes V_0$ has submodules X and Y with $X \supset Y$ and

$$(V_0 \otimes V_0)/X \simeq V_0^* \simeq Y .$$

So $V_0 \otimes V_0 \otimes V_0^*$ has submodules $X \otimes V_0^*$ and $Y \otimes V_0^*$ such that

$$(V_0 \otimes V_0 \otimes V_0^*)/(X \otimes V_0^*) \simeq V_0^* \otimes V_0^* \simeq Y \otimes V_0^* .$$

Also

$$V_0 \otimes V_0 \otimes V_0^* = C \oplus D$$

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where

$$C \simeq V_0 \ , \quad D \simeq V_0 \otimes W_0 \ .$$

Let π be the projection onto D. Since $Y \otimes V_0^*$ is indecomposable, $C = \ker \pi$ is not a submodule of $Y \otimes V_0^*$, and so π maps $Y \otimes V_0^*$ monomorphically into $V_0 \otimes W_0$ and $V_0 \otimes W_0$ has a submodule isomorphic to $V_0^* \otimes V_0^*$.

Again

$$(C \oplus D)/\left(X \otimes V_0^*\right) \simeq V_0^* \otimes V_0^*$$

is indecomposable. Since C is irreducible, $\left(C+X \otimes V_0^*\right)/\left(X \otimes V_0^*\right)$ is either 0 or irreducible, and so in either case must be contained in $\left(D+X \otimes V_0^*\right)/\left(X \otimes V_0^*\right)$. Hence

$$V_0^* \otimes V_0^* \simeq \left(D+X \otimes V_0^*\right)/\left(X \otimes V_0^*\right) \simeq D/\left(D \cap \left(X \otimes V_0^*\right)\right)$$

and $V_0^* \otimes V_0^*$ is a quotient of $D \simeq V_0 \otimes W_0$. //

COROLLARY 2.3. *For* $n \geq 2$, $\left(V_0 \otimes W_0, V_0 \otimes W_0\right) \geq 3$.

Proof. Let α, β be the endomorphisms mapping $V_0 \otimes W_0$ to its submodules V_0^* and $V_0^* \otimes W_0^*$ respectively, which exist by Lemma 2.2. Then α, β and the identity are linearly independent. //

LEMMA 2.4. *Assume* $n \geq 2$. *Then*

$$W_0 \otimes W_0 \simeq 2W_0 \oplus A_0$$

where A_0 *is a self-dual indecomposable module with composition factors (counting multiplicities)*

$$1,\ 1,\ 1,\ 1,\ V_0 \otimes V_1,\ V_0 \otimes V_1,\ V_0^* \otimes V_1^*,\ V_0^* \otimes V_1^*,\ W_1 \ .$$

1 *is the unique top factor and the unique bottom factor of* A_0.

Proof. The composition factors of $V_0 \otimes V_0 \otimes V_0^* \otimes V_0^*$ are found from those of $V_0 \otimes V_0$ and $V_0^* \otimes V_0^*$. But this module is $1 \oplus 2W_0 \oplus W_0 \otimes W_0$, giving the composition factors of $W_0 \otimes W_0$. Now, by Lemma 1.2,

$$\begin{aligned}(W_0 \otimes W_0, 1 \oplus W_0) &= (W_0 \otimes W_0, V_0 \otimes V_0^*) \\ &= (W_0 \otimes V_0, W_0 \otimes V_0) \geq 3\end{aligned}$$

by Corollary 2.3. Since $(W_0 \otimes W_0, 1) = 1$, $(W_0 \otimes W_0, W_0) \geq 2$. Thus $W_0 \otimes W_0$ has a submodule R such that $(W_0 \otimes W_0)/R \simeq 2W_0$. By self-duality $W_0 \otimes W_0$ also has a submodule Q isomorphic to $2W_0$. Since W_0 only appears twice as a composition factor of $W_0 \otimes W_0$, it follows that $Q \cap R = \{0\}$, so $W_0 \otimes W_0 \simeq Q \oplus R$ and we have the decompositon claimed. Clearly A_0 is self-dual since $W_0 \otimes W_0$ and W_0 are. Finally the top and bottom factors of A_0 are found using Lemma 1.2. //

Lemmas 2.2-2.4 fail for $n = 1$ because the modules given as composition factors are no longer irreducible. However one can immediately deduce the corresponding result.

