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ORDER-PRESERVING EXTENSIONS OF LIPSCHITZ MAPS

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Abstract

We study the problem of extending an order-preserving real-valued Lipschitz map defined on a subset of a partially ordered metric space without increasing its Lipschitz constant and preserving its monotonicity. We show that a certain type of relation between the metric and order of the space, which we call *radiality*, is necessary and sufficient for such an extension to exist. Radiality is automatically satisfied by the equality relation, so the classical McShane–Whitney extension theorem is a special case of our main characterization result. As applications, we obtain a similar generalization of McShane's uniformly continuous extension theorem, along with some functional representation results for radial partial orders.

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1. Introduction

The most important continuous extension theorem for order-preserving functions is *Nachbin's extension theorem* (of [27]). This theorem considers a partially ordered topological space, and gives conditions under which a continuous and order-preserving real-valued function defined on a compact subset of such a space can be extended to the entire space in such a way as to remain continuous and order preserving. It has found profound applications, especially in the field of decision theory. (The references for the present discussion are provided in the body of the paper.)

Another extension theorem of great importance is the famous *McShane–Whitney extension theorem* (of [22] and [32]), which shows that any Lipschitz map defined on a subset of a metric space can be extended to the entire space without increasing the Lipschitz constant of the original map. This theorem paved the way toward various types of Lipschitz extension theorems for Banach space-valued Lipschitz maps, presently a topic of active research in geometric functional analysis. In addition, it has recently been pivotally used in the literature on machine learning and metric data analysis.

The primary objective of this note is to understand to what extent a Nachbin-type generalization of the McShane–Whitney theorem is possible. To state our query precisely, consider a 1-Lipschitz real-valued function f on a subset S of a metric space X. Now suppose X is endowed with a partial order $\succcurlyeq$, and that f is order preserving (in the sense that $f(x) \geq f(y)$ for every $x, y \in S$ with $x \succcurlyeq y$). The problem is to determine under what conditions (that do not depend on S and f) one can extend f to an order-preserving 1-Lipschitz map on X. Our main result (Theorem 3.1, below) says that this is possible if, and only if, $\succcurlyeq$ satisfies a rather demanding condition, which we call *radiality*. (We actually prove a slightly more general result that covers $\ell_\infty(T)$ valued-maps as well, for any nonempty set T.) Radiality of $\succcurlyeq$ demands that if $x \succcurlyeq y$ while $z \succcurlyeq y$ does not hold, then the distance between x and z is larger than that between y and z (and similarly for the case where not $y \succcurlyeq x$ but $y \succcurlyeq z$). While it is obviously strong, this condition is necessary for the sought monotonic Lipschitz extension theorem. Moreover, when $\succcurlyeq$ is total, it reduces to *radial convexity*, which is commonly used in the field of topological order theory. Finally, the equality ordering is radial, so our extension theorem generalizes the McShane–Whitney extension theorem (just like Nachbin's theorem generalizes the Tietze extension theorem).

We also present some applications of our monotonic Lipschitz extension theorem. First, we show that every radial partial order on a (compact) metric space can be represented by means of a (compact) collection of order-preserving Lipschitz functions. An immediate corollary of this is that every radial partial order is closed. Second, we prove that on any radial partially ordered σ-compact metric space X, there is a strictly increasing Lipschitz map F (in the sense that $F(x) > F(y)$ for every distinct $x, y \in X$ with $x \succcurlyeq y$). Finally, we combine our extension theorem with the recent remetrization approach introduced by Beer [4] to show that if $\succcurlyeq$ is a radial partial order on a metric space X, then any bounded (or more generally, Lipschitz for large distances), order-preserving, and uniformly continuous map on a subset of X can be extended to a function on the entire space in such a way that it remains order preserving and uniformly continuous. Radiality can actually be relaxed substantially in this result, but characterizing those metric posets on which such an extension is possible is presently an open problem.

2. Preliminaries

2.1. Posets. Let X be a nonempty set. A *preorder* on X is a reflexive and transitive binary relation on X, while a *partial order* on X is an antisymmetric preorder on X. We refer to the ordered pair $(X, \succcurlyeq)$ as a *poset* if $\succcurlyeq$ is a partial order on X. (In this context, X is called the *carrier* of the poset.) A preorder on X is *total* if any two elements x and y of X are *$\succcurlyeq$-comparable*, that is, either $x \succcurlyeq y$ or $y \succcurlyeq x$ holds. A total partial order on X is said to be a *linear order* on X; in this case, we refer to $(X, \succcurlyeq)$ as a *loset*.

Let $(X, \succcurlyeq)$ be a poset. For any $x \in X$, we define $x^{\downarrow} := \{z \in X : x \succcurlyeq z\}$ and $x^{\uparrow} := \{z \in X : z \succcurlyeq x\}$. (A set of the former type is said to be a *principal ideal* in $(X, \succcurlyeq)$, and one of the latter type is called a *principal filter* in $(X, \succcurlyeq)$.) In turn, for any $S \subseteq X$,

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we define the *$\succcurlyeq$-decreasing closure* of S as $S^{\downarrow} := \bigcup_{x\in S} x^{\downarrow}$, and define the *$\succcurlyeq$-increasing closure* $S^{\uparrow}$ of S dually. In turn, S is said to be *$\succcurlyeq$-decreasing* if $S = S^{\downarrow}$ and *$\succcurlyeq$-increasing* if $S = S^{\uparrow}$.

Given any poset $(X, \succcurlyeq)$, we denote the asymmetric part of $\succcurlyeq$ by $\succ$, that is, $x \succ y$ means $y \neq x \succcurlyeq y$. We also define the binary relation $\succcurlyeq^{\bullet}$ on X as

$$x \succcurlyeq^{\bullet} y \quad \text{if and only if not } y \succcurlyeq x.$$

Thus, $x \succcurlyeq^{\bullet} y$ means that either $x \succ y$, or x and y are not $\succcurlyeq$-comparable. It is plain that $\succcurlyeq^{\bullet}$ is an irreflexive relation. In general, this relation is neither symmetric nor asymmetric, nor is it transitive. When $\succcurlyeq$ is total, however, $\succcurlyeq^{\bullet}$ equals $\succ$.

A function $f : X \to Y$ from a poset $X = (X, \succcurlyeq)$ to a poset $Y = (Y, \trianglerighteq)$ is said to be *order preserving* if for every $x, y \in X$,

$$x \succcurlyeq y \quad \text{implies } f(x) \trianglerighteq f(y).$$

If Y is $(\mathbb{R}, \geq)$, where $\geq$ is the usual order, we refer to f simply as *$\succcurlyeq$-increasing*. Note that the indicator function of any $\succcurlyeq$-increasing subset of X is an $\succcurlyeq$-increasing map.

