SOME PROPERTIES OF C-CONVEX SETS

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1. Introduction. The notion of convexity in $\mathfrak{R}_m$ (m-dimensional Euclidean space) can be generalized to apply to non-connected sets as follows.

DEFINITION 1. *A set is said to be C-convex if each of its components is convex. If the number of components of such a set is n, it is called a C_n-convex set.*

In order to determine the character of the complement of a C_n-convex set, we use the notion of L_n set, a concept studied by my colleague Alfred Horn and myself [2]. Although my original goal was to establish the fact that in the plane the complement of a bounded open C_n-convex set ($n > 1$) is an L_{n+1} set, the auxiliary concept of "Maximal families of disjoint open convex sets" almost preempted my original intention. For this reason, the latter concept has been studied in §**3** separately. In order to complete the terminology, I restate the definition given by Horn and myself [2].

DEFINITION 2. *A set S is called an L_n set if each pair of points in S can be joined by a polygonal arc in S having at most n segments.*

Throughout this paper we confine ourselves to sets in $\mathfrak{R}_2$.

2. Polygonal sets in the plane. In the following treatment the words vertex, edge and face are used in the usual sense [3, pp. 194-5]. An edge is always incident with a face, and a face may be bounded or unbounded. A linear edge is one which is contained in a straight line.

DEFINITION 3. *A polygonal set P_n is a connected closed set which has the following properties.*

(a) *It is the sum of a finite number of linear edges.*

(b) *Its complement consists of n components, and each of these is convex (called a face).*

(c) *Each vertex of P_n is incident with at least three edges.*

NOTATION. A polygonal arc P in P_n joining x and y is denoted by $xx_1 \ldots x_ty$, where $x_1, \ldots, x_t$ denote the vertices of P_n on P distinct from x and y. If no such vertices exist, then $P = xy$. The boundary of a face F of P_n is denoted by $B(F)$.

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DEFINITION 4. *An improper vertex of P_n is one which is incident with at least four edges of P_n. A segment of a polygonal arc P in P_n, as distinguished from an edge of P_n, is a maximal connected linear subset of P.*

DEFINITION 5. *If a polygonal arc in P_n joining x and y has a shortest length (a proper or improper minimum) relative to the arcs in P_n joining x and y, it is called a minimal polygonal arc, and we denote it by $P(x, y)$.*

LEMMA 1. *Let F be a face of P_n. If $x \in B(F)$, $y \in B(F)$, then any minimal polygonal arc $P(x, y) \subset B(F)$.*

Lemma 1 is an immediate consequence of the convexity of F.

LEMMA 2. *Let $P(x, y) = xx_1 \ldots x_t y$ be a minimal polygonal arc in P_n. Let $\mathfrak{F}_i = (F_{i1}, F_{i2}, \ldots, F_{im_i})$ denote the collection of faces of P_n which have x_i as a vertex, and which do not have $x_{i-1}x_i$ as an edge ($i = 1, \ldots, t$; $x_0 = x$). Then all of the faces in the collection $\sum_{i=1}^{t} \mathfrak{F}_i$ are distinct.*

Proof. Condition (c) in Definition 3 implies that $m_i \geqslant 1$ $(i = 1, \ldots, t)$. Suppose there exist two faces F_{is} and F_{kr} contained in $\sum_{i=1}^{t} \mathfrak{F}_i$ such that $F_{is} = F_{kr} (1 \leqslant i < k \leqslant t)$. By Lemma 1, we have then $P(x_i, x_k) = x_i x_{i+1} \ldots x_k \subset B(F_{kr})$. However, since by hypothesis, $x_{k-1}x_k \not\subset B(F_{kr})$, we have $x_{k-1}x_k \not\subset P(x, y)$, which is a contradiction. Hence Lemma 2 is clearly true.

