Continuum of allosteric actions for non-amenable surface groups

MATTHIEU JOSEPH

*Université Paris-Saclay, CNRS, Laboratoire de Mathématiques d'Orsay,
Orsay 91405, France
(e-mail: matthieu.joseph@universite-paris-saclay.fr)*

Abstract. Let Σ be a closed surface other than the sphere, the torus, the projective plane or the Klein bottle. We construct a continuum of probability measure preserving ergodic minimal profinite actions for the fundamental group of Σ that are topologically free but not essentially free, a property that we call allostery. Moreover, the invariant random subgroups we obtain are pairwise distincts.

Key words: measure-preserving actions, minimal actions, invariant random subgroups, uniformly recurrent subgroups, space of subgroups
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1. Introduction

Let Γ be a countable discrete group. Let α be a minimal action of Γ on a compact Hausdorff space C. The action α is *topologically free* if for every non-trivial element $\gamma \in \Gamma$, the set $\{x \in C \mid \alpha(\gamma)x = x\}$ has empty interior. This notion of freeness can be characterized by the triviality of the uniformly recurrent subgroup (hereafter URS) associated with the action α as follows. Let $\mathrm{Sub}(\Gamma)$ be the space of subgroups of Γ and let $\mathrm{Stab}_\alpha : C \rightarrow \mathrm{Sub}(\Gamma)$ be the Borel map defined by

$$\mathrm{Stab}_\alpha(x) := \{\gamma \in \Gamma \mid \alpha(\gamma)x = x\}.$$

Here $\mathrm{Sub}(\Gamma)$ is equipped with the topology of pointwise convergence which turns it into a compact totally disconnected topological space on which Γ acts continuously by conjugation. Glasner and Weiss proved in [**GW15**] that there exists a unique closed, Γ-invariant, minimal subset in the closure of $\{\mathrm{Stab}_\alpha(x) \mid x \in C\}$, called the *stabilizer URS*, associated with the minimal action α, which we denote by $\mathrm{URS}(\alpha)$. The stabilizer URS is trivial if it is equal to $\{\{1\}\}$. One of the features of the stabilizer URS associated with

a minimal action α is that its triviality is equivalent to the topological freeness of α, see Lemma 2.1.

Let (X, μ) be a standard probability measure space and let β be a probability measure preserving (hereafter p.m.p.) action of a countable group Γ on (X, μ). The action β is *essentially free* if for every non-trivial $\gamma \in \Gamma$, the set $\{x \in X \mid \alpha(\gamma)x = x\}$ is μ-negligible. The measurable counterpart of the stabilizer URS is the *stabilizer invariant random subgroup*, stabilizer IRS for short, associated with β. It is defined as the Γ-invariant Borel probability measure $(\mathrm{Stab}_\beta)_*\mu$ on $\mathrm{Sub}(\Gamma)$, and is denoted by $\mathrm{IRS}(\beta)$. A stabilizer IRS is the prototype of an *IRS*, which is a Borel probability measure on $\mathrm{Sub}(\Gamma)$ that is invariant under the conjugation action of Γ. The *trivial IRS* is the Dirac measure at the trivial subgroup. Observe that $\mathrm{IRS}(\beta)$ is trivial if and only if β is essentially free. Abért, Glasner and Virág proved that every IRS is in fact a stabilizer IRS for some p.m.p. action, see [AGV14].

An *ergodic minimal action* $\Gamma \curvearrowright (C, \mu)$ is a minimal action of Γ on a compact Hausdorff space C together with a Γ-invariant ergodic Borel probability measure μ. Thus, an ergodic minimal action has both a stabilizer URS and a stabilizer IRS. It is a classical result that the essential freeness of an ergodic minimal action implies its topological freeness, see Lemma 2.2. In other words, if the stabilizer IRS of an ergodic minimal action is trivial, then its stabilizer URS is trivial. The present article provides new counterexamples in the study of the converse.

Definition 1.1. An ergodic minimal action is *allosteric* (ἄλλος: other, στερεός: fix, firm, solid, rigid) if it is topologically free but not essentially free. A group is allosteric if it admits an allosteric action.

Question 1.1. What is the class of allosteric groups?

First, let us discuss examples of groups that do not belong to this class. It is the case for groups whose ergodic IRSs are all atomic, that is, equal to the uniform measure on the set of conjugates of a subgroup which admits only finitely many conjugates. Indeed, we prove in Proposition 2.3 that the IRS of an ergodic minimal action which is topologically free is either trivial, or has no atoms. Thus, if $\mathrm{Sub}(\Gamma)$ is countable, then Γ is not allosteric, see Corollary 2.4. Examples of groups with only countably many subgroups are finitely generated nilpotent groups, more generally polycyclic groups, extensions of Noetherian groups by groups with only countably many subgroups (e.g. solvable Baumslag-Solitar groups $\mathrm{BS}(1, n)$), see [BLT19] or Tarski monsters.

There are also groups whose ergodic IRSs are all atomic for other reasons. For instance, this is the case for lattices in simple higher rank Lie groups [SZ94], commutator subgroups of either a Higman–Thompson group or the full group of an irreducible shift of finite type [DM14], and a projective special linear group $\mathrm{PSL}_n(k)$ over an infinite countable field k [PT16]. See also [Cre17], [CP17] or [Bek20] for other examples of groups with few ergodic IRSs. Thus, none of these groups are allosteric, because of their lack of IRSs.

More surprisingly, there exists non-allosteric groups with plenty of ergodic IRSs, such as countable abelian groups which admit uncountably many subgroups. Indeed, if Γ is such a group, then any Borel probability measure on $\mathrm{Sub}(\Gamma)$ is an IRS, but Γ is not allosteric

since any minimal Γ-action which is topologically free is actually essentially free for any invariant measure, see Remark 4.4. Another example is given by the group FSym($\mathbb{N}$) of finitely supported permutations on $\mathbb{N}$, as well as its alternating subgroup Alt($\mathbb{N}$). They both admit many ergodic IRSs, see [**TTD18**, **Ver12**]. However, an argument similar to that of [**TTD18**, Lemma 10.4] implies that neither FSym($\mathbb{N}$) nor Alt($\mathbb{N}$) is allosteric.

Let us now discuss examples of allosteric groups. Bergeron and Gaboriau proved in [**BG04**] that if Γ is non-amenable and isomorphic to a free product of two non-trivial residually finite groups, then Γ is allosteric. We refer to Remark 2.12 for a more precise statement of their results. In [**AE07**], Abért and Elek independently proved that finitely generated non-abelian free groups are allosteric, and in [**AE12**], they proved that the free product of four copies of $\mathbb{Z}/2\mathbb{Z}$ admits an allosteric action whose orbit equivalence relation is measure hyperfinite. In all [**AE07**, **AE12**, **BG04**], the allosteric actions obtained are in fact profinite, see §2.2 for a definition. These were the first known examples answering a question of Grigorchuk, Nekrashevich and Sushchanskii in [**GNS00**, Problem 7.3.3] about the existence of profinite allosteric actions.

The main result of this article is to prove that non-amenable *surface groups*, that is, fundamental groups of closed surfaces other than the sphere, the torus, the projective plane or the Klein bottle, are allosteric. More precisely, we prove the following result.

