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IDEMPOTENTS IN COMPLEX BANACH ALGEBRAS

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1. Introduction. The concept of the spectrum of A relative to Q, where A and Q commute and are elements in a complex Banach algebra $\mathcal{B}$ with identity I, was developed in [**1**]. A complex number z is in the *Q-resolvent set of A* if and only if $A - zI - \bar{z}Q$ is invertible in $\mathcal{B}$; otherwise, z is in the *Q-spectrum of A*, or *spectrum of A relative to Q*. One result from [**1**] was the following.

THEOREM. *Suppose no points in the ordinary spectrum of Q have unit magnitude. Let C be a simple closed rectifiable curve which lies in the Q-resolvent of A, and let*

$$(*)\quad J := -P^{-1} \int_C (A - zI - \bar{z}Q)^{-1}(Idz + Qd\bar{z})$$

where P is defined as

$$(\dagger)\quad P := \int_{|z|=1} (zI + \bar{z}Q)^{-1}(Idz + Qd\bar{z}).$$

Then J is an idempotent which commutes with A and Q; moreover, $J = 0$ if and only if the interior of C belongs to the Q-resolvent set of A, and $J = I$ if and only if the Q-spectrum of A lies entirely interior to C.

(Here P plays the role of the constant $2\pi iI$ in the ordinary spectral theory. The ordinary spectrum of P is a subset of the set $\{2\pi i, -2\pi i\}$.)

The question arises as to whether all idempotents which commute with A can be obtained using this more general concept of the spectrum. It is well known that in the case of the usual spectrum, there exist elements A with connected spectrum and where many nontrivial idempotents commute with A, but where the functional calculus of the usual spectrum cannot retrieve these idempotents using integration of the resolvent.

In this paper we prove that for every idempotent J which commutes with A there exists an element Q which commutes with A such that J can be retrieved as in the above theorem. This result is satisfying in that it demonstrates that disconnected Q-spectra exist at least in the same abundance as there are nontrivial idempotents which commute with A.

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2. Generation of idempotents by integration. We let sp A denote the ordinary spectrum of A, $\mathrm{sp}_Q(A)$ the Q-spectrum of A, and $\mathrm{res}_Q(A)$ the Q-resolvent set of A.

THEOREM 1. *Let $\mathscr{B}$ be a complex Banach algebra with identity I, and let A be an invertible element in $\mathscr{B}$. Suppose J is any nontrivial idempotent in $\mathscr{B}$ which commutes with A. Then there exists an element Q in $\mathscr{B}$ which commutes with A and J, with no points in* sp Q *of unit magnitude, such that*

$$(*) \quad J = -P^{-1} \int_C (A - zI - \bar{z}Q)^{-1}(Idz + Qd\bar{z})$$

for some circle C about the origin which lies in $\mathrm{res}_Q(A)$.

Proof. Let us consider Q of the form $Q = qJ$ for some fixed complex number q. For λ in $\mathbf{C}$ we set

$$S(\lambda) := A - \lambda I - \bar{\lambda}Q = A - \lambda I - \bar{\lambda}qJ.$$

Let $\mathscr{B}^*$ be a maximal commutative subalgebra of $\mathscr{B}$ containing A and Q. Let $\Phi_{\mathscr{B}^*}$ denote the set of all algebra homomorphisms of $\mathscr{B}^*$ onto $\mathbf{C}$. Then for each σ in $\Phi_{\mathscr{B}^*}$ we have from $J^2 = J$ that $\sigma(J) = 0$ or $\sigma(J) = 1$. Now

$$\sigma(S(\lambda)) = \sigma(A) - \lambda - \bar{\lambda}q\sigma(J)$$

and $\lambda \in \mathrm{sp}_Q(A)$ if and only if $\sigma(S(\lambda)) = 0$ for some σ, which means that for some σ in $\Phi_{\mathscr{B}^*}$,

$$\lambda = \frac{\sigma(A) - q\overline{\sigma(A)}\sigma(J)}{1 - |q|^2|\sigma(J)|^2}.$$