LEMMA 2.5. *For* $n = 1$,

$$V_0 \otimes W_0 = W_0 \oplus P(V_0) , \quad V_0^* \otimes W_0 = W_0 \oplus P(V_0^*)$$

and

$$W_0 \otimes W_0 = 3W_0 \oplus P(V_0) \oplus P(V_0^*) \oplus P(1) .$$

Proof. From Lemma 2.2 we deduce that the composition factors of $V_0 \otimes W_0$ are

$$V_0, V_0^*, V_0, 1, W_0, V_0^*, V_0 .$$

For $n = 1$, W_0 is projective and so $V_0 \otimes W_0 = W_0 \oplus Q$. Using Lemma 1.2, $(Q, V_0) = 1$, while $(Q, U) = 0$ for all other irreducibles. Hence $Q = P(V_0)$. The other equations follow similarly. //

3. The principal indecomposable modules

In this section we decompose the projective modules $S \otimes U$, for U irreducible, to find the principal indecomposables. By Lemma 1.1,

$$S \otimes U \simeq \sum^{\oplus} (S \otimes U,\ V)P(V)$$

summing over the irreducibles V. But, by Lemma 1.2,

$$(S \otimes U,\ V) = (S,\ U^* \otimes V)\ .$$

The next few lemmas are devoted to calculating this.

DEFINITION 3.1. Let $U = U_0 \otimes U_1 \otimes \ldots \otimes U_{n-1}$ and $X = X_0 \otimes X_1 \otimes \ldots \otimes X_{n-1}$ be irreducibles where each U_i and X_i is 1, V_i, V_i^* or W_i. Then $Y = Y_0 \otimes Y_1 \otimes \ldots \otimes Y_{n-1}$ is a *cross-section* of $U \otimes X$ if each Y_i is a composition factor of $U_i \otimes X_i$.

LEMMA 3.2. *Let* U *and* X *be irreducibles and let* Y *be a cross-section of* $U \otimes X$. *If* S *is a composition factor of* Y *then* Y *is isomorphic to* S *or to* $V_N \otimes V_N^*$.

Proof. By Lemmas 2.1-2.4 each Y_i is isomorphic to one of

$$1,\ V_i,\ W_i,\ V_i \otimes V_{i+1},\ V_i^* \otimes V_{i+1},\ V_{i+1},\ W_{i+1}$$

or to the dual of one of these. Thus $\dim Y = 3^a 8^b$ and since Y has n tensor factors $a \le 2(n-b)$. If Y is reducible it has a reducible pair of tensor factors

$$V_i \otimes V_i,\ V_i^* \otimes V_i,\ V_i \otimes W_i,\ W_i \otimes W_i$$

or the dual of one of these. Call Y' a *standard reduction* of Y if Y' is obtained from Y by replacing a reducible pair by one of its composition factors. Call it *type* A if $V_i^* \otimes V_i$ is replaced by W_i.

If S is a composition factor of Y, it is obtained from Y by a sequence of standard reductions. The only one which increases the power of 8 in the dimension is type A, which decreases the power of 3 by two. If a reduction increases the power of 3, then the power of 8 is reduced by the same amount. Therefore to obtain S, of dimension 8^n, by standard reductions, we must have $a \ge 2(n-b)$. But then $a = 2(n-b)$ and $\dim Y = 9^{n-b} 8^b$, and there must be $n - b$ reductions of type A. If $b \ne n$ some Y_i has dimension 9, and then so must Y_{i-1} and Y_{i+1} and hence

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all the Y_i's , and so $Y \simeq V_N \otimes V_N^*$. Otherwise all Y_i have dimension 8 and $Y \simeq S$. //

DEFINITION 3.3. Let I, J and K be subsets of N . The ordered triple (I, J, K) is a *trio* if

(i) $\{I, J, K\}$ is a partition of N ,

(ii) $|K|$ is even,

(iii) if $i, j \in I \cup J$ then both belong to I or both belong to J if and only if $|\{k \in K \mid i < k < j\}|$ is even.