THEOREM 1.2. *Any non-amenable surface group admits a continuum of profinite allosteric actions that are pairwise topologically and measurably non-isomorphic.*

Moreover, we prove that the IRSs given by the non-isomorphic allosteric actions that we construct are pairwise distinct. We refer to Theorems 4.1 and 4.2 for a precise statement of our results. Let us mention that surface groups are known to have a large 'zoo' of IRSs. For instance, Bowen, Grigorchuk and Kravchenko proved in [**BGK17**] that any non-elementary Gromov hyperbolic group admits a continuum of IRSs which are weakly mixing when considered as dynamical systems on Sub(Γ). In an upcoming work (personal communication), Carderi, Le Maître and Gaboriau prove that non-amenable surface groups admit a continuum of IRSs whose support coincides with the *perfect kernel* of Γ, that is, the largest closed subset without isolated points in Sub(Γ). However, our IRSs are drastically different from the latter ones: we show that they are not weakly mixing and that their support is strictly smaller than the perfect kernel, see Remarks 4.4 and 4.5.

We develop in §2 the preliminary results needed about profinite actions and allosteric actions. In particular, we prove that allostery is invariant under commensurability. To build ergodic profinite allosteric actions of non-amenable surface groups, we rely on a residual property of non-amenable surface groups to prove in §3 that they admit special kinds of finite index subgroups. The proof of Theorem 1.2 is completed in §4.

2. *Preliminaries*

2.1. *Topological dynamic and URS/IRS.* Let C be a compact Hausdorff space and let α be an action by homeomorphisms of a countable discrete group Γ on C. The action α is *minimal* if the orbit of every $x \in C$ is dense. Recall that α is topologically free if for every non-trivial element $\gamma \in \Gamma$, the closed set

$$\text{Fix}_\alpha(\gamma) := \{x \in C \mid \alpha(\gamma)x = x\}$$

has empty interior. Since C is a Baire space, this is equivalent to saying that the set $\{x \in C \mid \text{Stab}_\alpha(x) \neq \{1\}\}$ is meagre, that is, a countable union of nowhere dense sets.

The set $\text{Sub}(\Gamma)$ of subgroups of Γ naturally identifies with a subset of $\{0, 1\}^\Gamma$. It is closed for the product topology. Thus, the induced topology on $\text{Sub}(\Gamma)$ turns it into a compact totally disconnected space, on which Γ acts continuously by conjugation. A *URS* of Γ is a closed minimal Γ-invariant subset of $\text{Sub}(\Gamma)$. The *trivial URS* is the URS that only contains the trivial subgroup. Recall that the stabilizer URS of a minimal action α of Γ on C is the unique closed, Γ-invariant minimal subset in the closure of $\{\text{Stab}_\alpha(x) \mid x \in C\}$. It is straightforward to check that the map Stab_α is upper-semi continuous. Therefore, it is continuous on a G_δ dense set. If $C_0 \subset C$ denotes the locus of continuity of $\text{Stab}_\alpha : C \to \text{Sub}(\Gamma)$, then one can prove that $\text{URS}(\alpha)$ is equal to the closure of the set $\{\text{Stab}_\alpha(x) \mid x \in C_0\}$, see [**GW15**].

A proof of the following classical result can be found in [**LBMB18**, Proposition 2.7].

LEMMA 2.1. *Let α be a minimal Γ-action on a compact Hausdorff space C. Then α is topologically free if and only if its stabilizer URS is trivial, if and only if there exists $x \in C$ such that $\text{Stab}_\alpha(x)$ is trivial.*

The following lemma clarifies the relation between the stabilizer URS and the stabilizer IRS. Recall that the support of a Borel probability measure is the intersection of all closed subsets of full measure.

LEMMA 2.2. *Let α be a minimal Γ-action on a compact Hausdorff space C and μ be a Γ-invariant Borel probability measure on C. Then $\text{URS}(\alpha)$ is contained in the support of $\text{IRS}(\alpha)$. In particular, if $\text{IRS}(\alpha)$ is trivial, then $\text{URS}(\alpha)$ is trivial.*

Proof. Let F be a closed subset of $\text{Sub}(\Gamma)$ such that $\mu(\text{Stab}_\alpha^{-1}(F)) = 1$. By minimality of α, every non-empty open subset U of C satisfies $\mu(U) > 0$. Thus, $\text{Stab}_\alpha^{-1}(F)$ is dense in C. Let $x \in C$ be a continuity point of Stab_α. Let $(x_n)_{n \geq 0}$ be a sequence of elements in $\text{Stab}_\alpha^{-1}(F)$ that converges to x. Then $\text{Stab}_\alpha(x) \in F$ and we thus obtain that $\text{URS}(\alpha) \subset F$. By definition of the support of $\text{IRS}(\alpha)$, this implies that $\text{URS}(\alpha) \subset \text{supp}(\text{IRS}(\alpha))$. $\square$

The following proposition gives a partial converse to Lemma 2.2.

PROPOSITION 2.3. *Let α be a minimal Γ-action on a compact Hausdorff space C and μ be a Γ-invariant Borel probability measure on C. If $\text{URS}(\alpha)$ is trivial, then $\text{IRS}(\alpha)$ is either trivial or atomless.*

Proof. Assume that $\text{IRS}(\alpha)$ has a non-trivial atom $\{\Lambda\}$. By invariance, the atoms $\{\gamma \Lambda \gamma^{-1}\}$ have equal measure for all $\gamma \in \Gamma$. Thus, Λ has only finitely many conjugates. Thus, the closure in $\text{Sub}(\Gamma)$ of the set $\{\text{Stab}_\alpha(x) \mid x \in C\}$ contains the finite set $\{\gamma \Lambda \gamma^{-1} \mid \gamma \in \Gamma\}$, which is closed, Γ-invariant and minimal. Thus, $\text{URS}(\alpha)$ is non-trivial. $\square$

This last result implies that the converse of Lemma 2.2 is actually true for groups admitting only countably many subgroups.

COROLLARY 2.4. *Let α be a minimal Γ-action on a compact Hausdorff space C and μ a Γ-invariant Borel probability measure on C. If $\mathrm{Sub}(\Gamma)$ is countable, then $\mathrm{IRS}(\alpha)$ is trivial if and only if $\mathrm{URS}(\alpha)$ is trivial.*

Thus, groups Γ such that $\mathrm{Sub}(\Gamma)$ is countable are not allosteric.

2.2. *Profinite actions and their URS/IRS.* Let Γ be a countable group. For every $n \geq 0$, let α_n be a Γ-action on a finite set X_n, and assume that for every $n \geq 0$, α_n is a quotient of α_{n+1}, that is, there exists a Γ-equivariant onto map $q_n : X_{n+1} \twoheadrightarrow X_n$. The inverse limit of the finite spaces X_n is the space

$$\varprojlim X_n := \left\{ (x_n)_{n \geq 0} \in \prod_{n \geq 0} X_n \mid \text{for all } n \geq 0, q_n(x_{n+1}) = x_n \right\}.$$

This space is closed, thus compact, and totally disconnected in the product topology. Let α be the Γ-action by homeomorphisms on $\varprojlim X_n$ defined by

$$\alpha(\gamma)(x_n)_{n \geq 0} := (\alpha_n(\gamma)x_n)_{n \geq 0}.$$

If each X_n is endowed with a Γ-invariant probability measure μ_n, we let μ be the unique Borel probability measure on $\varprojlim X_n$ that projects onto μ_k via the canonical projection $\pi_k : \varprojlim X_n \to X_k$, for every $k \geq 0$. The Γ-action α preserves μ and is called the *inverse limit* of the p.m.p. Γ-actions α_n. A p.m.p. action of Γ is *profinite* if it is measurably isomorphic to an inverse limit of p.m.p. Γ-actions on finite sets. A proof of the following Lemma can be found in [**Gri11**, Proposition 4.1].