So from the above we conclude that

$$\mathrm{sp}_Q(A) = \left\{\lambda = \frac{\sigma(A) - q\overline{\sigma(A)}\sigma(J)}{1 - |q|^2|\sigma(J)|^2}\middle|\sigma \in \Phi_{\mathscr{B}^*}\right\}.$$

By Corollary 1 [**1**], $\mathrm{sp}_Q(A)$ is a nonempty compact subset of $\mathbf{C}$. We choose q so that $|q| \neq 1$; then since

$$\mathrm{sp}\, A = \{\sigma(A)|\sigma \in \Phi_{\mathscr{B}^*}\}$$

(see [**2**], Theorem 3.1.6), and $0 \notin \mathrm{sp}\, A$, we have $\mathrm{sp}_Q(A)$ is bounded away from 0 in $\mathbf{C}$.

We now break $\mathrm{sp}_Q(A)$ into two sets, corresponding to $\sigma(J) = 0$ and $\sigma(J) = 1$; namely, we define

$$S_0 := \{\lambda = \sigma(A)|\sigma(J) = 0 \text{ and } \sigma \in \Phi_{\mathscr{B}^*}\},$$

$$S_1 := \left\{\lambda = \frac{\sigma(A) - q\overline{\sigma(A)}}{1 - |q|^2}\middle|\sigma(J) = 1 \text{ and } \sigma \in \Phi_{\mathscr{B}^*}\right\}.$$

Then clearly $\mathrm{sp}_Q(A)$ is the union of S_0 and S_1, and since J is a nontrivial idempotent in $\mathscr{B}$ it follows that both S_0 and S_1 are nonempty subsets of $\mathrm{sp}_Q(A)$.

Now since sp A is bounded away from 0 in $\mathbf{C}$ and $S_0 \subset \mathrm{sp}\, A$, it is clear that if we choose q large enough and positive, $q > 1$, then S_1 can be made so close to 0 in $\mathbf{C}$ that $S_1 \cap S_0 = \emptyset$, and also such that there exists a circle C about 0 in $\mathbf{C}$ with S_1 and S_0 lying interior and exterior to C, respectively.

Having chosen such a fixed q, we will now show that if we parametrize C as a simple closed rectifiable curve in a counterclockwise direction, then J can be reproduced by formula (*). By Theorem 12 [**1**] we know that the element K as defined by the right side of (*) is a nontrivial idempotent in $\mathscr{B}$ which commutes with A and Q; therefore if we write

$$K := -P^{-1} \int_C (A - zI - \bar{z}qJ)^{-1}(Idz + qJd\bar{z}),$$

we have two cases to consider:

Case 1. Suppose $\sigma \in \Phi_{\mathscr{B}^*}$ and $\sigma(J) = 0$; then $\sigma(A) \in S_0$ and $\sigma(A)$ lies exterior to $\mathbf{C}$, implying that

$$\sigma(K) = -\sigma(P^{-1}) \int_C (\sigma(A) - z)^{-1} dz = 0.$$

Case 2. Suppose $\sigma \in \Phi_{\mathscr{B}^*}$ and $\sigma(J) = 1$; then we have

$$\sigma(K) = -\sigma(P^{-1}) \int_C [\sigma(A) - z - \bar{z}q]^{-1}(dz + qd\bar{z}).$$

We let $\phi(z) := z + \bar{z}q$ and note that $\sigma(A) - z - \bar{z}q = 0$ only at

$$h_0 := (\sigma(A) - q\overline{\sigma(A)})/(1 - |q|^2),$$

with h_0 lying in S_1 and hence in the interior of C. Since q is real we have $\phi(h_0) = \sigma(A)$, and so we can write

$$\sigma(K) = -\sigma(P^{-1}) \int_C [\sigma(A) - z - \bar{z}q]^{-1}(dz + qd\bar{z})$$

$$= \sigma(P)^{-1} \int_C \frac{d\phi(z)}{\phi(z) - \phi(h_0)},$$

where from (†)

$$\sigma(P) = \int_{|z|=1} [\phi(z)]^{-1} d\phi(z).$$

Therefore, from the scalar version of Theorem 7 [**1**] (with $\mathscr{B} = \mathbf{C}$, $Q = q$, $f(z) \equiv 1$), we obtain $\sigma(K) = 1$.