Note that (ii) ensures the consistency of (iii) under the obvious circular ordering of N . Also given K and $i \in N\backslash K$, i may be assigned either to I or to J and then I and J are determined by (iii), so each $K \neq N$ with $|K|$ even determines a unique pair of trios (I, J, K) and (J, I, K) . $(\emptyset, \emptyset, N)$ is a trio if and only if n is even.

LEMMA 3.4. *Let U and $V(I, J, K)$ be irreducibles such that $U \otimes V(I, J, K)$ has a cross-section Y isomorphic to S or to $V_N \otimes V_N^*$.*

(i) If $I \cup J \cup K \neq N$ then $Y \simeq S$ and $U \simeq V(J, I, L)$ where $I \cup J \cup K \cup L = N$.

(ii) Let $I \cup J \cup K = N$. If $Y \simeq S$ then $U \simeq V(J, I, L)$ where $L \subseteq K$; if $Y \simeq V_N \otimes V_N^*$ then $U \simeq V(X, Y, I \cup J \cup Z)$ where $\{X, Y, Z\}$ is a partition of K and $(I \cup X, J \cup Y, Z)$ is a trio.

Proof. Let the ith tensor factor of $V(I, J, K)$ be X_i .

In case *(i)* we have $Y_i \simeq U_i$ for some $i \notin I \cup J \cup K$ and so for this i , $Y_i \simeq W_i$ and consequently $Y \simeq S$. Then

$$U_i \otimes X_i \simeq \text{one of } V_i \otimes V_i^*,\ W_i \text{ or } W_i \otimes W_i$$

for each i , so U has the form claimed.

In case *(ii)* first suppose $Y \simeq S$. If $Y_i \simeq W_i$ for all i , then $U \simeq V(J, I, L)$ with $L \subseteq K$. Otherwise $Y_i \simeq W_{i+1}$ for all i , and then both irreducibles must be isomorphic to S .

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Now suppose $Y \simeq V_N \otimes V_N^*$. Each Y_i is isomorphic to $V_i \otimes V_{i+1}$ or $V_i^* \otimes V_{i+1}$ or the dual of one of these. Define

$$E = \{i \in N \mid Y_i \simeq V_i^* \otimes V_{i+1}\} ,$$

$$F = \{i \in N \mid Y_i \simeq V_i \otimes V_{i+1}^*\} ,$$

and

$$H = \{i \in N \mid Y_i \simeq V_i \otimes V_{i+1} \text{ or } V_i^* \otimes V_{i+1}^*\} .$$

Then (E, F, H) is a trio. Also

$$i \in E \text{ implies } U_i \otimes X_i \simeq V_i \otimes W_i ,$$

$$i \in F \text{ implies } U_i \otimes X_i \simeq V_i^* \otimes W_i ,$$

$$i \in H \text{ implies } U_i \otimes X_i \simeq W_i \otimes W_i ,$$

by Lemmas 2.2 and 2.4. Thus U_i is never trivial and $U_i \simeq W_i$ for $i \in I \cup J$.