LEMMA 2.5. *Let $\Gamma \curvearrowright^\alpha \varprojlim(X_n, \mu_n)$ be the inverse limit of the p.m.p. finite actions $\Gamma \curvearrowright^{\alpha_n} (X_n, \mu_n)$ and let μ denotes the inverse limit of the μ_n. Then the following are equivalent.*
(1) *For every $n \geq 0$, α_n is transitive and μ_n is the uniform measure on X_n.*
(2) *The action α is minimal.*
(3) *The action α is μ-ergodic.*
(3) *The action α is uniquely ergodic, that is, μ is the unique Γ-invariant Borel probability measure on $\varprojlim X_n$.*

With the above notation, the following lemma is useful to compute the measure of a closed subset in an inverse limit (here, no group action is involved).

LEMMA 2.6. *Let A be a closed subset of $\varprojlim X_n$. Then, $A = \bigcap_{n \geq 0} \pi_n^{-1}(\pi_n(A))$. Thus,*

$$\mu(A) = \lim_{n \to +\infty} \mu_n(\pi_n(A)).$$

Proof. First, A is contained in $\bigcap_{n \geq 0} \pi_n^{-1}(\pi_n(A))$ since it is contained in each $\pi_n^{-1}(\pi_n(A))$. Conversely, let x be in $\bigcap_{n \geq 0} \pi_n^{-1}(\pi_n(A))$. For every $n \geq 0$, there exists $y_n \in A$ such that $\pi_n(x) = \pi_n(y_n)$. By compactness of A, let $y \in A$ be a limit of some subsequence of $(y_n)_{n \geq 0}$. By definition of the product topology, for every $n \geq 0$, $\pi_n(x) = \pi_n(y)$, thus $x = y$ and x belongs to A. $\square$

Let $(\Gamma_n)_{n\geq 0}$ be a *chain* in Γ, that is, an infinite decreasing sequence $\Gamma = \Gamma_0 \geq \Gamma_1 \geq \ldots$ of finite index subgroups. If $X_n = \Gamma/\Gamma_n$ and μ_n is the uniform probability measure on X_n, then we get a profinite action that is ergodic by Lemma 2.5. Conversely, any ergodic (equivalently minimal) profinite Γ-action $\Gamma \curvearrowright \varprojlim X_n$ is measurably isomorphic to a profinite action of the form $\Gamma \curvearrowright \varprojlim \Gamma/\Gamma_n$ for some chain $(\Gamma_n)_{n\geq 0}$, by fixing a point $x \in \varprojlim X_n$ and letting Γ_n be the stabilizer of $\pi_n(x) \in X_n$.

LEMMA 2.7. *Let $(\Gamma_n)_{n\geq 0}$ be a chain in Γ and let α be the corresponding ergodic profinite Γ-action. Then $\mathrm{URS}(\alpha)$ is trivial if and only if there exists $(\gamma_n \Gamma_n)_{n\geq 0} \in \varprojlim \Gamma/\Gamma_n$ such that*

$$\bigcap_{n\geq 0} \gamma_n \Gamma_n \gamma_n^{-1} = \{1\}.$$

Proof. For all $x \in \varprojlim \Gamma/\Gamma_n$, if $x = (\gamma_n \Gamma_n)_{n\geq 0}$, then

$$\mathrm{Stab}_\alpha(x) = \bigcap_{n\geq 0} \gamma_n \Gamma_n \gamma_n^{-1}.$$

Thus, the result is a direct consequence of Lemma 2.1. $\square$

PROPOSITION 2.8. *Let $(\Gamma_n)_{n\geq 0}$ be a chain in Γ and let α be the corresponding ergodic profinite Γ-action. If $\mathrm{URS}(\alpha)$ is trivial, then either $\mathrm{IRS}(\alpha)$ is trivial or there exists a finite index subgroup $\Lambda \leq \Gamma$ such that the p.m.p. Λ-action by conjugation on $(\mathrm{Sub}(\Gamma), \mathrm{IRS}(\alpha))$ is not ergodic.*

Proof. Assume that the p.m.p. Γ-action by conjugation on $(\mathrm{Sub}(\Gamma), \mathrm{IRS}(\alpha))$ remains ergodic under any finite index subgroup of Γ. Since $\mathrm{URS}(\alpha)$ is trivial, there exists by Lemma 2.7 a sequence $(\gamma_n)_{n\geq 0}$ of elements in Γ such that

$$\bigcap_{n\geq 0} \gamma_n \Gamma_n \gamma_n^{-1} = \{1\}.$$

For every $k \geq 0$, if $\pi_k : \varprojlim \Gamma/\Gamma_n \to \Gamma/\Gamma_k$ denotes the projection onto the kth coordinate, then the set

$$\{\mathrm{Stab}_\alpha(x) \mid x \in \varprojlim \Gamma/\Gamma_n, \pi_k(x) = \gamma_k \Gamma_k\} \subset \mathrm{Sub}(\Gamma)$$

has positive measure for $\mathrm{IRS}(\alpha)$, is contained in $\mathrm{Sub}(\gamma_k \Gamma_k \gamma_k^{-1})$ and is invariant under the finite index subgroup $\mathrm{Stab}_{\alpha_k}(\gamma_k \Gamma_k) = \gamma_k \Gamma_k \gamma_k^{-1}$. By ergodicity, it is a full measure set. Thus, for almost every (a.e.) $x \in \varprojlim \Gamma/\Gamma_n$, $\mathrm{Stab}_\alpha(x)$ is a subgroup of $\gamma_k \Gamma_k \gamma_k^{-1}$. Since this is true for every $k \geq 0$, we conclude that $\mathrm{IRS}(\alpha)$ is trivial. $\square$

2.3. *Allostery and commensurability.* Two groups Γ_1 and Γ_2 are *commensurable* if there exist finite index subgroups $\Lambda_1 \leq \Gamma_1$ and $\Lambda_2 \leq \Gamma_2$ such that Λ_1 is isomorphic to Λ_2. In this section, we prove the following result.

THEOREM 2.9. *Allostery is invariant under commensurability.*

We prove Theorem 2.9 in two steps, by showing that allostery is inherited by finite index overgroups in Proposition 2.10 and by finite index subgroups in Proposition 2.11. Let Γ be a countable group and $\Lambda \leq \Gamma$ a finite index subgroup. Let $\alpha : \Lambda \curvearrowright (C, \mu)$ be an action by homeomorphisms on a compact Hausdorff space C with a Λ-invariant Borel probability measure μ on C. Let $\beta : \Gamma \curvearrowright C \times \Gamma$ be the action defined by $\beta(\gamma)(x, \delta) = (x, \gamma\delta)$. Define an equivalence relation $\sim$ on a $C \times \Gamma$ by declaring that $(x, \gamma) \sim (x', \gamma')$ if and only if there exists $\lambda \in \Lambda$ such that $(x', \gamma') = (\alpha(\lambda)x, \gamma\lambda)$. The Γ-action β projects onto a Γ-action by homeomorphisms on $C \times \Gamma/\sim$ which is called the Γ-action *induced* by α. We denote it by $\mathrm{Ind}_\Lambda^\Gamma \alpha$. The product of μ with the counting measure on Γ projects onto a Borel probability measure on $C \times \Gamma/\sim$ which is invariant by $\mathrm{Ind}_\Lambda^\Gamma \alpha$.