2.2. Normally ordered topological posets. A *topological poset* is an ordered pair $(X, \succcurlyeq)$ where X is a topological space and $\succcurlyeq$ is a partial order on X such that $\succcurlyeq$ is closed in $X \times X$ (relative to the product topology). In turn, a *normally ordered topological space* is an ordered pair $(X, \succcurlyeq)$, where X is a topological space and $\succcurlyeq$ is a partial order on X such that for every pair of disjoint closed subsets A and B such that A is $\succcurlyeq$-decreasing and B is $\succcurlyeq$-increasing, there exist disjoint open subsets O and U of X such that O is $\succcurlyeq$-decreasing and contains A, and U is $\succcurlyeq$-increasing and contains B. If, in addition, $\succcurlyeq$ is closed in $X \times X$, we refer to $(X, \succcurlyeq)$ as a *normally ordered topological poset.*

In his seminal work, Nachbin [27] studied such spaces and obtained the following generalization of the classical Tietze extension theorem.

THE NACHBIN EXTENSION THEOREM. *Let $(X, \succcurlyeq)$ be a normally ordered topological poset. Then for every compact subset S of X, and every $\succcurlyeq$-increasing and continuous $f : S \to \mathbb{R}$, there is an $\succcurlyeq$-increasing and continuous $F : X \to \mathbb{R}$ with $F|_S = f$.*

This is a truly powerful extension theorem, which holds true also when $\succcurlyeq$ is not antisymmetric (see [25]). It is used extensively in decision theory; see, for instance, [8, 14, 15, 29], and references cited therein. We should also emphasize that a similar order-theoretic generalization of the Gillman–Jerison theorem on the characterization of C^*-embeddings was recently obtained by Yamazaki [33].

2.3. Partially ordered metric spaces. A *partially* (respectively, *linearly*) *ordered metric space* is an ordered triplet $(X, d, \succcurlyeq)$ such that (X, d) is a metric space and $(X, \succcurlyeq)$ is a poset (respectively, loset). If, in addition, $\succcurlyeq$ is a closed subset of $X \times X$, we refer to $(X, d, \succcurlyeq)$ as a *metric poset* (respectively, *metric loset*).

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A partially ordered metric space $(X, d, \succcurlyeq)$ is said to be *radially convex* (or the partial order $\succcurlyeq$ on (X, d) is *radially convex*) if

$$x \succ y \succ z \quad \text{implies} \quad d(x, z) \geq \max\{d(x, y), d(y, z)\} \tag{2-1}$$

for every $x, y, z \in X$. This concept builds an appealing connection between the order and metric structures to be imposed on a given set. Indeed, such partially ordered metric spaces have received some attention in topological order theory (see [5, 10, 30], among many others), and are often used in the topological analysis of smooth dendroids (see [17]).

In what follows, we need to work with a strengthening of radial convexity. We say that a partially ordered metric space $(X, d, \succcurlyeq)$ is *radial* (or that the partial order $\succcurlyeq$ on (X, d) is *radial*) if

$$x \succcurlyeq^{\bullet} y \succ z \quad \text{implies} \quad d(x, z) \geq d(x, y) \tag{2-2}$$

and

$$x \succ y \succcurlyeq^{\bullet} z \quad \text{implies} \quad d(x, z) \geq d(y, z). \tag{2-3}$$

While radiality is more demanding than radial convexity, these concepts coincide when the partial order at hand is total.

LEMMA 2.1. *A linearly ordered metric space is radial if and only if it is radially convex.*

Indeed, if $(X, d, \succcurlyeq)$ is a linearly ordered metric space, then $\succcurlyeq^{\bullet} = \succ$ by definition of $\succcurlyeq^{\bullet}$, so in this case, (2-1) is equivalent to (2-2) and (2-3) put together.

2.4. Examples of radial metric posets. If we order and metrize any nonempty subset of $\mathbb{R}$ in the usual way, we obtain a radial metric loset. In addition, it is plain that every partially ordered discrete metric space is radial, and the equality relation on any metric space is radial. However, easy examples show that ordering $\mathbb{R}^2$ coordinate-wise and endowing it with the Euclidean metric yields a radially convex metric poset that is not radial.

Before proceeding further, we present a few more examples.

EXAMPLE 2.2. Consider the poset $(X, \succcurlyeq)$ where $X := \{x_1, x_2, x_3, x_4\}$, $x_1 \succ x_2 \succ x_4$, $x_1 \succ x_3 \succ x_4$, and x_2 and x_3 are not $\succcurlyeq$-comparable. (This poset is isomorphic to $(2^S, \supseteq)$ for any doubleton S.) For any $a, b \in (0, 1)$, define $d_{a,b} : X \times X \to [0, 1]$ by the matrix

$$\begin{bmatrix} 0 & a & a & 1 \\ a & 0 & b & 1-a \\ a & b & 0 & 1-a \\ 1 & 1-a & 1-a & 0 \end{bmatrix}$$

whose ijth term is $d_{ab}(x_i, x_j), i, j = 1, \ldots, 4$. Then, $d_{a,b}$ is a metric on X if and only if $\min\{a, 1-a\} \geq \frac{1}{2}b$. In fact, under this parametric restriction, $(X, d_{a,b}, \succcurlyeq)$ is a radially convex metric poset. In addition, if $1 - a < b < a$, this metric poset

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satisfies condition (2-3), but not (2-2), while if $a < b < 1 - a$, then the opposite situation ensues. (In particular, this shows that there is no redundancy in our definition of radiality.) Consequently, $(X, d_{a,b}, \succcurlyeq)$ is a radial metric poset, provided that $\min\{a, 1 - a\} \geq b$.

EXAMPLE 2.3. Let T be a tree with a finite set X of vertices and root $x_0 \in X$. The *path-metric* on X (induced by T) is defined as

$$\rho_T(x, y) := \text{the length of the path between } x \text{ and } y \text{ in } T.$$

(Since T is a tree, there is a unique path between any two of its vertices.) We define $d_T : X \times X \to \{0, 1, 2\}$ by setting $d_T(x, y) := \min\{\rho_T(x, y), 2\}$ if x and y are on the same path whose one endpoint is x_0, and $d_T(x, y) := 1$ otherwise. It is readily checked that d_T is a metric on X. Finally, we define the partial order $\succcurlyeq$ on X by

$$x \succcurlyeq y \quad \text{if and only if} \quad y \text{ is on the path between } x_0 \text{ and } x.$$

Then, $(X, \rho_T, \succcurlyeq)$ is a radially convex metric poset (which need not be radial), while $(X, d_T, \succcurlyeq)$ is a radial metric poset.