THEOREM 1. *Let $P(x, y) = xx_1 \ldots x_t y$ be a minimal polygonal arc in P_n. Then there exists a collection $\mathfrak{F} = (F_0, F_1, F_2, \ldots, F_t)$ of distinct faces of P_n such that the edge $x_i x_{i+1} \subset B(F_i)$ ($i = 0, \ldots, t$; $x_0 = x$, $x_{t+1} = y$). Let p denote the number of faces in $\mathfrak{G} = \sum_{i=1}^{t} \mathfrak{F}_i - \mathfrak{F}$, and let v be the number of faces in P_n not incident with any part of $P(x, y)$. Then $p + t + v \leqslant n - 2$.*

Proof. Theorem 1 follows from Lemma 2. Let F_0 and F'_0 be the faces of P_n incident with xx_1. As in the proof of Lemma 2, F_0 non $\in \sum_{i=1}^{t} \mathfrak{F}_i$, F'_0 non $\in \sum_{i=1}^{t} \mathfrak{F}_i$, since $P(x, y)$ is minimal. Define F_k to be a member of $\mathfrak{F}_k$ having $x_k x_{k+1}$ as an edge ($k = 1, \ldots, t$). Hence $\mathfrak{F}$ has been defined, and it contains distinct members. Moreover, since F'_0 non $\in \mathfrak{F}$, F'_0 non $\in \mathfrak{G}$, by counting distinct faces, we get $p + t + 1 + v \leqslant n - 1$.

COROLLARY 1. *A polygonal set P_n ($n \geqslant 2$) is an L_{n-1} set.*

3. Maximal families of convex sets in the plane.

DEFINITION 6. *A family of disjoint open convex sets is said to be maximal if no member of the family is a proper subset of an open convex set which is disjoint with the rest of the family.*

A family of this type containing exactly n members is called an M_n set.

LEMMA 3. *Each member of an open C_n-convex set ($n > 1$) can be enclosed in an open convex set which has a polygonal boundary, and which is disjoint with the rest of C_n. The boundary of this set need not be connected.*

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This lemma was proved by Stoelinga. See Bonnesen and Fenchel [1, p. 5].

THEOREM 2. *The boundary of a maximal family M_n ($n > 1$) of disjoint open convex sets is the sum of a finite number of line segments, lines and half-lines.*

Proof. Each member of M_n must be a two-dimensional convex plane polygon, otherwise by Lemma 3, it would not be maximal. Since there are a finite number of members in M_n, each of which has a finite number of linear elements in its boundary, the boundary of M_n is the sum of a finite number of line segments, lines and half-lines.

DEFINITION 7. *A component of the complement (face) of a polygonal set is called a pinwheel R provided:*

(i) *It is a bounded convex set.*

(ii) *The vertices of $\bar{R}$ can be ordered consecutively ($x_1, x_2, \ldots, x_t; x_t = x_1$) so that for each vertex x_i there exists an edge E_i of the polygonal set which abuts R externally at x_i, and which is a linear extension of $x_{i-1}x_i$ ($i = 2, \ldots, t$).* (See Figure 1; $E_t = E_1$).

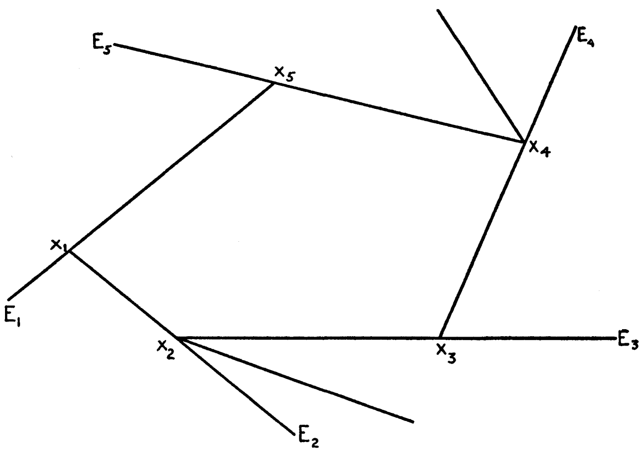

FIGURE 1. A pinwheel

THEOREM 3. *Each component of the complement of the closure of a maximal family M_n is a pinwheel.*

Proof. Let K be any component of the complement of $\bar{M}_n$. By Theorem 2, K has a boundary consisting solely of line segments, lines or half-lines. Let $B(K)$ be a *component* of the boundary of K. Among the finite number of vertices of $B(K)$ we include corners as well as vertices of the boundary of M_n. Since M_n is maximal, there exists a finite edge $x_1x_2 \subset B(K)$.