PROPOSITION 2.10. *Let Γ be a countable group and $\Lambda \leq \Gamma$ a finite index subgroup. Then the Γ-action induced by any allosteric Λ-action is allosteric.*

Proof. Let $\alpha : \Lambda \curvearrowright (C, \mu)$ be an allosteric action. It is an exercise to prove that $\mathrm{Ind}_\Lambda^\Gamma \alpha$ is ergodic and minimal. Moreover, $\mathrm{IRS}(\mathrm{Ind}_\Lambda^\Gamma \alpha)$ is non-trivial since the restriction of $\mathrm{Ind}_\Lambda^\Gamma \alpha$ to Λ is not essentially free. Finally, $\mathrm{URS}(\alpha)$ is trivial, thus there exists by Lemma 2.1 a point $x \in C$ such that $\mathrm{Stab}_\alpha(x) = \{1\}$. Let y be the projection of $(x, 1)$ onto the quotient $(C \times \Gamma)/\sim$, then $\mathrm{Stab}_{\mathrm{Ind}_\Lambda^\Gamma \alpha}(y) = \{1\}$. Since $\mathrm{Ind}_\Lambda^\Gamma \alpha$ is minimal, this implies by Lemma 2.1 that $\mathrm{URS}(\mathrm{Ind}_\Lambda^\Gamma \alpha)$ is trivial. Thus, $\mathrm{Ind}_\Lambda^\Gamma \alpha$ is allosteric. $\qquad\square$

PROPOSITION 2.11. *Any finite index subgroup of an allosteric group is allosteric.*

Proof. Let $\Lambda \leq \Gamma$ be a finite index subgroup. We recall the following two facts. If $\Gamma \curvearrowright (X, \mu)$ is an ergodic action, then any Λ-invariant measurable set $A \subset X$ of positive measure satisfies $\mu(A) \geq 1/[\Gamma : \Lambda]$. Moreover, for any Λ-invariant measurable set $B \subset X$ of positive measure, there exists a Λ-invariant measurable set $A \subset B$ of positive measure on which Λ acts ergodically.

Let Γ be an allosteric group, and let $\Lambda \leq \Gamma$ be a finite index subgroup. Let N be the normal core of Λ (the intersection of the conjugates of Λ). It is a finite index normal subgroup of Γ which is contained in Λ. We will prove that N is allosteric. Proposition 2.10 will then imply that Λ is allosteric. We let $d = [\Gamma : N]$ and we fix $\gamma_1, \ldots, \gamma_d \in \Gamma$, a coset representative system for N in Γ. Let $\Gamma \curvearrowright^\alpha (C, \mu)$ be an allosteric action. For all $x \in C$, we define $\mathcal{O}_N(x) = \overline{\{\alpha(\gamma)x \mid \gamma \in N\}}$. This is a closed, N-invariant subset of C. By minimality of α, for all $x \in C$,

$$X = \bigcup_{i=1}^{d} \mathcal{O}_N(\alpha(\gamma_i)x).$$

Moreover, since N is normal in Γ, for all $x \in C$ and $\gamma \in \Gamma$, we have $\mathcal{O}_N(\alpha(\gamma)x) = \alpha(\gamma)\mathcal{O}_N(x)$. This implies that $\mu(\mathcal{O}_N(\alpha(\gamma)x)) = \mu(\mathcal{O}_N(x))$ and that $\mu(\mathcal{O}_N(x)) > 0$. Let y be a point in some closed, N-invariant and N-minimal set. Then $N \curvearrowright \mathcal{O}_N(y)$ is minimal. Let $A \subset \mathcal{O}_N(y)$ be an N-invariant measurable set of positive measure on which N acts ergodically. Let μ_A be the Borel probability measure on A induced by μ. Then $N \curvearrowright (\mathcal{O}_N(y), \mu_A)$ is an ergodic minimal action, which is still topologically free. Let us prove

that it is not essentially free. Since α is allosteric, $\mathrm{IRS}(\alpha)$ is atomless, see Proposition 2.3. Thus, for μ-a.e. $x \in C$, $\mathrm{Stab}_\alpha(x)$ is infinite. Since N has finite index in Γ, this implies that for μ-a.e. $x \in C$, $\mathrm{Stab}_\alpha(x) \cap N$ is infinite. Thus, $N \curvearrowright (\mathcal{O}_N(y), \mu_A)$ is not essentially free and thus is allosteric. $\qquad\square$

Remark 2.12. It is proved in [**BG04**, Théorème 4.1] that if Γ is isomorphic to a free product of two *infinite* residually finite groups, then Γ admits a continuum of profinite allosteric actions. Let Γ' be a non-amenable group which is isomorphic to a free product of two non-trivial residually finite groups. Then Kurosh's theorem [**Ser77**, §5.5] implies that Γ' admits a finite index subgroup Γ isomorphic to a free product of finitely many (and at least two) residually finite infinite groups. Proposition 2.10 then implies that Γ' is allosteric.

3. *Finite index subgroups of surface groups*

3.1. *Residual properties of surface groups.*
A surface group is the fundamental group of a closed connected surface. If the surface is orientable, then its fundamental group is called an *orientable surface group* and a presentation is given by

$$\langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1] \cdots [x_g, y_g] = 1 \rangle,$$

for some $g \geq 1$ called the genus of the surface (if $g = 0$, then the surface is a sphere and its fundamental group is trivial). If the surface is non-orientable, we call its fundamental group a *non-orientable surface group*. It has a presentation given by

$$\langle x_1, \ldots, x_g \mid x_1^2 \ldots x_g^2 = 1 \rangle,$$

for some $g \geq 1$, called the genus of the surface. A surface group is amenable if and only if it is the fundamental group of the sphere, the torus (orientable surfaces of genus 0 and 1), the projective plane or the Klein bottle (non-orientable surfaces of genus 1 and 2). These groups are respectively isomorphic to the trivial group for the sphere, $\mathbb{Z}^2$ for the torus, $\mathbb{Z}/2\mathbb{Z}$ for the projective plane and the non-trivial semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}$ for the Klein bottle.

Definition 3.1. Let p be a prime number. A group Γ is a *residually finite p-group* if for every non-trivial element $\gamma \in \Gamma$, there exists a normal subgroup $N \trianglelefteq \Gamma$ such that Γ/N is a finite p-group and $\gamma \notin N$. Equivalently, Γ is a residually finite p-group if and only if there exists a chain $(\Gamma_n)_{n \geq 0}$ in Γ consisting of normal subgroups such that for every $n \geq 0$, the quotient Γ/Γ_n is a finite p-group and

$$\bigcap_{n \geq 0} \Gamma_n = \{1\}.$$

Baumslag proved in [**Bau62**] that orientable surface groups are residually free, that is, for every non-trivial element γ, there exists a normal subgroup $N \trianglelefteq \Gamma$ such that Γ/N is a free group and $\gamma \notin N$. Free groups are residually finite p-groups for every prime p, a result independently proved by Takahasi [**Tak51**] and by Gruenberg in [**Gru57**]. This implies the following well-known result: orientable surface groups are residually finite p-groups for every prime p.
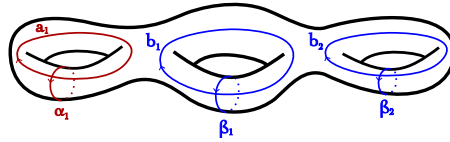
FIGURE 1. Illustration of $\Gamma_{A,B}$ with $|A| = 1$ and $|B| = 2$ as an orientable surface group.