EXAMPLE 2.4. Let A and B be two disjoint bounded subsets of a metric space (Y, d). Let $\succcurlyeq_A$ and $\succcurlyeq_B$ be radially convex linear orders on (A, d) and (B, d), respectively. Let $\succcurlyeq$ be the disjoint sum of $\succcurlyeq_A$ and $\succcurlyeq_B$, that is, $\succcurlyeq$ is the partial order on $X := A \sqcup B$ with $x \succcurlyeq y$ if and only if either $x \succcurlyeq_A y$ or $x \succcurlyeq_B y$.

Now pick any number $\theta \geq \max\{\mathrm{diam}(A), \mathrm{diam}\,(B)\}$, and consider the function $D : X \times X \to [0, \infty)$ with

$$D(x, y) := \begin{cases} d(x, y) & \text{if } (x, y) \in A^2 \text{ or } (x, y) \in B^2, \\ \frac{1}{2}\theta & \text{otherwise.} \end{cases}$$

It is easily checked that D is a metric on X. In fact, $(X, D, \succcurlyeq)$ is a radial partially ordered metric space.

EXAMPLE 2.5. Let I stand for the unit interval $[0, 1]$, and take any set J that does not intersect I. Define the partial order on $X := I \sqcup J$ with $x \succcurlyeq y$ if and only if either $(x, y) \in J \times I$ or $\{x, y\} \subseteq I$ and $x \geq y$. (In other words, $\succcurlyeq$ agrees with the usual order on I, and puts anything in J above all numbers in I. No two distinct elements of J are $\succcurlyeq$-comparable.) Define $d : X \times X \to [0, \infty)$ as follows: (i) $d|_{I\times I}$ is the absolute value metric on I; (ii) $d|_{J\times J}$ is the discrete metric on J; (iii) $d(x, y) := 1 + y$ if $(x, y) \in J \times I$; and (iv) $d(x, y) := 1 + x$ if $(x, y) \in I \times J$. Then, $(X, d, \succcurlyeq)$ is a radial partially ordered metric space.

In passing, we note that it may be a mistake to think of the radiality property as prohibitively strong. In the context of metric data analysis and machine learning (see [9, 13, 16]), one often works with finite metric spaces or metric graphs (relative to which the Lipschitz extension problems are by no means trivial). As shown by Examples 2.2 and 2.3 above, the radiality property may turn out to be considerably less demanding in those sorts of environments.

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2.5. Lipschitz functions. For any real number $K \geq 0$, a function $f : X \to Y$ from a partially ordered metric space $X = (X, d_X, \succcurlyeq_X)$ to a partially ordered metric space $Y = (Y, d_Y, \succcurlyeq_Y)$ is said to be *K-Lipschitz* if for every $x, y \in X$,

$$d_Y(f(x), f(y)) \leq K d_X(x, y). \tag{2-4}$$

We say that f is *Lipschitz* if it is K-Lipschitz for some $K \geq 0$. The smallest $K \geq 0$ such that (2-4) holds for every $x, y \in X$, is called the *Lipschitz constant* of f. For excellent treatments of the general theory of Lipschitz functions, see [11, 31].

We denote the set of all K-Lipschitz maps from X to Y as $\mathrm{Lip}_K(X, Y)$, but write $\mathrm{Lip}_K(X)$ for $\mathrm{Lip}_K(X, \mathbb{R})$. In turn, the sets of all order-preserving members of $\mathrm{Lip}_K(X, Y)$ and $\mathrm{Lip}_K(X)$ are denoted as $\mathrm{Lip}_{K,\uparrow}(X, Y)$ and $\mathrm{Lip}_{K,\uparrow}(X)$, respectively. Throughout this note, we consider these as metric spaces relative to the uniform metric. This makes these spaces complete, but, in general, not separable.

2.6. The monotone Lipschitz extension property. We say that a partially ordered metric space $(X, d, \succcurlyeq)$ has the *monotone Lipschitz extension property* if for every nonempty $S \subseteq X$, every $K > 0$, and $f \in \mathrm{Lip}_{K,\uparrow}(S)$, there exists an $F \in \mathrm{Lip}_{K,\uparrow}(X)$ with $F|_S = f$. In this terminology, the classical *McShane–Whitney extension theorem* can be viewed as saying that $(X, d, =)$ has the monotone Lipschitz extension property. Our primary objective in this note is to see exactly to what extent we can replace $=$ with a partial order on X in this statement.

REMARK 2.6. When $(X, d, \succcurlyeq)$ has the monotone Lipschitz extension property, we can always ensure the achieved extension has the same range as the function to be extended, provided that the range of the function is closed. To see this, take any $F \in \mathrm{Lip}_{K,\uparrow}(X)$ and $S \subseteq X$, and assume $F(S)$ is closed. Where $m := \inf_{x \in S} F(x)$ and $M := \sup_{x \in S} F(x)$, the map $G : X \to [m, M]$ defined by

$$G(x) := \max\{\min\{F(x), M\}, m\},$$

is an $\succcurlyeq$-increasing K-Lipschitz map with $G|_S = F|_S$.

3. Monotone Lipschitz extensions

Unless a partially ordered metric space is totally ordered, or it is finite, its radiality seems like a fairly demanding condition. Nevertheless, our main finding in this note shows that this condition is necessary and sufficient for any such space to possess the monotone Lipschitz extension property.

THEOREM 3.1. *A partially ordered metric space $(X, d, \succcurlyeq)$ has the monotone Lipschitz extension property if and only if it is radial.*

PROOF. Suppose $(X, d, \succcurlyeq)$ is not radial. Then, there exist three points x, y, z in X such that either

$$x \succcurlyeq^{\bullet} y \succ z \quad \text{and} \quad d(x, z) < d(x, y), \tag{3-1}$$

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or

$$x \succ y \succcurlyeq^{\bullet} z \quad \text{and} \quad d(x,z) < d(y,z). \tag{3-2}$$