Since M_n contains a finite number of components, let C_j be the member of M_n abutting x_1x_2. The straight line through x_1x_2 determines two open half-

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planes $\mathfrak{R}_1^+$ and $\mathfrak{R}_1^-$ where $C_j \subset \mathfrak{R}_1^+$ by definition. Let E_1 and E_2 be the edges of C_j which abut K at x_1 and x_2 respectively. Since M_n is a maximal family of convex sets, $E_1 + E_2 - x_1 - x_2 \not\subset \mathfrak{R}_1^+$. Moreover, since C_j is convex, $E_1 + E_2 - x_1 - x_2 \not\subset \mathfrak{R}_1^-$. Hence at least one of the edges E_1 and E_2 is a linear extension of x_1x_2. Without loss of generality suppose E_2 is an extension of x_1x_2. Hence, x_2 must be a vertex of the boundary of M_n, so that at least *three* edges of the boundary of M_n are incident with x_2. Hence, the interior angle θ_2 of $\bar{K}$ at x_2 is less than π. Let x_2x_3 denote the edge (finite or infinite) of $\bar{K}$ which together with x_1x_2 makes the angle θ_2. The edge x_2x_3 must be *finite*, otherwise the member of M_n abutting x_2x_3 would not be maximal relative to M_n. By induction, we get a finite polygonal line $x_1x_2 \ldots x_s$ and a set of extensions $E_i (i = 2, \ldots, s)$ such that the interior angle of $\bar{K}$ at x_i is less than π, and such that E_i is an extension of $x_{i-1}x_i$. Since $B(K)$ has a finite number of vertices including corners, it is clear that this sequence $x_1x_2 \ldots x_s$ can only be continued until we get $x_1x_2 \ldots x_{t-1}x_t$ where $x_1, \ldots x_{t-1}$ are all distinct, and where x_t is one of the vertices $x_1, x_2, \ldots, x_{t-2}$. One can prove that $x_t = x_1$, otherwise all the extensions E_i would not exist, which is a contradiction. Since M_n is maximal, the interior angle of the simple closed polygon $x_1x_2 \ldots x_t$ $(x_t = x_1)$ at x_1 is also less than π. Hence $x_1x_2 \ldots x_t$ is a closed convex polygonal curve. Since K is connected, $B(K)$ is contained in the closed convex set bounded by $x_1x_2 \ldots x_t$, and it follows by an argument of the type just given for $x_1x_2 \ldots x_t$ that $B(K) = x_1x_2 \ldots x_t$.

Finally, we show that the set bounded by $B(K)$ is K. Suppose a component $B_1(K)$ of the boundary of K exists which is interior to the convex set bounded by $B(K)$. By virtue of the previous paragraph, $B_1(K)$ would bound a convex set, at least part of which would belong to K. But this would make K disconnected, which is a contradiction. Thus K satisfies (i) and (ii).

THEOREM 4. *Each component of the boundary of a maximal family M_n $(n > 1)$ of disjoint open convex convex sets is a polygonal set.*

The complement of the boundary of M_n is a maximal family M_r $(r \geqslant n)$, where $r - n$ is the number of pinwheels in the complement of $\bar{M}_n$.

Proof. Property (a) in Definition 3 holds by virtue of Theorem 2. Properties (b) and (c) hold since each member of M_n is convex, and since each residual domain of $\bar{M}_n$ is convex. The concluding statement follows from Theorem 3.

THEOREM 5. *The boundary of a maximal family M_n $(n > 1)$ has α components if and only if $\alpha - 1$ members of M_n are slabs (A slab is an open convex set bounded by two parallel lines).*

Proof. Let T be a component of the boundary of M_n. The set T must be unbounded, otherwise the unbounded component of M_n abutting T externally would not be convex. If the boundary of each member of M_n incident[1] with T

[1]A member of M_n is said to be incident with T if its boundary contains at least one edge of T.

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is connected, then the boundary of M_n is in T, and T is the only component of the boundary of M_n. *If a member of M_n has a disconnected boundary, then it must be a slab, since it is convex.* The set T can have at most two slabs abutting it, since two disjoint slabs must be parallel. All the slabs in M_n then must be parallel, and between two consecutive slabs there can be at most one component of the boundary of M_n. These facts clearly imply the conclusions of Theorem 5.

In the following treatment it should be recalled that in the definition of an L_n set, the word segment was used, and not edge (See Definitions 2 and 4).