Remark 3.1. By a result of Baumslag [**Bau67**], non-amenable non-orientable surface groups are also residually p-finite groups for every prime p. However, we leave as an exercise to the interested reader the fact that the fundamental group of a Klein bottle is not residually p for some prime p. We will not require these results.

3.2. *Special kind of finite index subgroups in surface groups.* Let A, B be two non-empty totally ordered finite sets. We let $\Gamma_{A,B}$ be the group defined by the generators $a_i, \alpha_i, b_j, \beta_j$ for $i \in A$ and $j \in B$ and the relation

$$[a_{i_1}, \alpha_{i_1}] \ldots [a_{i_n}, \alpha_{i_n}] = [b_{j_1}, \beta_{j_1}] \ldots [b_{j_m}, \beta_{j_m}],$$

where $i_1 \leq \cdots \leq i_n$ is an ascending enumeration of the elements in A and $j_1 \leq \cdots \leq j_m$ an ascending enumeration of the elements in B. Then $\Gamma_{A,B}$ is isomorphic to a non-amenable orientable surface group, and every non-amenable orientable surface group is isomorphic to $\Gamma_{A,B}$ for some non-empty totally ordered finite sets A and B (see Figure 1). The group $\Gamma_{A,B}$ naturally splits as an amalgamated product

$$\Gamma_{A,B} = \Gamma_A *_{\mathbb{Z}} \Gamma_B,$$

where $\Gamma_A = \langle a_i, \alpha_i, i \in A \rangle$ and $\Gamma_B = \langle b_j, \beta_j, j \in B \rangle$ are free groups of rank $2|A|$ and $2|B|$.

If $A' \subset A$ and $B' \subset B$, there is a natural surjection $\Gamma_{A,B} \twoheadrightarrow \Gamma_{A',B'}$ defined on the generators by

$$
\begin{array}{llll}
a_i \mapsto a_i' & \text{for every } i \in A', & b_j \mapsto b_j' & \text{for every } j \in B', \\
\alpha_i \mapsto \alpha_i' & \text{for every } i \in A', & \beta_j \mapsto \beta_j' & \text{for every } j \in B', \\
a_i \mapsto 1 & \text{for every } i \in A \setminus A', & b_j \mapsto 1 & \text{for every } j \in B \setminus B', \\
\alpha_i \mapsto 1 & \text{for every } i \in A \setminus A', & \beta_j \mapsto 1 & \text{for every } j \in B \setminus B'.
\end{array}
$$

We say that this morphism *erases* the generators $a_i, \alpha_i, b_j, \beta_j$ for $i \in A \setminus A'$ and $j \in B \setminus B'$, see Figure 2. Algebraically, $\Gamma_{A',B'}$ is isomorphic to the quotient of $\Gamma_{A,B}$ by the normal closure of the set $\{a_i, \alpha_i, b_j, \beta_j \mid i \in A \setminus A', j \in B \setminus B'\}$ in $\Gamma_{A,B}$, and the homomorphism $\Gamma_{A,B} \twoheadrightarrow \Gamma_{A',B'}$ corresponds to the quotient group homomorphism.

Here is the main theorem of this section. In what follows, $\mathbb{Z}[1/p]$ denotes the set of rational numbers of the form k/p^n for $k, n \in \mathbb{Z}$.

THEOREM 3.2. *Let Γ be a non-amenable orientable surface group and fix a decomposition $\Gamma = \Gamma_A *_{\mathbb{Z}} \Gamma_B$ as above. Let p be a prime number and $r \in]0, 1[\cap \mathbb{Z}[1/p]$. Let $\langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$ be the normal closure of the amalgamated subgroup $\mathbb{Z}$ in Γ_B. For every non-trivial $\gamma \in \Gamma$*
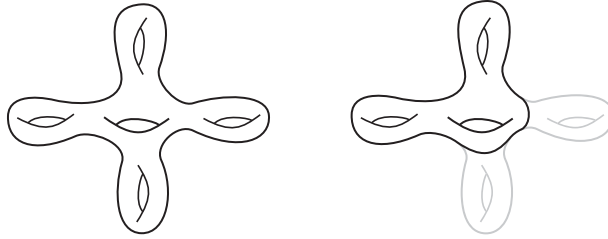
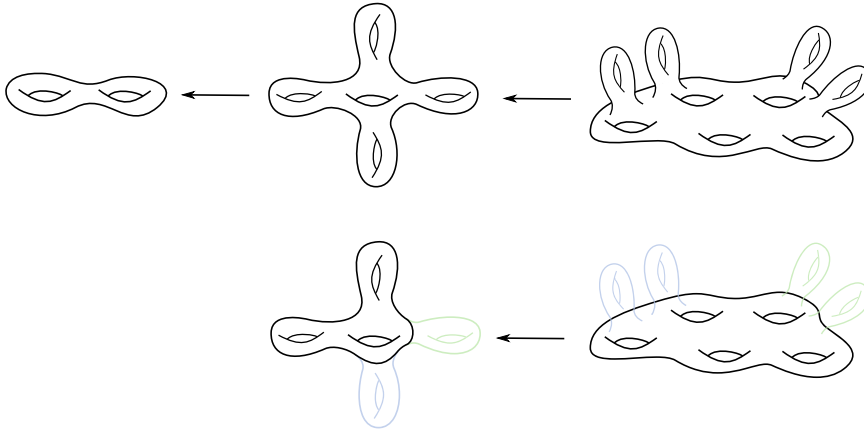FIGURE 2. An illustration of the morphism that erases generators.

FIGURE 3. Illustrations of the proof of Theorem 3.2. The above line illustrates the coverings corresponding to the inclusions $\Lambda \leq \Lambda_d \leq \Gamma$. The bottom line illustrates the covering corresponding to the inclusion $N' \leq \Lambda_d/N$.

and for every element $\delta \in \Gamma_B \setminus \langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$, there exists a finite index subgroup $\Lambda \leq \Gamma$ such that:

(1) $\gamma \notin \Lambda$;

(2) *the index $[\Gamma : \Lambda]$ is a power of p;*

(3) *the number of left cosets $x \in \Gamma/\Lambda$ that are fixed by every element in Γ_A is equal to $r[\Gamma : \Lambda]$;*

(4) *none of the left coset $x \in \Gamma/\Lambda$ is fixed by δ.*

Proof. Fix A, B two non-empty totally ordered finite sets, such that Γ is isomorphic to $\Gamma_{A,B}$. Let $S = \{a_i, \alpha_i, b_j, \beta_j \mid i \in A, j \in B\}$ be the set of generators. Let j_0 be the smallest element in B. Let $\gamma \in \Gamma \setminus \{1\}$ and $\delta \in \Gamma_B \setminus \langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$. Let p be a prime number and $r \in\,]0, 1[\cap \mathbb{Z}[1/p]$.