Assume first the case (3-1), set $S := \{x, y\}$, and define $f : S \to \mathbb{R}$ by $f(x) := d(x, y)$ and $f(y) := 0$. Then, $f \in \mathrm{Lip}_{1,\uparrow}(S)$, but for any 1-Lipschitz extension $F : X \to \mathbb{R}$ of f,

$$F(z) \geq F(x) - d(x,z) > f(x) - d(x,y) = 0 = F(y),$$

which means F is not $\succcurlyeq$-increasing. If, however, (3-2) holds, we set $S := \{y, z\}$, and define $f : S \to \mathbb{R}$ by $f(y) := d(y, z)$ and $f(z) := 0$. Then, $f \in \mathrm{Lip}_{1,\uparrow}(S)$, but for any 1-Lipschitz extension $F : X \to \mathbb{R}$ of f,

$$F(x) \leq F(z) + d(x,z) < d(y,z) = F(y),$$

which means F is not $\succcurlyeq$-increasing. This proves the necessity part of the assertion. The sufficiency part is a special case of a more general result that we establish below. □

There does not seem to be an easy way of getting around the radiality requirement for the monotonic Lipschitz extension problem. For a partially ordered metric space $(X, d, \succcurlyeq)$ that is not radial, the argument above shows that it may not be possible to extend an $\succcurlyeq$-increasing 1-Lipschitz map on a compact and $\succcurlyeq$-increasing (or $\succcurlyeq$-decreasing) set $S \subseteq X$ to an $\succcurlyeq$-increasing 1-Lipschitz map on X.

Setting $\succcurlyeq$ as the equality relation in Theorem 3.1 yields the classical McShane–Whitney extension theorem. The following is another straightforward corollary.

COROLLARY 3.2. *A linearly ordered metric space $(X, d, \succcurlyeq)$ has the monotone Lipschitz extension property if and only if it is radially convex.*

REMARK 3.3. It was shown by Mehta [23] that every topological loset $(X, \succcurlyeq)$ is a normally ordered topological space. Therefore, specializing the Nachbin extension theorem to the context of metric spaces, we find: *Given any metric loset $(X, d, \succcurlyeq)$ and any compact $S \subseteq X$, every $\succcurlyeq$-increasing $f \in C(S)$ extends to an $\succcurlyeq$-increasing $F \in C(X)$*. Corollary 3.2 can be thought of as the reflection of this result in the context of Lipschitz functions. It says that if we add radial convexity to its hypotheses, we get an order-preserving Lipschitz extension of any order-preserving Lipschitz function defined on any (possibly noncompact) subset of X.

The Lipschitz extension problem for Banach space-valued maps on a metric space is a rather deep one, and is the subject of ongoing research in metric space theory and geometric functional analysis. However, there is one special case of the problem that is settled by the McShane–Whitney theorem in a routine manner. This is when the Lipschitz maps to be extended take values in the Banach space $\ell_\infty(T)$ of all bounded real functions on some nonempty set T. (This generalization is of interest, because every metric space can be isometrically embedded in $\ell_\infty(T)$ for some T.) Precisely the same holds for the monotone Lipschitz extension problem as well where we consider $\ell_\infty(T)$ as partially ordered coordinate-wise. (For any $u, v \in \ell_\infty(T)$, we write

$u \geq v$ whenever $u(t) \geq v(t)$ for every $t \in T$). We now prove the sufficiency part of Theorem 3.1 in this more general context.

THEOREM 3.4. *Let $(X, d, \succcurlyeq)$ be a radial partially ordered metric space. For any $K \geq 0$, let S be a nonempty subset of X and $f : S \to \ell_\infty(T)$ an order-preserving K-Lipschitz map. Then, there exists an order-preserving K-Lipschitz map $F : X \to \ell_\infty(T)$ with $F|_S = f$.*

PROOF. We assume $S \neq X$, for otherwise, there is nothing to prove. Similarly, the claim is trivially true when $K = 0$, so we may assume $K > 0$. Moreover, it is enough to prove the assertion for $K = 1$, for then the general case obtains by applying what is established to the map $K^{-1} f$.

The following proof is patented after the typical way one proves the Hahn–Banach theorem. In the initial stage of the argument, we take an arbitrary $x \in X \backslash S$ and extend f to an order-preserving 1-Lipschitz function on $S \cup \{x\}$. To this end, consider the functions $a_x : T \to [-\infty, \infty]$ and $b_x : T \to [-\infty, \infty]$ defined as

$$a_x(t) := \sup\{f(z)(t) : z \in S \cap x^{\downarrow}\}$$

and

$$b_x(t) := \inf\{f(y)(t) : y \in S \cap x^{\uparrow}\}.$$

If $S \cap x^{\downarrow} = \emptyset$, then $a_x(t) = -\infty$ for every $t \in T$, while $S \cap x^{\uparrow} = \emptyset$ implies $b_x(t) = \infty$ for every $t \in T$. On the other hand, if both $S \cap x^{\downarrow}$ and $S \cap x^{\uparrow}$ are nonempty, monotonicity of f yields $-\infty < a_x(t) \leq b_x(t) < \infty$ for all $t \in T$. In all contingencies, then, $[a_x(t), b_x(t)]$ is a nonempty interval in the set of all extended reals.

We next define the functions $\alpha_x : T \to [-\infty, \infty]$ and $\beta_x : T \to [-\infty, \infty]$ by

$$\alpha_x(t) := \sup\{f(z)(t) - d(x, z) : z \in S\}$$

and

$$\beta_x(t) := \inf\{f(y)(t) + d(x, y) : y \in S\}.$$

(These are the McShane and Whitney extensions of f, respectively.) In this case, both $\alpha_x(t)$ and $\beta_x(t)$ are real numbers for every $t \in T$. In fact, as f is 1-Lipschitz, for every $y, z \in S$,

$$f(z)(t) - f(y)(t) \leq \|f(z) - f(y)\|_\infty \leq d(z, y) \leq d(x, y) + d(x, z),$$

whence $f(z)(t) - d(x, z) \leq f(y)(t) + d(x, y)$, for all $t \in T$. Conclusion: $-\infty < \alpha_x(t) \leq \beta_x(t) < \infty$ for all $t \in T$.

We claim that

$$\alpha_x(t) \leq b_x(t) \quad \text{and} \quad a_x(t) \leq \beta_x(t) \tag{3-3}$$

for every $t \in T$. To see this, suppose $\alpha_x(t) > b_x(t)$ for some $t \in T$. Then, there exist $y \in S \cap x^{\uparrow}$ and $z \in S$ such that $f(y)(t) < f(z)(t) - d(x, z)$. It follows that $f(y)(t) < f(z)(t)$,

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so $y \succcurlyeq z$ does not hold (because f is $\succcurlyeq$-increasing). Thus: $z \succcurlyeq^\bullet y \succ x$. Since $(X, d, \succcurlyeq)$ is radial, therefore, $d(x, z) \geq d(y, z)$. This entails

$$f(y)(t) < f(z)(t) - d(x, z) \leq f(z)(t) - d(y, z),$$

and hence, $\|f(z) - f(y)\|_\infty \geq f(z)(t) - f(y)(t) > d(z, y)$, contradicting f being 1-Lipschitz. We conclude that $\alpha_x(t) \leq b_x(t)$ for all $t \in T$, as claimed. The second inequality in (3-3) is established analogously.