By cases 1 and 2 we now have $\sigma(K) = 1$ when $\sigma(J) = 1$ and $\sigma(K) = 0$ when $\sigma(J) = 0$. Therefore

$$\sigma(J - K) = 0 \quad \text{for all } \sigma \text{ in } \Phi_{\mathscr{B}^*},$$

which implies $\mathrm{sp}(J - K) = \{0\}$ and

$$\|(J - K)^n\|^{1/n} \to 0 \quad \text{as } n \to \infty.$$

Now

$$(J - K)^2 = J - 2JK + K, \quad (J - K)^3 = J - K,$$

and by induction, $(J - K)^{2n+1} = J - K$ for all positive integers n. But then

$$\|J - K\|^{1/2n+1} \to 0 \quad \text{as } n \to \infty,$$

which means $\|J - K\| = 0$, $J = K$.

The case of a singular element A is handled in the following.

THEOREM 2. *Let $\mathcal{B}$ be a complex Banach algebra with identity I, and let $A \in \mathcal{B}$ with $0 \in \mathrm{sp}\, A$. Suppose J is a nontrivial idempotent in $\mathcal{B}$ which commutes with A. Then there exists a scalar β and an element Q in $\mathcal{B}$ which commutes with A and J, with no points in* sp *Q of unit magnitude, such that*

$$J = -P^{-1} \int_C (A - \beta I - zI - \bar{z}Q)^{-1}(Idz + Qd\bar{z})$$

for some circle C about the origin which lies in $\mathrm{res}_Q(A - \beta I)$.

Proof. Choose a scalar β such that $0 \notin \mathrm{sp}(A - \beta I)$ and then apply Theorem 1 to $A - \beta I$.

A generalization of Theorem 1 now follows.

THEOREM 3. *Let $\mathcal{B}$ be a complex Banach algebra with identity I, and let A be an invertible element in $\mathcal{B}$. Suppose that*

$$J_1 + J_2 + \dots + J_n = I$$

is a finite resolution of the identity in $\mathcal{B}$; that is, each J_i is a nontrivial idempotent, and $J_iJ_j = 0$ for $i \neq j$. Further suppose that each J_i commutes with A. Then there exists an element Q in $\mathcal{B}$ which commutes with A and each J_i, with no points in sp *Q of unit magnitude, and a sequence of concentric circles $C_1, C_2, \dots, C_n$, of decreasing radii and each centered at* 0, *such that*

$$J_i = -P^{-1} \int_{C_i - C_{i+1}} (A - zI - \bar{z}Q)^{-1}(Idz + Qd\bar{z}),$$
$$i = 1, 2, \dots, n - 1,$$

$$J_n = -P^{-1} \int_{C_n} (A - zI - \bar{z}Q)^{-1}(Idz + Qd\bar{z}).$$

Proof. Let

$$Q = q_1J_1 + q_2J_2 + \dots + q_nJ_n$$

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for fixed scalars $q_1, q_2, \ldots, q_n$, with no q_i having unit magnitude. Then, as in the proof of Theorem 1,

$$S(\lambda) := A - \lambda I - \bar{\lambda} Q = A - \lambda I - \bar{\lambda}(q_1 J_1 + \ldots + q_n J_n).$$