Step 1: Cyclic covering. Let $\varphi : \Gamma_{A,B} \twoheadrightarrow \mathbb{Z}$ be the surjective homomorphism defined on the generators of $\Gamma_{A,B}$ by

$$\varphi(b_{j_0}) = 1, \quad \varphi(\beta_{j_0}) = 0,$$
$$\varphi(a_i) = \varphi(\alpha_i) = \varphi(b_j) = \varphi(\beta_j) = 0 \text{ for every } i \in A, j \in B \setminus \{j_0\}.$$
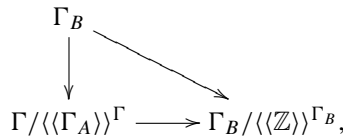
For every $d \geq 1$, we let Λ_d be the kernel of the homomorphism $\Gamma \to \mathbb{Z}/d\mathbb{Z}$ obtained by composing φ with the homomorphism of reduction modulo d. Then Λ_d is a surface group.

Let us describe a generating set for Λ_d. For every $0 \le k \le d-1$ and $i \in A$, let $a_{i,k}$ and $\alpha_{i,k}$ be the conjugates of a_i and α_i respectively, by $b_{j_0}^k$. Similarly, let $b_{j,k}$ and $\beta_{j,k}$ be the conjugates of b_j and β_j respectively, by $b_{j_0}^k$. Then Λ_d is generated by the set

$$\bigcup_{k=0}^{d-1} \{a_{i,k}, \alpha_{i,k} \mid i \in A\} \cup \bigcup_{k=0}^{d-1} \{b_{j,k}, \beta_{j,k} \mid j \in B \setminus \{j_0\}\} \cup \{b_{j_0}^d, \beta_{j_0}\}.$$

So far, every left coset $x \in \Gamma/\Lambda_d$ is fixed by every element of Γ_A, and either every or none of the left coset $x \in \Gamma/\Lambda_d$ is fixed by δ, depending on whether $\delta \in \Lambda_d$ or not.

Step 2: Erasing the right amount of generators. Let n be the length of $\gamma \in \Gamma \setminus \{1\}$ in the generating set S. In the following, we let d be a (large enough) power of the prime p such that rd is an integer and $rd + n \le d$. Let $E \subset \{n+1, \ldots, d-1-n\}$ be a subset of cardinality rd, so that γ does not belong to the normal closure N of the set $\bigcup_{k \in E} b_{j_0}^k \Gamma_A b_{j_0}^{-k}$ in Λ_d. Let us prove that none of the conjugate of δ by a power of b_{j_0} belongs to N. Assume this is not the case, then this would imply that δ belongs to the normal closure of $\bigcup_{k=0}^{d-1} b_{j_0}^k \Gamma_A b_{j_0}^{-k}$ in Λ_d, which is easily seen to be equal to the normal closure $\langle\langle \Gamma_A \rangle\rangle^\Gamma$ of Γ_A in Γ. However, the group $\Gamma/\langle\langle \Gamma_A \rangle\rangle^\Gamma$ is naturally isomorphic to $\Gamma_B/\langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$, in such a way that the following diagram commutes:

$$
\begin{array}{ccc}
\Gamma_B & & \\
\downarrow & \searrow & \\
\Gamma/\langle\langle \Gamma_A \rangle\rangle^\Gamma & \longrightarrow & \Gamma_B/\langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B},
\end{array}
$$

which implies that $\Gamma_B \cap \langle\langle \Gamma_A \rangle\rangle^\Gamma$ is equal to $\langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$. This would thus imply that $\delta \in \langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$, which is a contradiction.

Step 3: The group Λ_d/N is a residually finite p-group. We let $\pi : \Lambda_d \twoheadrightarrow \Lambda_d/N$ be the quotient group homomorphism. As orientable surface groups are p-residually finite and since Λ_d/N is an orientable surface group, there exists a normal subgroup $N' \trianglelefteq \Lambda_d/N$ whose index is a power of p, such that for every $k \in \{0, \ldots, d-1\} \setminus E$, for every $i \in A$, $\pi(a_{i,k}) \notin N'$ and $\pi(\alpha_{i,k}) \notin N'$. If $\gamma \in \Lambda_d$, we also assume that $\pi(\gamma) \notin N$, and if $\delta \in \Lambda_d$, we also assume that for all $k \in \{0, \ldots, d-1\}$, $\pi(b_{j_0}^k \delta b_{j_0}^{-k}) \notin N'$. Let us prove that the subgroup $\Lambda := \pi^{-1}(N')$ of Γ satisfies the four conclusions of the theorem.

Proof of (1). Either $\gamma \notin \Lambda_d$ and thus $\gamma \notin \Lambda$, or $\gamma \in \Lambda_d$ and $\pi(\gamma) \notin N$.

Proof of (2). Since the index of N' in Λ_d/N is a power of p, $[\Lambda_d : \Lambda]$ is also a power of p. Thus $[\Gamma : \Lambda] = [\Gamma : \Lambda_d][\Lambda_d : \Lambda]$ is a power of p.

Proof of (3). By construction, $x \in \Gamma/\Lambda$ is fixed by every element in Γ_A if and only its image under the canonical $[\Lambda_d : \Lambda]$-to-one map $\Gamma/\Lambda \mapsto \Gamma/\Lambda_d$ is equal to $b_{j_0}^k \Lambda_d$ for some $k \in E$. Since $|E| = rd$, there are exactly $rd[\Lambda_d : \Lambda] = r[\Gamma : \Lambda]$ such $x \in \Gamma/\Lambda$.

Proof of (4). If $\delta \notin \Lambda_d$, then none of the coset $x \in \Gamma/\Lambda$ is fixed by δ. If $\delta \in \Lambda_d$, then for all $k \in \{0, \ldots, d-1\}$, we have $\pi(b_{j_0}^k \delta b_{j_0}^{-k}) \notin N'$, and thus $\delta b_{j_0}^{-k} \Lambda \ne b_{j_0}^{-k} \Lambda$.

By normality of Λ in Λ_d, we deduce that none of the coset $x \in \Gamma/\Lambda$ is fixed by δ (see Figure 3). $\qquad\square$

4. *Proof of the main theorem*

In this section, we give the proof of Theorem 1.2. More precisely, we prove the following results.

THEOREM 4.1. (Orientable case) *Let Γ be a non-amenable orientable surface group and fix a decomposition $\Gamma = \Gamma_A *_{\mathbb{Z}} \Gamma_B$ as above. Let $\langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$ be the normal closure of the amalgamated subgroup $\mathbb{Z}$ in Γ_B. Then there exists a continuum $(\alpha^t)_{0<t<1}$ of ergodic profinite allosteric actions of Γ such that for all $0 < t < 1$:*

(1)   *the set of points whose stabilizer for α^t contains Γ_A has measure t;*

(2)   *each element of $\Gamma_B \setminus \langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$ acts essentially freely for α^t.*

In particular, for all $0 < s < t < 1$, the actions α^s and α^t are neither topologically nor measurably isomorphic, and the probability measures $\mathrm{IRS}(\alpha^s)$ and $\mathrm{IRS}(\alpha^t)$ are distinct.