In view of these observations, we conclude that the intervals $[a_x(t), b_x(t)]$ and $[\alpha_x(t), \beta_x(t)]$ overlap for every $t \in T$. We define $F : S \cup \{x\} \to \ell_\infty(T)$ as

$$F(w)(t) := \begin{cases} f(w)(t) & \text{if } w \in S, \\ \theta(t) & \text{if } w = x, \end{cases}$$

where $\theta(t)$ is an arbitrarily picked real number in $[a_x(t), b_x(t)] \cap [\alpha_x(t), \beta_x(t)]$ for any $t \in T$. Then, F is 1-Lipschitz, because for any $y \in S$,

$$f(y)(t) - d(x, y) \leq \alpha_x(t) \leq F(x)(t) \leq \beta_x(t) \leq f(y)(t) + d(x, y),$$

and hence $|F(x)(t) - F(y)(t)| \leq d(x, y)$ for all $t \in T$, that is, $\|F(y) - F(x)\|_\infty \leq d(x, y)$. In addition, for every $y \in S$ with $y \succcurlyeq x$, we have $f(y)(t) \geq b_x(t) \geq F(x)(t)$, and similarly, for every $z \in S$ with $x \succcurlyeq z$, we have $f(x)(t) \geq a_x(t) \geq F(z)(t)$, for all $t \in T$. Thus, F is order-preserving as well.

The proof is completed by a standard transfinite induction argument. Let $\mathcal{F}$ stand for the set of all (A, F) such that $S \subseteq A \subseteq X$ and $F \in \mathrm{Lip}_{1,\uparrow}(A, \ell_\infty(T))$ with $F|_S = f$. Since it includes (S, f), this collection is not empty. In addition, it is easily verified that $(\mathcal{F}, \trianglerighteq)$ is an inductive poset where $(A, F) \trianglerighteq (B, G)$ if and only if $A \supseteq B$ and $F|_B = G$. So, by Zorn's lemma, there is a $\trianglerighteq$-maximal element (A, F) in $\mathcal{F}$. In view of the first part of the proof, we must have $A = X$. □

REMARK 3.5. By setting $\theta(t) := \max\{a_x(t), \alpha_x(t)\}$ for all $t \in T$ in the proof above, and modifying the transfinite induction part of the proof in the obvious way, we find that there is a smallest order-preserving K-Lipschitz map $F : X \to \ell_\infty(T)$ with $F|_S = f$ in the context of Theorem 3.4. That there is also a largest such F is established analogously.

REMARK 3.6. There are various generalizations of the Lipschitz property, and the construction above adapts to some of these. To wit, Miculescu [24] considers (K, g)-Lipschitz functions that are functions f from a metric space (X, d_X) to another metric space (Y, d_Y) such that $d_X(f(x), f(y)) \leq K d_Y(g(x), g(y))$ for every $x, y \in X$. Theorems 3.1 and 3.4 may be modified in the obvious way to account for such functions as well.

REMARK 3.7. Given Theorem 3.1, it is natural to inquire if the monotonic Lipschitz extensions of real functions can be carried out locally. To state the problem, we recall that a real map on a metric space $X = (X, d)$ is called *pointwise Lipschitz* if for every $y \in X$, there exist $K_y \geq 0$ and $\delta_y > 0$ such that $|f(x) - f(y)| \leq K_y d(x, y)$ for all

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$x \in X$ with $d(x, y) < \delta_y$. The question is: if $(X, d, \succcurlyeq)$ is a radial partially ordered metric space, S a nonempty closed subset of X, and $f : S \to \mathbb{R}$ is an $\succcurlyeq$-increasing pointwise Lipschitz map, does there exist an $\succcurlyeq$-increasing pointwise Lipschitz map $F : X \to \mathbb{R}$ with $F|_S = f$? If $\succcurlyeq$ is the equality relation, the answer is known to be yes; see, for instance, [12, 19]. The first part of the proof above also adapts to show that the answer is yes so long as we add only finitely many points in the extension. That is, minor modifications of that part of the proof yield the following fact.

Let $(X, d, \succcurlyeq)$ be a radial partially ordered metric space and S a nonempty closed subset of X with $|X\backslash S| < \infty$. Then, every $\succcurlyeq$-increasing pointwise Lipschitz map on S can be extended to an $\succcurlyeq$-increasing pointwise Lipschitz map on X.

Unfortunately, the transfinite inductive step of the proof above fails to deliver this result without the requirement $|X\backslash S| < \infty$.

4. Functional representations of radial orders

For any nonempty X, $\mathcal{F} \subseteq \mathbb{R}^X$, and $x, y \in X$, we write $\mathcal{F}(x) \geq \mathcal{F}(y)$ to mean $f(x) \geq f(y)$ for every $f \in \mathcal{F}$. For any such collection $\mathcal{F}$, the binary relation $\gtrsim$ on X defined by $x \gtrsim y$ if and only if $\mathcal{F}(x) \geq \mathcal{F}(y)$, is a preorder on X. Conversely, for every preorder $\gtrsim$ on X, there is a family $\mathcal{F}$ with $x \gtrsim y$ if and only if $\mathcal{F}(x) \geq \mathcal{F}(y)$ for every $x, y \in X$. (For any $z \in X$, define $f_z : X \to \{0, 1\}$ by $f_z(x) := 1$ if $x \gtrsim z$ and $f_z(x) := 0$ otherwise. The claim follows by setting $\mathcal{F} := \{f_z : z \in X\}$.) In this case, we say that $\mathcal{F}$ *represents* $\gtrsim$. In several applied mathematical fields, such as decision theory and the theory of optimal transportation, it is important to determine the structure of the families of real functions that may represent a given preorder in this sense.

As an easy consequence of Theorem 3.1, we find that any radial partial order on any metric space can be represented by a family of order-preserving 1-Lipschitz real-valued functions.

PROPOSITION 4.1. *Let $(X, d, \succcurlyeq)$ be a radial partially ordered metric space. Then, there exists an $\mathcal{F} \subseteq \mathrm{Lip}_{1,\uparrow}(X)$ that represents $\succcurlyeq$. If (X, d) is compact, we can choose $\mathcal{F}$ in such a way that it is compact and* $\sup_{F\in\mathcal{F}} \|F\|_\infty \leq \mathrm{diam}(X)$.