Define a partition $\{X, Y, Z\}$ of K by

$$U_i \simeq \begin{cases} V_i & \text{for } i \in X , \\ V_i^* & \text{for } i \in Y , \\ W_i & \text{for } i \in Z . \end{cases}$$

Then $U \simeq V(X, Y, I \cup J \cup Z)$. Also

$$E = I \cup X , \quad F = J \cup Y , \quad H = Z .$$

So $(I \cup X, J \cup Y, Z)$ is a trio, and the proof is complete. //

PROPOSITION 3.5. *Let U and $V(I, J, K)$ be irreducibles and suppose the dimension*

$$(S, U \otimes V(I, J, K)) \neq 0 .$$

(i) If $I \cup J \cup K \neq N$ then $U \simeq V(J, I, L)$ where $I \cup J \cup K \cup L = N$ and the dimension is $2^{|K \cap L|}$.

(ii) If $I \cup J \cup K = N$ then either $U \simeq V(J, I, L)$ for some $L \subseteq K$ and the dimension is $2^{|L|}$, or $U \simeq V(X, Y, I \cup J \cup Z)$ where $\{X, Y, Z\}$

is a partition of K *and* $(I \cup X, J \cup Y, Z)$ *is a trio, and the dimension is* $2^{|Z|}$ *, except that*

$$(S, S \otimes S) = \begin{cases} 2^n + 1 & \text{if } n \text{ is odd,} \\ 3.2^n + 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof. This dimension is the number of times that S appears as a composition factor of $U \otimes V(I, J, K)$. Since each composition factor occurs in some cross-section, by Lemma 3.2 this is the number of cross-sections Y isomorphic to S or to $V_N \otimes V_N^*$. By Lemma 3.4, U has one of the stated forms.

In case *(i)*, $Y \simeq S$ by Lemma 3.4 *(i)*. There is a unique choice for each Y_i except for $i \in K \cap L$ when there are two, since $W_i \otimes W_i$ has two composition factors W_i and so the dimension is $2^{|K \cap L|}$.

In case *(ii)* first note that by Lemma 3.4 *(ii)* it is impossible to both choose $Y \simeq S$ and $Y \simeq V_N \otimes V_N^*$ except when $I \cup J = \emptyset$, both irreducibles are S , and $(\emptyset, \emptyset, N)$ is a trio so n is even. There are 2^n cross-sections Y of $S \otimes S$ with each $Y_i \simeq W_i$, and one with each $Y_i \simeq W_{i+1}$. When n is even there are also 2.2^n with each $Y_i \simeq V_i \otimes V_{i+1}$ or $V_i^* \otimes V_{i+1}^*$. This gives the stated dimensions for $(S, S \otimes S)$.

Otherwise at least one of U and $V(I, J, K)$ is not isomorphic to S . If $U \simeq V(J, I, L)$ with $L \subseteq K$ then (by Lemma 3.4 *(ii)*) Y_i must be chosen isomorphic to W_i for all i . For $i \in L$ there are 2 choices for Y_i (as a composition factor of $U_i \otimes W_i \simeq W_i \otimes W_i$), and otherwise only one choice. This gives $2^{|L|}$ choices for Y .

If $U \simeq V(X, Y, I \cup J \cup Z)$ then Y must be chosen isomorphic to $V_N \otimes V_N^*$. For $i \in X, Y, I$ or J there is a unique choice for Y_i , isomorphic to one of $V_i^* \otimes V_{i+1}$ and $V_i \otimes V_{i+1}^*$. For $i \in Z$, Y must be chosen compatibly as $V_i \otimes V_{i+1}$ or $V_i^* \otimes V_{i+1}^*$ from $W_i \otimes W_i$. This gives

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$2^{|Z|}$ choices for Y. The proof is complete. //

We can now decompose the modules $S \otimes U$. For a subset K of N let $\overline{K}$ be the complement of K in N. Then from Lemma 2.4,

$$(3.6) \qquad S \otimes W_L = \sum_{K \cup L = N}^{\oplus} 2^{|K \cap L|} W_K \otimes A_{\overline{K}}$$

where $A_R = \prod_{r \in R}^{\otimes} A_r$. On the other hand from Lemmas 1.1 and 1.2 and Proposition 3.5, for $L \neq N$ we have

$$(3.7) \qquad S \otimes W_L = \sum_{K \cup L = N}^{\oplus} 2^{|K \cap L|} P(W_K) .$$

So, for $K \neq \emptyset$,

$$(3.8) \qquad P(W_K) = W_K \otimes A_{\overline{K}} .$$

NOTATION 3.9. Let B_R denote $V_R \otimes W_R$, for $R \subseteq N$.