THEOREM 6. *Let T be a component of the boundary of a maximal family M_n ($n > 1$) of disjoint open convex sets. Let s be the number of members of M_n which are incident with T. Then T is an L_{s-1} set.*

Proof. Replace each slab abutting T (if any exist) by the half-plane which contains that slab, and which abuts T. The thus modified s sets of M_n incident with T form a maximal family M_s. The complement of T is a maximal family M_r. By Theorem 4, $r - s \equiv q$ is the number of pinwheels in $M_r - M_s$. We designate the closures of these by $R_k (k = 1, \ldots, q)$. Choose $x \in T$, $y \in T$. If $P(x, y) = xy$, then it contains at most $s - 1$ segments. Let $P(x, y) = xx_1 \ldots x_t y$ and $\mathfrak{F}$ and $\mathfrak{G}$ denote the quantities described in Theorem 1.

Case 1. Suppose x non $\in R_k$, y non $\in R_k$ ($k = 1, \ldots, q$). First, let S_β ($\beta = 1, \ldots, q_1$) denote the closures of the pinwheels in $M_r - M_s$ each of which has one and only one vertex in common with $P(x, y) - x - y$. Since each of these vertices is then improper, we have $S_\beta \in \mathfrak{G}$ ($\beta = 1, \ldots, q_1$). Set up an order on $P(x, y)$ from x to y, and let $Q_j (j = 1, \ldots, q_2)$ denote in succession the closures of the pinwheels in $M_r - M_s$ for which $Q_1 \cdot P(x, y)$ contains at least one edge of T. Each set $Q_j \cdot P(x, y)$ is connected, and $Q_1 \cdot P(x, y)$ precedes $Q_2 \cdot P(x, y)$, on $P(x, y)$ etc. Let x_j^1 and x_j^2 denote the vertices of T where $P(x, y)$ enters and leaves Q_j respectively. *If a vertex of T is an interior point of a segment of $P(x, y)$, it is called a removable vertex of $P(x, y)$.* If x_1^1 and x_1^2 are both proper vertices of Q_1, then since Q_1 is a pinwheel, either x_1^1 or x_1^2 is a removable vertex of $P(x, y)$. If either x_1^1 or x_1^2 is an improper vertex of Q_1, the set $\mathfrak{G}$ in Theorem 1 contains at least one face corresponding to that vertex. Hence Q_1 corresponds either to a face of $\mathfrak{G}$ or to a removable vertex of $P(x, y)$. If $x_1^2 \neq x_2^1$, then Q_1 is isolated from Q_2. If $x_1^2 = x_2^1$, then x_2^1 is improper. Moreover Q_1 and Q_2 then have opposite orientations in the sense that the vertices of one of them are ordered clockwise and the vertices of the other counterclockwise. (See Figure 1.) One can show that this implies the following. If x_2^1 is *not* a removable vertex of $P(x, y)$, then either x_1^1 or x_2^2 must be an improper vertex or a removable vertex of $P(x, y)$. This is true whether the sense in which the directed $P(x, y)$ meets Q_1 and Q_2 coincides with their proper orientations or not. Hence, we can assign to each Q_1 and Q_2 either a member of $\mathfrak{G}$ or a removable vertex of $P(x, y)$, and the faces and vertices involved are all distinct. Suppose $Q_f, Q_{f+1}, \ldots, Q_{f+\sigma}$

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are a subset of consecutive sets from $Q_j (j = 1, \ldots, q_2)$ for which $x_f^2 = x_{f+1}^1$, $x_{f+1}^2 = x_{f+2}^1, \ldots, x_{f+\sigma-1}^2 = x_{f+\sigma}^1$. Then all of these vertices are improper. If none of these vertices is also a removable vertex of $P(x, y)$, then since each consecutive pair of $Q_f, Q_{f+1}, \ldots, Q_{f+\sigma}$ have opposite orientations, one can show that either x_f^1 or $x_{f+\sigma}^2$ is a removable vertex of $P(x, y)$ or an improper vertex. Hence, to each set in the above consecutive sets we can assign either a distinct face in $\mathfrak{G}$ or a removable vertex of $P(x ,y)$. Moreover, one can choose the faces of $\mathfrak{G}$ just mentioned distinct from $\sum_{\beta=1}^{q_1} S_\beta$. *Now*, by separating $P(x, y) \,.\, \sum_{j=1}^{q_2} Q_j$ into disjoint parts, the above type of argument implies the following. There is a subset of faces in $\mathfrak{G} - \sum_{\beta=1}^{q_1} S_\beta$ and a set of distinct removable vertices of $P(x, y)$ which together are in $1 - 1$ correspondence with $Q_1, Q_2, \ldots, Q_{q_2}$. Hence, if we let m equal the number of *segments* in $P(x, y)$, the above together with the fact $S_\beta \subset \mathfrak{G}$ $(\beta = 1, \ldots, q_1)$ implies that $m + q_1 + q_2 \leqslant t + 1 + p$. Theorem 1 implies that $p + t + 1 + v \leqslant r - 1$. Since $q_1 + q_2 \leqslant q$, $q - q_1 - q_2 \leqslant v$, and since $r = s + q$, we have $m \leqslant s - 1$. Thus $P(x, y)$ contains at most $s - 1$ segments.