PROOF. Assume $|X| > 1$, which implies $\succcurlyeq^\bullet \neq \varnothing$, for otherwise, there is nothing to prove. For any $x, y \in X$ with $x \succcurlyeq^\bullet y$, define $f_{x,y} \in \mathbb{R}^{\{x,y\}}$ by $f_{x,y}(x) := d(x, y)$ and $f_{x,y}(y) := 0$, and note that $f \in \mathrm{Lip}_{1,\uparrow}(\{x, y\})$. We apply Theorem 3.1 to extend $f_{x,y}$ to an $\succcurlyeq$-increasing 1-Lipschitz real-valued map $F_{x,y}$ on X. Next, define $\mathcal{F} := \{F_{x,y} : x \succcurlyeq^\bullet y\}$. Then, $x \succcurlyeq y$ implies $F(x) \geq F(y)$ for all $F \in \mathcal{F}$ simply because every member of $\mathcal{F}$ is $\succcurlyeq$-increasing. Conversely, if $x \succcurlyeq y$ does not hold, we have $F(x) < F(y)$ for some $F \in \mathcal{F}$, namely, $F = F_{y,x}$.

Now suppose (X, d) is compact. Put $K := \mathrm{diam}(X)$, and note that $K \in (0, \infty)$. Next, for any fixed $e \in X$, define

$$\mathcal{G} := \left\{\frac{1}{K}(F - F(e)) : F \in \mathcal{F}\right\}.$$

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Then, $|G(x)| = |G(x) - G(e)| \leq K^{-1}d(x, e) \leq 1$ for every $x \in X$, so $\|G\|_\infty \leq 1$, for every $G \in \mathcal{G}$. Moreover, $\mathcal{G} \subseteq \mathrm{Lip}_{\,1/K,\uparrow}(X)$ and $\mathcal{G}$ represents $\succcurlyeq$. Then, $\mathcal{H} := K\,\mathrm{cl}(\mathcal{G})$ is a closed and bounded set of 1-Lipschitz bounded functions that represents $\succcurlyeq$. Since any subset of $\mathrm{Lip}_1(X)$ is equicontinuous, applying the Arzelà–Ascoli theorem yields the second claim of the proposition. □

As an immediate consequence of Proposition 4.1, we obtain the somewhat surprising fact that every radial partially ordered metric space is, per force, a metric poset.

COROLLARY 4.2. *Every radial partial order on a metric space X is a closed subset of $X \times X$.*

The concepts of 'radial partially ordered metric space' and 'radial metric poset' are thus identical. We adopt the latter terminology in the remainder of the paper.

We next apply our main extension theorem to show that a radially convex linear order on a σ-compact metric space can be represented by a Lipschitz function. The main ingredient of the argument is contained in the following observation.

LEMMA 4.3. *Let $(X, d, \succcurlyeq)$ be a radial metric poset. Then, for any compact subset S of X, there is a $G \in \mathrm{Lip}_{1,\uparrow}(X)$ such that $\|G\|_\infty \leq \mathrm{diam}(S)$ and*

$$G(x) > G(y) \quad \textit{for every } x, y \in S \textit{ with } x \succ y.$$

PROOF. Take any compact $S \subseteq X$, and use Proposition 4.1 to find a compact, and hence, separable, $\mathcal{F} \subseteq \mathrm{Lip}_{1,\uparrow}(S)$ such that (i) $\sup_{F\in\mathcal{F}} \|F\|_\infty \leq \mathrm{diam}(S)$ and (ii) $x \succcurlyeq y$ if and only if $\mathcal{F}(x) \geq \mathcal{F}(y)$ for every $x, y \in X$. Let (F_m) be a sequence in $\mathcal{F}$ such that $\{F_1, F_2, \ldots\}$ is dense in $\mathcal{F}$. We define $G := \sum_{n\geq 1} 2^{-n}F_n$. It is readily checked that $G \in \mathrm{Lip}_{1,\uparrow}(S)$. In addition, if $x, y \in S$ satisfy $x \succ y$, then $F(x) > F(y)$ for some $F \in \mathcal{F}$ (because $\mathcal{F}$ represents $\succcurlyeq$). Consequently, since $\{F_1, F_2, \ldots\}$ is dense in $\mathcal{F}$ relative to the uniform metric, there exists an $n \in \mathbb{N}$ with $F_n(x) > F_n(y)$, which implies $G(x) > G(y)$. To complete the proof, we extend G to X by using Theorem 3.1, and recall Remark 2.6. □

THEOREM 4.4. *Let $(X, d, \succcurlyeq)$ be a radial metric poset such that (X, d) is σ-compact. Then, there is a Lipschitz function $F : X \to \mathbb{R}$ with*

$$F(x) > F(y) \quad \textit{for every } x, y \in S \textit{ with } x \succ y. \tag{4-1}$$

PROOF. By hypothesis, there exists a sequence (S_m) of compact subsets of X such that $S_1 \subseteq S_2 \subseteq \cdots$ and $S_1 \cup S_2 \cup \cdots = X$. We may assume $|S_1| > 1$. Put $K_n := \mathrm{diam}$ (S_n), and note that $K_n \in (0, \infty)$ for each n. By Lemma 4.3, for every $n \in \mathbb{N}$, there is a $G_n \in \mathrm{Lip}_{1,\uparrow}(X)$ such that $\|G_n\|_\infty \leq K_n$ and $G_n(x) > G_n(y)$ for every $x, y \in S_n$ with $x \succ y$. We define $F \in \mathbb{R}^X$ by $F(x) := \sum_{n\geq 1} 2^{-n}K_n^{-1}G_n$. It is plain that $F(x) > F(y)$ for every $x, y \in X$ with $x \succ y$. Moreover, F is K_1^{-1}-Lipschitz. Indeed, for any $x, y \in X$,

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$$|F(x) - F(y)| \leq \sum_{n\geq 1} \frac{1}{2^n K_n} |G_n(x) - G_n(y)| \leq \sum_{n\geq 1} \frac{1}{2^n K_1} d(x, y) \leq \frac{1}{K_1} d(x, y)$$

since $K_1 \leq K_n$ for each n. □

COROLLARY 4.5. *Let $(X, d, \succcurlyeq)$ be a radially convex metric loset such that (X, d) is σ-compact. Then, there is a Lipschitz function $F : X \to \mathbb{R}$ with*

$$x \succcurlyeq y \quad \textit{if and only if} \quad F(x) \geq F(y)$$

for every $x, y \in X$.