PROPOSITION 3.10. *Let I, J and K be pairwise disjoint subsets of N with $K \neq \emptyset$. Let $L = \overline{I \cup J \cup K}$. Then*

$$P(I, J, K) = B_I \otimes B_J^* \otimes W_K \otimes A_L .$$

Proof. Since

$$B_I \otimes B_J^* \otimes W_K \otimes A_L = V_I \otimes V_J^* \otimes P(W_{I \cup J \cup K})$$

it is projective. We show that $V(I, J, K)$ is the only top factor. Now because $W_{I \cup J \cup K} \otimes A_L$ is a direct summand of $S \otimes W_L$ it follows that $B_I \otimes B_J^* \otimes W_K \otimes A_L$ is a direct summand of $S \otimes V(I, J, L)$. Consider, for an irreducible U,

$$(S \otimes V(I, J, L), U) = (S, V(J, I, L) \otimes U) .$$

Since $K \neq \emptyset$, $I \cup J \cup L \neq N$ so by Proposition 3.5 if this space is non-zero then $U \simeq V(I, J, K \cup Z)$ with $Z \subseteq L$. Hence these U's are the only candidates for top factors of $B_I \otimes B_J^* \otimes W_K \otimes A_L$. However

$$(B_I \otimes B_J \otimes W_K \otimes A_L, V(I, J, K \cup Z)) = (W_{\overline{L}} \otimes A_L, V_I^* \otimes V_J \otimes V(I, J, K \cup Z)) ,$$

and

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$$V_I^* \otimes V_J \otimes V(I,\ J,\ K \cup Z) \simeq \sum_{\substack{X \subseteq I \\ Y \subseteq J}}^{\oplus} W_X \otimes W_Y \otimes W_{K \cup Z}$$

which is a direct sum of irreducibles. Since, by (3.8),

$$W_{\bar{L}} \otimes A_L = P\left(W_{\bar{L}}\right) = P\left(W_{I \cup J \cup K}\right)$$

the dimension is non-zero only if $Z = \emptyset$ and then it is one. This proves the proposition. //

An expression for $P(1)$ comes from decomposing $S \otimes S$. Let T be the set of trios *not* including $(\emptyset,\ \emptyset,\ N)$. Using $\subset$ to mean proper inclusion, from Lemma 1.1 and Proposition 3.5 we have

$$S \otimes S = \sum_{L \subset N} 2^{|L|} P\left(W_L\right) \oplus \sum_{(I,J,K) \in T} 2^{|K|} P(I,\ J,\ K) \oplus m'S$$

where

$$m' = \begin{cases} 2^n + 1 & \text{if } n \text{ is odd, and} \\ 3.2^n + 1 & \text{if } n \text{ is even.} \end{cases}$$

From (3.6) on the other hand

$$\begin{aligned} S \otimes S &= \sum_{K \subseteq N}^{\oplus} 2^{|K|} W_K \otimes A_{\bar{K}} \\ &= A_N \oplus \sum_{\emptyset \neq K \subset N}^{\oplus} 2^{|K|} W_K \otimes A_{\bar{K}} \oplus 2^n S . \end{aligned}$$

Using (3.8) then we have

$$A_N = P(1) \oplus \sum_{(I,J,K) \in T}^{\oplus} 2^{|K|} P(I,\ J,\ K) \oplus \left(m'-2^n\right) S .$$

Recall that $(\emptyset,\ \emptyset,\ N)$ is a trio if and only if n is even. Hence we may rewrite this as

$$(3.11) \qquad P(1) = A_N - \sum^{\oplus} 2^{|K|} P(I,\ J,\ K) - mS ,$$

where

$$m = \begin{cases} 1 & \text{if } n \text{ is odd, and} \\ 2^n + 1 & \text{if } n \text{ is even,} \end{cases}$$

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and the sum is now over all trios (I, J, K) .