Case 2. Suppose x non $\in R_k$ $(k = 1, \ldots, q)$, $y \in R_i$(i fixed). Let $y \in x_{a-1}x_a$, an edge of R_i. Choose y' in the interior of E_a (see Figure 1). If $E_a \not\subset R_k$ $(k = 1, \ldots, q)$, then by Case 1 $P(x, y')$ has at most $s - 1$ segments. It is easy to see that x and y can be joined by a polygonal arc having at most $s - 1$ segments. Secondly, if $E_a \subset R_j$(j fixed), then $x_a \in R_i$, $x_a \in R_j$. Let $P(x, x_a)$ and $\sum_{i=1}^{t} \mathfrak{F}_i$ be the quantities in Lemma 2. Since x_a is an improper vertex which is an end-point of $P(x, x_a)$, and since $P(x, x_a)$ is minimal. there exist at least two faces of T having x_a as a vertex, not belonging to $\sum_{i=1}^{t} \mathfrak{F}_i$, and distinct from F_0 and F'_0 (see Theorem 1). This together with a proof similar to Case 1 implies the following. If x_a is a removable vertex of $P(x, x_a) + x_a y$ or if $x_a y \subset P(x, x_a)$, then $P(x, x_a)$ contains at most $s - 1$ segments. If $P(x, x_a)$ and $x_a y$ are not so related, then $P(x, x_a)$ contains at most $s - 2$ segments. In any case, x and y can be joined by a polygonal arc in T having at most $s - 1$ segments. The same proof holds if x and y are interchanged. If both x and y are contained in the boundaries of pinwheels of $M_r - M_s$, a similar proof applied to x and y simultaneously yields the same conclusions.

THEOREM 7. *Let T be a component of the boundary of a maximal family M_n, and let s be the number of faces of M_n incident with T. Suppose that $s \geqslant 3$, and suppose a slab or half-plane B exists which is incident with T. Then through any point $x \in T$ there passes an infinite polygonal ray in T having at most $s - 2$ segments.*

Proof. If $x \in T \cdot \overline{B}$, then any half-line in $T \cdot \overline{B}$ having x as endpoint will suffice. If $x \in T - T \cdot \overline{B}$, choose a point $y \in T - T \cdot \overline{B}$ which is contained in the interior of an infinite half-line of T. By Theorem 6, there exists a minimal polygonal arc $P(x, y) \subset T$ having at most $s - 1$ segments. If $P(x, y) \cdot \overline{B} = 0$, then B non $\in \mathfrak{F}$, B non $\in \mathfrak{G}$ (see Theorem 1), and it is clear by the arguments

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given for Theorem 6, with $v \geqslant 1$, that $P(x, y)$ will contain at most $s - 2$ segments. If $P(x, y) \cdot \bar{B}$ contains an edge of T, then clearly x can be joined to infinity via a portion of $P(x, y)$ and a suitable half-line in $T \cdot \bar{B}$ which together contain at most $s - 2$ segments. If $P(x, y) \cdot \bar{B}$ contains a vertex x' of T which is not incident with an edge of $P(x, y) \cdot \bar{B}$, then x' is improper. Since x' non $\in R_k$ ($k = 1, \ldots, q$), defined in the proof of Theorem 6, that proof implies that $P(x, y)$ will contain at most $s - 2$ segments. Hence in all cases x can be joined to infinity by an at most $s - 2$ sided polygonal ray in T.