This result has the flavor of the continuous utility representation theorems of decision theory. Indeed, it provides a rather easy proof of the following well-known result of that literature.

COROLLARY 4.6. *Let $\succsim$ be a closed total preorder on a compact metric space $X = (X, d)$. Then, there exists a continuous map $u : X \to \mathbb{R}$ such that*

$$x \succsim y \quad \textit{if and only if} \quad u(x) \geq u(y)$$

for every $x, y \in X$.

PROOF. Define $\mathbf{x} := \{y \in X : x \succsim y \succsim x\}$ for any $x \in X$, and note that $\mathcal{X} := \{\mathbf{x} : x \in X\}$ is a partition of X. Then, the binary relation $\succcurlyeq \subseteq \mathcal{X} \times \mathcal{X}$ defined by $\mathbf{x} \succcurlyeq \mathbf{y}$ if and only if $x \succsim y$, is a partial order on $\mathcal{X}$. Let H_d stand for the Hausdorff metric on $\mathcal{X}$. Then, $(\mathcal{X}, H_d, \succcurlyeq)$ is a compact metric loset. By the Carruth metrization theorem (of [10]), there exists a metric D on $\mathcal{X}$ such that H_d and D are equivalent, and $D(\mathbf{x}, \mathbf{z}) = D(\mathbf{x}, \mathbf{y}) + D(\mathbf{y}, \mathbf{z})$ for every $x, y, z \in X$ with $x \succ y \succ z$. We may thus apply Corollary 4.5 to obtain an $\succcurlyeq$-increasing and 1-Lipschitz F map on $(\mathcal{X}, D, \succcurlyeq)$ such that $\mathbf{x} \succcurlyeq \mathbf{y}$ if and only if $F(\mathbf{x}) \geq F(\mathbf{y})$ for every $x, y \in X$. The map $u : X \to \mathbb{R}$ with $u(x) := F(\mathbf{x})$ fulfills the requirements of the assertion. □

5. Monotone uniformly continuous extensions

5.1. A Monotone version of McShane's uniformly continuous extension theorem. As another application of Theorem 3.1, we prove a uniformly continuous extension theorem in the context of radial metric posets. A special case of this theorem corresponds to the monotonic version of McShane's famous uniformly continuous extension theorem for bounded functions.

For any metric spaces $X = (X, d_X)$ and $Y = (Y, d_Y)$, a function $f : X \to Y$ is said to be *Lipschitz for large distances* if for every $\delta > 0$, there is a $K_\delta > 0$ such that $d_Y(f(x), f(y)) \leq K_\delta d_X(x, y)$ whenever $d_X(x, y) \geq \delta$. This concept often arises with extension and approximation problems concerning uniformly continuous functions; see, for instance, [7, 18, 21]. In fact, a basic result of this literature says that every uniformly continuous map on a Menger-convex metric space is, per force, Lipschitz for large distances (see [7, Proposition 1.11]).

We need to make two observations about real-valued functions that are Lipschitz for large distances. The first one is basic, and was noted explicitly in [18].

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LEMMA 5.1. *Every bounded real-valued function on a metric space is Lipschitz for large distances.*

PROOF. For any bounded real-valued function f on a metric space $X = (X, d)$, and $\delta > 0$,

$$|f(x) - f(y)| \le \left(\frac{2}{\delta}\|f\|_\infty\right) d(x, y)$$

for all $x, y \in X$ with $d(x, y) \ge \delta$. □

Our second observation provides a characterization of uniformly continuous real-valued maps that are Lipschitz for large distances. This characterization seems new, but we should note that Beer and Rice [6] work out several related results. In the statement of the result, and henceforth, ω_f stands for the *modulus of continuity* of any given real-valued function f on $X = (X, d)$, that is, $\omega_f : [0, \infty) \to [0, \infty]$ is the function defined by

$$\omega_f(t) := \sup\{|f(x) - f(y)| : x, y \in X \text{ and } d(x, y) \le t\}.$$

LEMMA 5.2. *Let $X = (X, d)$ be a metric space and $f \in UC(X)$. Then, f is Lipschitz for large distances if and only if there exist nonnegative real numbers a and b such that $\omega_f(t) \le at + b$ for every $t \ge 0$.*

PROOF. For any $a, b \in \mathbb{R}$, let $h_{a,b}$ denote the map $t \mapsto at + b$ on $[0, \infty)$. Suppose first that $\omega_f \le h_{a,b}$ for some $a, b \ge 0$. Then, for any $\delta > 0$, setting $K_\delta := a + b/\delta$ yields

$$|f(x) - f(y)| \le \omega_f(d(x, y)) \le ad(x, y) + b \le ad(x, y) + b\left(\frac{d(x, y)}{\delta}\right) = K_\delta d(x, y)$$

for every $x, y \in X$ with $d(x, y) \ge \delta$. Conversely, suppose f is Lipschitz for large distances. Note first that uniform continuity of f entails that there is a $\delta > 0$ with $\omega_f(\delta) \le 1$. In turn, by the Lipschitz property of f, there exists a $K := K_\delta > 0$ such that

$$|f(x) - f(y)| \le Kd(x, y) \quad \text{for all } x, y \in X \text{ with } d(x, y) \ge \delta.$$

We wish to show that $\omega_f \le h_{K,1}$. To this end, fix an arbitrary $t \ge 0$ and take any $x, y \in X$ with $d(x, y) \le t$. If $d(x, y) < \delta$, then $|f(x) - f(y)| \le \omega_f(\delta) \le 1 \le h_{K,1}(t)$. Otherwise, $|f(x) - f(y)| \le Kd(x, y) \le Kt \le h_{K,1}(t)$. Conclusion: $|f(x) - f(y)| \le h_{K,1}(t)$ for any $x, y \in X$ with $d(x, y) \le t$. Taking the sup over all such x and y yields $\omega_f(t) \le h_{K,1}(t)$. □

A map $f : X \to \mathbb{R}$ for which there exist $a, b \ge 0$ with $\omega_f(t) \le at + b$ for all $t \ge 0$ is sometimes called a *function with an affine majorant*. [6] investigates such functions in detail. In fact, this concept already plays a prominent role in McShane's original article [22, page 841] where it is emphasized that when X is a normed linear space, a uniformly extendable real function on a subset of X must have an affine majorant.

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We now proceed to show that the uniformly continuous extension theorem of McShane [22] also generalizes to the context of radial metric posets. This is proved most easily by adopting the remetrization technique of Beer [4, pages 23–25], which derives the said extension from the McShane-Whitney theorem. For the sake of completeness, we provide the details of Beer's technique within the proof.