THEOREM 4.2. (Non-orientable case) *Let Γ' be a non-amenable non-orientable surface group. Then there exists an index-two subgroup $\Gamma \leq \Gamma'$ which is isomorphic to an orientable surface group and which decomposes as $\Gamma = \Gamma_A *_{\mathbb{Z}} \Gamma_B$, and a continuum $(\beta^t)_{0<t<1}$ of ergodic profinite allosteric actions of Γ' such that for all $0 < t < 1$, the set of points whose stabilizer for β^t contains Γ_A has measure $t/2$. In particular, for all $0 < s < t < 1$, the actions β^s and β^t are neither topologically nor measurably isomorphic, and the probability measures $\mathrm{IRS}(\beta^s)$ and $\mathrm{IRS}(\beta^t)$ are distinct.*

During the proof of these theorems, we will need the following lemma.

LEMMA 4.3. *Let Γ be a group, and $\Lambda_1, \ldots, \Lambda_n$ be finite index subgroups of Γ. If the indices $[\Gamma : \Lambda_i]$, $i \in \{1, \ldots, n\}$, are pairwise coprime integers, then the left coset action $\Gamma \curvearrowright \Gamma/(\Lambda_1 \cap \cdots \cap \Lambda_n)$ is isomorphic to the diagonal action $\Gamma \curvearrowright \Gamma/\Lambda_1 \times \cdots \times \Gamma/\Lambda_n$ of the left coset actions $\Gamma \curvearrowright \Gamma/\Lambda_i$.*

Proof. The kernel of the group homomorphism $\Gamma \to \Gamma/\Lambda_1 \times \cdots \times \Gamma/\Lambda_n$ defined by $\gamma \mapsto (\gamma\Lambda_1, \ldots, \gamma\Lambda_n)$ is equal to $\Lambda_1 \cap \cdots \cap \Lambda_n$. Thus, $\Gamma/(\Lambda_1 \cap \cdots \cap \Lambda_n)$ is isomorphic to a subgroup of $\Gamma/\Lambda_1 \times \cdots \times \Gamma/\Lambda_n$. Moreover, for every $1 \leq i \leq n$,

$$[\Gamma : \Lambda_1 \cap \cdots \cap \Lambda_n] = [\Gamma : \Lambda_i][\Lambda_i : \Lambda_1 \cap \cdots \cap \Lambda_n],$$

and since the indices $[\Gamma : \Lambda_i]$ are pairwise coprime, this implies that $[\Gamma : \Lambda_1 \cap \cdots \cap \Lambda_n]$ is divisible by $[\Gamma : \Lambda_1] \cdots [\Gamma : \Lambda_n]$. Thus, the group homomorphism $\Gamma/\Lambda_1 \cap \cdots \cap \Lambda_n \to \Gamma/\Lambda_1 \times \cdots \times \Gamma/\Lambda_n$ is an isomorphism and it is Γ-equivariant. $\qquad\square$

We are now ready to prove Theorems 4.1 and 4.2.

Proof of Theorem 4.1. Let Γ be a non-amenable orientable surface group, and we fix a decomposition $\Gamma = \Gamma_A *_{\mathbb{Z}} \Gamma_B$. Let $0 < t < 1$ be a real number. Let $(p_n)_{n \geq 1}$ be a sequence of pairwise distinct prime numbers. We fix a sequence $(r_n)_{n \geq 1}$ such that each r_n belongs to $]0, 1[\cap \mathbb{Z}[1/p_n]$ and $\prod_{n \geq 1} r_n = t$. Such a sequence exists because each $\mathbb{Z}[1/p_n]$ is

dense in $\mathbb{R}$. Finally, let $(\gamma_n)_{n \geq 0}$ be an enumeration of the elements in Γ with $\gamma_0 = 1$, and $(\delta_n)_{n \geq 1}$ be an enumeration of the elements in $\Gamma_B \setminus \langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$. For every $n \geq 1$, there exists by Theorem 3.2 a finite index subgroup $\Lambda_n^t \leq \Gamma$ which does not contain γ_n, whose index $[\Gamma : \Lambda_n^t]$ is a power of p_n, such that the number of left cosets $x \in \Gamma/\Lambda_n^t$ that are fixed by any element of Γ_A is equal to $r_n[\Gamma : \Lambda_n^t]$, and such that none of the left coset $x \in \Gamma/\Lambda_n^t$ is fixed by δ_n. For every $n \geq 1$, let $\Gamma_n^t := \Lambda_1^t \cap \cdots \cap \Lambda_n^t$. The sequence $(\Gamma_n^t)_{n \geq 1}$ forms a chain in Γ, and we denote by α^t the corresponding ergodic profinite action and by μ_t the profinite Γ-invariant probability measure on $\varprojlim \Gamma/\Gamma_n^t$. This is a p.m.p. ergodic minimal action and we will prove that it is allosteric. By construction of Λ_n^t, we have that

$$\bigcap_{n \geq 1} \Gamma_n^t = \{1\}.$$

This implies by Lemma 2.7 that $\mathrm{URS}(\alpha^t)$ is trivial. Let us prove that each element of $\Gamma_B \setminus \langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$ acts essentially freely for α^t. Let $\delta \in \Gamma_B \setminus \langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$. By Lemma 4.3, the number of $x \in \Gamma/\Gamma_n^t$ such that $\delta x = x$ is equal to the number of $(x_1, \ldots, x_n) \in \Gamma/\Lambda_1^t \times \cdots \times \Gamma/\Lambda_n^t$ such that $(\delta x_1, \ldots, \delta x_n) = (x_1, \ldots, x_n)$. If n is large enough, then this last number is zero by construction of Λ_n^t. Thus, Lemma 2.6 implies that $\mathrm{Fix}_{\alpha^t}(\delta)$ is μ_t-negligible.

Finally, let us prove that the actions α^t are not essentially free. By construction, the indices $[\Gamma : \Lambda_i^t]$ are pairwise coprime. Thus, Lemma 4.3 implies that the number of $x \in \Gamma/\Gamma_n^t$ that are fixed by every element in Γ_A is equal to the number of $(y_1, \ldots, y_n) \in \Gamma/\Lambda_1^t \times \cdots \times \Gamma/\Lambda_n^t$ that are fixed for the diagonal action by every element in Γ_A. By construction of Λ_i^t, this number is equal to $r_1[\Gamma : \Lambda_1^t] \times \cdots \times r_n[\Gamma : \Lambda_n^t]$ which is equal to $r_1 \ldots r_n[\Gamma : \Gamma_n^t]$. Thus, Lemma 2.6 implies that the μ_t-measure of the set of points whose stabilizer for α^t contains Γ_A is t. In particular, this implies that $\mathrm{IRS}(\alpha^t)$ is non-trivial. Thus, α^t is allosteric. Moreover, this also implies that for all $0 < s < t < 1$, the actions α^s and α^t are not measurably isomorphic and thus not topologically isomorphic since every α^t is uniquely ergodic by Lemma 2.5, and this finally implies that the measures $\mathrm{IRS}(\alpha^s)$ and $\mathrm{IRS}(\alpha^t)$ are distinct. $\square$

Proof of Theorem 4.2. Let Σ' be a non-orientable surface of genus $g \geq 3$. Consider the usual embedding of an orientable surface Σ of genus $g - 1$ into $\mathbb{R}^3$ in such a way that the reflexions in all three coordinate planes map the surface to itself, and let ι to be the fixed-point free antipodal map $x \mapsto -x$. Then Σ' is homeomorphic to the quotient of Σ by ι, and the covering $\Sigma \mapsto \Sigma/\iota \approx \Sigma'$ is called the orientation covering. We decompose Σ as the union of two surfaces Σ_A and Σ_B with one boundary, of genus $|A|$ and $|B|$ respectively, with $|A| \leq |B|$, so that $\iota(\Sigma_A) \subset \Sigma_B$. Fix a point $p \in \Sigma_A \cap \Sigma_B$, then Van Kampen's theorem implies that the fundamental group Γ of the surface Σ based at p is isomorphic to $\Gamma_A *_{\mathbb{Z}} \Gamma_B$ with $\Gamma_A = \pi_1(\Sigma_A, p)$, $\Gamma_B = \pi_1(\Sigma_B, p)$ and $\mathbb{Z} \approx \pi_1(\Gamma_A \cap \Gamma_B, p)$. The fundamental group Γ' of Σ' based at $p' = \iota(p)$ naturally contains the subgroup Γ as an index-two subgroup. Fix a curve contained in Σ_B that joins p to $\iota(p)$. This produces an element $\gamma_0 \in \Gamma' \setminus \Gamma$ that satisfies $\gamma_0 \Gamma_A \gamma_0^{-1} \leq \Gamma_B$.