4. C_n-convex sets in the plane. In this section we investigate the complement of an open bounded C_n-convex set.

DEFINITION 8. *A maximal family of disjoint open convex sets M_n is said to be a maximal extension of an open C_n-convex set C_n if $M_n \supset C_n$, and if each member of M_n contains a unique member of C_n.*

THEOREM 8. *The complement of an open bounded C_n-convex set is an L_{n+1} set if $n > 1$. If $n = 1$, the complement is an L_3 set.*

Proof. Let M_n be a maximal extension of C_n, and let M_r be the family defined in Theorem 4, so that $M_r \supset M_n$. Let x_1 and x_2 be any two points in $\bar{M}_r - C_n$, and let K_1 and K_2 be components of M_r such that $x_1 \in \bar{K}_1$, $x_2 \in \bar{K}_2$. The sets K_1 and K_2 need not be distinct. When $n = 1$, the proof is trivial. When $n = 2$, there exist only two components in C_2, so that the boundary of M_2 is a straight line. The proof that x_1 and x_2 can be joined by a polygonal arc L_3 not intersecting C_2 is trivial.

Proof for $n \geqslant 3$. *Case* 1. Suppose the boundary of M_r has no slabs or half-planes incident with it. In this case the boundary of M_r, denoted by T, must be connected (see Theorem 5). If $x_i \in T$ ($i = 1, 2$), relabel it y_i. If x_i non $\in T$, then since each member of C_n is convex, and since each K_i is not a slab or a half-plane there exists a line segment $x_i y_i \subset \bar{M}_r - C_n$ such that $y_i \in T$. By Theorem 6, y_1 and y_2 can be joined by an L_{n-1} polygonal arc in T. Hence, x_1 and x_2 can be joined by an L_{n+1} polygonal arc in $\bar{M}_r - C_n$.

Case 2. Suppose the boundary of T has at least one slab or half-plane incident with it, and suppose that K_1 and K_2 are incident with the same component T_1 of T. Let s denote the number of faces of M_n incident with T_1, If $s = 2$, x_1 and x_2 can be joined by an at most 3-sided polygonal arc in $\bar{M}_r - C_n$. Hence, suppose $s \geqslant 3$. Then either a line passes through x_i not intersecting C_n, or a segment $x_i y_i$ exists such that $y_i \in T_1$, $x_i y_i \cdot C_n = 0$. If both y_1 and y_2 exist, the remainder of the proof is the same as in Case 1. Suppose a line L exists through x_1 not intersecting C_n, and suppose y_2 exists. Then by Theorem 7, a polygonal ray $Q \subset T$ exists through y_2 having at most $s - 2$ segments. Since C_n is bounded, points $z_1 \in L$, $z_2 \in Q$ exist such that $z_1 z_2 \cdot C_n = 0$. Hence, it is clear that x_1 and x_2 can be joined by an L_{s+1} ($s \leqslant n$) polygonal arc in $\bar{M}_r - C_n$. The same proof holds if x_1 and x_2 are interchanged.

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Case 3. Suppose T is disconnected, and let K_i be incident with T_i $(i = 1, 2)$, where T_i are components of T with $T_1 \neq T_2$. Let s_i be the number of faces of M_n incident with T_i. Theorem 5 implies $4 \leqslant s_1 + s_2 \leqslant n + 1$. If $s_i = 2$, then x_i can be joined to infinity by a half-line not intersecting C_n. If $s_i \geqslant 3$, then x_i can be joined to infinity by a half-line not intersecting C_n, or a segment $x_i y_i$ exists such thst $y_i \in T_i$, $x_i y_i \cdot C_n = 0$. By applying Theorem 7, then x_i can be joined to infinity in all subcases by a polygonal ray R_i containing at most $s_i - 1$ segments. Since C_n is bounded, there exist points $z_i \in R_i$ such that z_1 and z_2 can be joined by an at most two-sided polygonal arc not intersecting C_n. Hence x_1 and x_2 can be joined by an at most μ-sided polygonal arc in $\overline{M}_r - C_n$, where, by counting, $\mu \leqslant (s_1 - 1) + (s_2 - 1) + 2 = s_1 + s_2 \leqslant n + 1$. This completes the proof.

The expression "C-convex set" was suggested to me by Professor Max Zorn some years ago.

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