THEOREM 5.3. *Let $(X, d, \succcurlyeq)$ be a radial metric poset and S a subset of X. Then, for every $\succcurlyeq$-increasing $f \in UC(S)$ that is Lipschitz for large distances, there exists an $\succcurlyeq$-increasing $F \in UC(X)$ with $F|_S = f$.*

PROOF. Let $\mathcal{H}$ stand for the set of all increasing affine self-maps h on $[0, \infty)$ with $\omega_f \leq h$. By Lemma 5.2, $\mathcal{H} \neq \emptyset$. We may thus define the map $\varphi : [0, \infty) \to \mathbb{R}$ by $\varphi(t) := \inf_{h \in \mathcal{H}} h(t)$. Clearly, φ is an increasing and concave (hence, subadditive) self-map on $[0, \infty)$. Since f is not constant, $\varphi(t) > 0$ for some $t > 0$, so concavity of φ entails $\varphi(t) > 0$ for all $t > 0$. We claim that φ is continuous at 0 (whence $\varphi \in C([0, \infty))$) and $\varphi(0) = 0$. To prove this, take any $\varepsilon > 0$. Since f is uniformly continuous, there exists a $\delta > 0$ with $\omega_f(t) \leq \varepsilon$ for every $t \in [0, \delta)$. In turn, as f is Lipschitz for large distances, there exists a $K > 0$ such that $|f(x) - f(y)| \leq Kd(x, y)$ whenever $d(x, y) \geq \delta$. Now consider the self-map h on $[0, \infty)$ with $h(t) := Kt + \varepsilon$. Clearly, $\omega_f(t) \leq h(0) \leq h(t)$ for all $t \in [0, \delta)$, while $\omega_f(t) \leq \max\{\varepsilon, Kt\} \leq h(t)$ for all $t \geq \delta$. It follows that $h \in \mathcal{H}$. Besides, $\varphi(t) \leq Kt + \varepsilon$ for all $t \geq 0$, which implies $\inf_{t>0} \varphi(t) \leq \varepsilon$. In view of the arbitrary choice of ε, we conclude that $\inf_{t>0} \varphi(t) = 0 = \varphi(0)$.

With these preparations in place, we now turn to the task at hand. Define $D : X \times X \to \mathbb{R}$ by $D(x, y) := \varphi(d(x, y))$. Since $\varphi(t) > 0$ for all $t > 0$, it is obvious that $D(x, y) > 0$ for every distinct $x, y \in X$, while $\varphi(0) = 0$ implies $D(x, x) = 0$ for all $x \in X$. Moreover, D is clearly symmetric and it satisfies the triangle inequality (because φ is increasing and subadditive). Thus, $(X, D, \succcurlyeq)$ is a partially ordered metric space. As φ is increasing, this space is radial. In addition, $|f(x) - f(y)| \leq h(d(x, y))$ for every $x, y \in S$ and $h \in \mathcal{H}$, and it follows that $|f(x) - f(y)| \leq D(x, y)$ for every $x, y \in S$, that is, f is 1-Lipschitz on the metric space $(S, D|_{S \times S})$. By Theorem 3.1, therefore, there exists an $\succcurlyeq$-increasing $F : X \to \mathbb{R}$ that is 1-Lipschitz on (X, D) with $F|_S = f$. Besides, for every $\varepsilon > 0$, continuity of φ at 0 ensures that there is a $\delta > 0$ small enough that $\varphi(t) < \varepsilon$ for all $t \in (0, \delta)$, which means $|F(x) - F(y)| < \varepsilon$ for all $x, y \in X$ with $d(x, y) \leq \delta$. It follows that F is uniformly continuous on the metric space (X, d). □

Since every bounded map on a metric space is Lipschitz for large distances (Lemma 5.1), the following is a special case of Theorem 5.3. When $\succcurlyeq$ is taken as the equality relation in its statement, this result reduces to McShane's uniformly continuous extension theorem for bounded real-valued functions.

COROLLARY 5.4. *Let $(X, d, \succcurlyeq)$ be a radial metric poset. Every $\succcurlyeq$-increasing, bounded, and uniformly continuous map on a subset S of X can be extended to an $\succcurlyeq$-increasing and uniformly continuous map on X.*

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As every continuous map on a compact metric space is uniformly continuous, an immediate consequence of Corollary 5.4 is the following observation, which provides a companion to Nachbin's extension theorem.

COROLLARY 5.5. *Let $(X, d, \succcurlyeq)$ be a radial metric poset, and S a nonempty compact subset of X. Then, for every $\succcurlyeq$-increasing $f \in C(S)$, there is an $\succcurlyeq$-increasing $F \in UC(X)$ with $F|_S = f$.*

At the cost of imposing the radiality property, this result drops the topological requirement of being normally ordered in Nachbin's extension theorem and, in addition, it guarantees the uniform continuity of the extension as opposed to its mere continuity.

5.2. The monotone uniform extension property. It should be noted that the similarity of the statements of Theorem 3.4 and Corollary 5.4 is misleading. To clarify this point, let us say that a partially ordered metric space $(X, d, \succcurlyeq)$ has the *monotone uniform extension property* if for every closed $S \subseteq X$, and $\succcurlyeq$-increasing and bounded $f \in UC(S)$, there is an $\succcurlyeq$-increasing $F \in UC(X)$ with $F|_S = f$. The point we wish to make is that this property is categorically different from the monotone Lipschitz extension property. After all, the proof of Theorem 3.1 shows that a finite metric poset has the monotone Lipschitz extension property if and only if that metric poset is radial. In other words, finiteness of the carrier does not allow us to improve Theorem 3.1. By contrast, one can inductively prove that every finite metric poset has the monotone uniform extension property.

Recall that a *UC-space* (also known as an *Atsuji space*) is a metric space such that every real-valued continuous function on it is uniformly continuous. (These spaces were originally considered by [1, 26, 28], and were later studied extensively by [2, 3, 20], among others.) Various characterizations of UC-spaces are known. For instance, a metric space $X = (X, d)$ is a UC-space if and only if every open cover of it has a Lebesgue number, which holds if and only if $d(A, B) > 0$ for every pair of nonempty disjoint closed subsets A and B of X.) Now, an immediate application of Nachbin's extension theorem shows that if endowing a UC-space with a closed preorder yields a normally ordered metric poset, then that poset has the monotone uniform extension property.

The family of all partially ordered metric spaces with the monotone uniform extension property is thus much larger than that of radial metric posets. Characterization of this family remains as an interesting open problem.

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