PROPOSITION 3.12. *If* $I \cup J \neq \emptyset$ *and* $K = \overline{I \cup J}$ *then*

$$B_I \otimes B_J^* \otimes A_K = P(I, J, \emptyset) \oplus \sum_Z^{\oplus} 2^{|Z|} P(X, Y, I \cup J \cup Z)$$

where the sum is over partitions $\{X, Y, Z\}$ *of* K *such that* $(I \cup X, J \cup Y, Z)$ *is a trio.*

Proof. Since

$$\left(B_I \otimes B_J^* \otimes W_K, U\right) = \left(S, V(J, I, K) \otimes U\right) ,$$

using Proposition 3.5 and Lemma 1.1 as before in the proof of Proposition 3.10, we have

$$B_I \otimes B_J^* \otimes W_K \otimes W_K = \sum_{L \subseteq K}^{\oplus} 2^{|L|} P(I, J, L) \oplus \sum_Z^{\oplus} 2^{|Z|} P(X, Y, I \cup J \cup Z)$$

where the second sum is over the partitions stated. On the other hand from Lemma 2.5 we have

$$B_I \otimes B_J^* \otimes W_K \otimes W_K = \sum_{L \subseteq K}^{\oplus} 2^{|L|} B_I \otimes B_J^* \otimes W_L \otimes A_{K-L} .$$

Since by Proposition 3.10,

$$B_I \otimes B_J^* \otimes W_L \otimes A_{K-L} = P(I, J, L)$$

for $L \neq \emptyset$, the result follows. //

In stating Theorem 1 we use the notation that $A = B - C$ means $B \simeq A \oplus C$.

THEOREM 1. *For an irreducible* $V(I, J, K)$ *let* $L = \overline{I \cup J \cup K}$. *Let* A_R, B_R *be the modules defined in* (3.6) *and Notation* 3.9 *respectively.*

(i) If $K \neq \emptyset$ *then*

$$P(I, J, K) = B_I \otimes B_J^* \otimes W_K \otimes A_L .$$

(ii) If $K = \emptyset$, $I \cup J \neq \emptyset$ *then*

$$P(I, J, \emptyset) = B_I \otimes B_J^* \otimes A_L - \sum_Z^{\oplus} 2^{|Z|} B_X \otimes B_Y^* \otimes W_{I \cup J \cup Z} ,$$

where the sum runs over partitions $\{X, Y, Z\}$ *of* L *such that*

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$(I \cup X, J \cup Y, Z)$ is a trio.

(iii) $P(1) = A_N - \sum^{\oplus} 2^{|Z|} B_X \otimes B_Y^* \otimes W_Z - dS$, *where*

$$d = \begin{cases} -1 & \text{if } n \text{ is odd, and} \\ 2^n - 1 & \text{if } n \text{ is even,} \end{cases}$$

and the sum runs over all trios (X, Y, Z) .

Proof. *(i)* and *(ii)* are from Propositions 3.10 and 3.12. *(iii)* follows from (3.11) and Proposition 3.10, with the observation that by *(ii)*, $P(N, \emptyset, \emptyset) = B_N - S$, and similarly for its dual. //

To give the dimensions of the principal indecomposables we need the following notion.

DEFINITION. Let I and J be disjoint subsets of N and let the elements of $I \cup J$ be $i_1 < i_2 < \ldots < i_r$. Let

$$k_j = |\{k \in N \mid i_j < k < i_{j+1}\}|$$

under the circular ordering of N , and for $j = 1, \ldots, r$ let

$$e_j = \begin{cases} 1 & \text{if both } i_j \text{ and } i_{j+1} \text{ belong to } I \text{ or both belong to } J \text{ , and} \\ -1 & \text{otherwise} \end{cases}$$

for $j = 1, \ldots, r$. Then the *type* of $\{I, J\}$ is the sequence $(e_1, k_1, e_2, k_2, \ldots, e_r, k_r)$.