Let $(\alpha^t)_{0 < t < 1}$ be a continuum of allosteric Γ-actions on (X_t, μ_t) given by Theorem 4.1. The actions $\beta^t : \Gamma' \curvearrowright (Y_t, \nu_t)$ induced by the Γ-actions α^t are allosteric, see

Proposition 2.10. Let us prove that the set of points in Y_t whose stabilizer for β^t contains Γ_A has ν_t-measure $t/2$. Since β^t is an induced action and $[\Gamma' : \Gamma] = 2$, the Γ'-action β^t is measurably isomorphic to a p.m.p. Γ'-action on $(X_t \times \{0, 1\}, \mu_t \times \text{ unif})$, still denoted by β^t, that satisfies the following two properties:

(1) for every $\gamma \in \Gamma' \setminus \Gamma$, the sets $X_t \times \{0\}$ and $X_t \times \{1\}$ are switched by $\beta^t(\gamma)$;

(2) for every $\gamma \in \Gamma$, for every $x \in X_t$, we have $\beta^t(\gamma)(x, 0) = (\alpha^t(\gamma)x, 0)$ and $\beta^t(\gamma)(x, 1) = (\alpha^t(\gamma_0 \gamma \gamma_0^{-1})x, 1)$.

This implies that for all $(x, \varepsilon) \in X_t \times \{0, 1\}$, the subgroup Γ_A is contained in $\text{Stab}_{\beta^t}(x, \varepsilon)$ if and only if either $\varepsilon = 0$ and Γ_A is contained in $\text{Stab}_{\alpha^t}(x)$, or $\varepsilon = 1$ and $\gamma_0 \Gamma_A \gamma_0^{-1}$ is contained in $\text{Stab}_{\alpha^t}(x)$. Thus, the set of points whose stabilizer for β^t contains Γ_A has ν_t-measure

$$\frac{t + \mu_t(\{x \in X_t \mid \gamma_0 \Gamma_A \gamma_0^{-1} \leq \text{Stab}_{\alpha^t}(x)\})}{2}.$$

To finish the proof, it is enough to prove that the intersection of $\gamma_0 \Gamma_A \gamma_0^{-1}$ and $\Gamma_B \setminus \langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$ is non-trivial, since any element in $\Gamma_B \setminus \langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$ acts essentially freely for α^t. The conjugation by γ_0 induces a group automorphism $\varphi : \Gamma \mapsto \Gamma$, such that $\varphi(\Gamma_A) \leq \Gamma_B$. Since Γ_A is not contained in the derived subgroup $D(\Gamma)$, so is $\varphi(\Gamma_A)$. However, the amalgamated subgroup $\mathbb{Z}$ is contained in $D(\Gamma)$, thus so is $\langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B}$. This implies that the intersection $\varphi(\Gamma_A) \cap (\Gamma_B \setminus \langle\langle \mathbb{Z} \rangle\rangle^{\Gamma_B})$ is non-empty. We deduce that the set of points whose stabilizer for β^t contains Γ_A has ν_t-measure $t/2$. We conclude that the actions β^t are neither measurably nor topologically pairwise isomorphic and that their IRS are pairwise disjoint as in Theorem 4.1. □

Remark 4.4. Let $\alpha : \Gamma \curvearrowright (C, \mu)$ be an allosteric action. Then we have

$$\text{supp}(\text{IRS}(\alpha)) \subset \overline{\{\text{Stab}_\alpha(x) \mid x \in C\}}.$$

This implies that the support of $\text{IRS}(\alpha)$ does not contain any non-trivial subgroup with only finitely many conjugates, because otherwise, the closure of the set $\{\text{Stab}_\alpha(x) \mid x \in C\}$ would contain a closed minimal Γ-invariant set $\neq \{\{1\}\}$. Carderi, Gaboriau and Le Maître proved (personal communication) that the perfect kernel of a surface group coincides with the set of its infinite index subgroups. This implies that allosteric actions of surface groups are not totipotent (a p.m.p. action is *totipotent* if the support of its IRS coincides with the perfect kernel of the group, see [CGLM20]).

Remark 4.5. A p.m.p. action $\Gamma \curvearrowright (X, \mu)$ is *weakly mixing* if for every $\varepsilon > 0$ and every finite collection Ω of measurable subsets of X, there exists a $\gamma \in \Gamma$ such that for every $A, B \in \Omega$,

$$|\mu(\gamma A \cap B) - \mu(A)\mu(B)| < \varepsilon.$$

With this definition, it is easily seen that the restriction of a weakly mixing action to a finite index subgroup remains weakly mixing. Thus, Proposition 2.8 implies that the IRSs of non-amenable surface groups we have constructed are not weakly mixing.

Remark 4.6. The proof of our main theorem applies mutatis mutandis to branched orientable surface groups, that is, fundamental groups of closed orientable branched
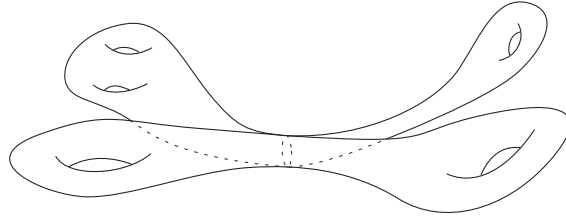
FIGURE 4. A branched surface.

surfaces (see Figure 4). These groups can be written as amalgams. Fix an integer $g \geq 2$ as well as $2g$ letters $x_1, y_1, \ldots, x_g, y_g$. Fix a partition of $\{1, \ldots, g\}$ into n non-empty intervals $A_1, \ldots, A_n$. Let Γ_k be the free group generated by x_i and y_i for every $i \in A_k$, and let $\mathbb{Z} \to \Gamma_k$ be the injective homomorphism defined by sending the generator of $\mathbb{Z}$ to the product $\prod_{i \in A_k} [x_i, y_i]$. Then the amalgam $*_{\mathbb{Z}} \Gamma_i$ is a branched orientable surface group, and any branched orientable surface group can be obtained this way. The fundamental group of a closed orientable branched surface of genus ≥ 2 is a residually p-finite group for every prime p, see [**KM93**, Theorem 4.2]. Thus, our method of proof applies to branched orientable surface groups, with any Γ_k in the role played by Γ_A during the proof of Theorem 4.1.

Question 4.7. Is the fundamental group of a compact hyperbolic 3-manifold allosteric? More generally, is the fundamental group of a compact orientable aspherical 3-manifold allosteric?

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