THEOREM 2. *(i) If $K \neq \emptyset$ then*

$$\dim P(I, J, K) = 8^n . 3^{n-|K|} . 2^{n-|I \cup J \cup K|} .$$

(ii) If $I \cup J \neq \emptyset$ let the type of $\{I, J\}$ be $(e_1, k_1, \ldots, e_r, k_r)$. Then

$$\dim P(I, J, \emptyset) = 8^n \left(3^r . 6^{n-r} - (1/2^r) \prod_{j=1}^{r} \left(5^{k_j} + e_j \right) \right) .$$

(iii) $\dim P(1) = 8^n (6^n - 5^n)$.

Proof. *(i)* is immediate from Theorem 1 *(i)*. For *(ii)* let the elements of $I \cup J$ be $i_1 < \ldots < i_r$ and let

$$K_j = \{k \in N \mid i_j < k < i_{j+1}\}$$

so that $k_j = |K_j|$. From Theorem 1 *(ii)* we have to sum the dimensions of the summands $B_X \otimes B_Y^* \otimes W_{I \cup J \cup Z}$ of $B_I \otimes B_J^* \otimes A_L$. Suppose we have subsets $L_j \subseteq K_j$, $j = 1, \ldots, r$, such that if $Z = \cup L_j$ then there is a trio $(I \cup X, J \cup Y, Z)$. By the definition of a trio $|L_j|$ must be even if $e_j = 1$ and odd if $e_j = -1$, and provided these conditions are fulfilled a trio exists. Now let $l_j = |L_j|$. The corresponding principal indecomposable $B_X \otimes B_Y^* \otimes W_{I \cup J \cup Z}$ has dimension

$$8^n.3^{k_1-l_1}.3^{k_2-l_2} \ldots 3^{k_r-l_r}$$

and occurs $2^{|Z|} = 2^{l_1+l_2+\ldots+l_r}$ times, so the total dimension of these summands is

$$8^n.2^{l_1}.3^{k_1-l_1} \ldots 2^{l_r}.3^{k_r-l_r} .$$

The sum of the dimensions of all the summands corresponding to trios $(I \cup X, J \cup Y, Z)$ is

$$8^n\left(\sum_{l_1} \binom{k_1}{l_1} 2^{l_1}.3^{k_1-l_1}\right) \ldots \left(\sum_{l_r} \binom{k_r}{l_r} 2^{l_r}.3^{k_r-l_r}\right) ,$$

where the sum is over l_j odd or l_j even according as e_j is -1 or 1 . By the binomial theorem this is

$$8^n\left(\left(5^{k_1}+e_1\right)/2\right) \ldots \left(\left(5^{k_r}+e_r\right)/2\right) .$$

Since $|I \cup J| = r$,

$$\dim B_I \otimes B_J^* \otimes A_L = 8^n.3^r.6^{n-r}$$

giving the result.

For *(iii)* note that $\dim A_N = 8^n.6^n$ and

$$\dim 2^{|Z|}B_X \otimes B_Y^* \otimes W_Z = 2^{|Z|}.3^{n-|Z|}.8^n .$$

For each $Z \subset N$ with $|Z|$ even there are two trios so

$$\begin{aligned}\dim P(1) &= 8^n\left(6^n-2 . \sum_{\substack{Z\subset N\\|Z|\text{ even}}} 2^{|Z|}.3^{n-|Z|}-d\right)\\ &= 8^n\left(6^n+1-2 . \sum_{\substack{k=0\\k\text{ even}}}^{n} \binom{n}{k}2^k3^{n-k}\right)\\ &= 8^n\left(6^n+1-2.\left(\left(5^n+1\right)/2\right)\right)\\ &= 8^n\left(6^n-5^n\right) . \qquad //\end{aligned}$$

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