Let $\mathscr{B}^*$ be a maximal commutative subalgebra of $\mathscr{B}$ containing A and J_1, $J_2, \ldots, J_n$, and again let $\Phi_{\mathscr{B}^*}$ denote the set of all algebra homomorphisms of $\mathscr{B}^*$ onto $\mathbf{C}$. For each σ in $\Phi_{\mathscr{B}^*}$ we have

$$1 = \sigma(I) = \sigma(J_1 + \ldots + J_n) = \sigma(J_1) + \ldots + \sigma(J_n);$$

moreover, since each J_i is idempotent, $\sigma(J_i) = 0$ or 1 for each J_i. Thus for any σ in $\Phi_{\mathscr{B}^*}$, σ is 1 at one and only one J_i, and is 0 at all other J_i's. Since each J_k is nontrivial, there exists for each k in $1 \leqq k \leqq n$ at least one σ_k in $\Phi_{\mathscr{B}^*}$ such that

$$\sigma_k(J_k) = 1 \quad \text{and} \quad \sigma_k(J_i) = 0 \quad \text{for } i \neq k.$$

Furthermore, for such σ_k, $\sigma_k(S(\lambda)) = 0$ if and only if

$$\sigma_k(A) - \lambda - \bar{\lambda} q_k \sigma_k(J_k) = 0,$$

$$\lambda = \frac{\sigma_k(A) - q_k \overline{\sigma_k(A)}}{1 - |q_k|^2}.$$

Recall that

$$\{\sigma(A) | \sigma \in \Phi_{\mathscr{B}^*}\} = \operatorname{sp} A,$$

and that sp A is compact and bounded away from 0 since A is invertible. We now look at mappings F_k, $k = 1, \ldots, n$, from $\mathbf{C}$ into $\mathbf{C}$, defined by

$$F_k(a) := \frac{a - q_k \bar{a}}{1 - |q_k|^2}.$$

We would like to choose the scalars $q_1, \ldots, q_n$ so that $F_1(\operatorname{sp} A)$, $F_2(\operatorname{sp} A), \ldots, F_n(\operatorname{sp} A)$ lie in disjoint annuli centered at 0.

Suppose sp A is contained in the annulus about 0 with boundaries

$$z_1(\theta) = r_1 e^{i\theta} \text{ and } z_2(\theta) = r_2 e^{i\theta},$$

and with $0 < r_1 < r_2$, $0 \leqq \theta < 2\pi$. An analysis of F_k shows that the image of such an annulus under the map F_k is an elliptical annulus about 0 whose boundaries are the images of the boundaries of the original annulus. Therefore,

$$F_k(r_1 e^{i\theta}) = \frac{r_1 e^{i\theta} - q_k r_1 e^{-i\theta}}{1 - |q_k|^2}$$

$$= \frac{r_1}{1 - |q_k|^2}(e^{i\theta} - q_k e^{-i\theta}),$$

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$$|F_k(r_1e^{i\theta})| = \frac{r_1}{|1 - |q_k|^2|}|e^{i\theta} - q_ke^{-i\theta}|$$
$$\geqq \frac{r_1}{|1 - |q_k|^2|}|1 - |q_k|| = \frac{r_1}{1 + |q_k|}.$$

Also, for $q_k > 1$,

$$|F_k(r_2e^{i\theta})| \leqq \frac{r_2}{|1 - |q_k|^2|}(1 + |q_k|) = \frac{r_2}{q_k - 1}.$$

We choose $q_1 = 2$ so that $F_1(\text{sp } A)$ lies inside the annulus between $\frac{r_1}{3}e^{i\theta}$ and $r_2e^{i\theta}$ by the above analysis.

Next we choose $q_2 > q_1$ where q_2 is the smallest positive integer such that

$$r_2/(q_2 - 1) < r_1/3;$$

then $F_2(\text{sp } A)$ lies in an annulus which is closer to 0 than the annulus which contains $F_1(\text{sp } A)$. Continuing in this manner we choose $q_3 > q_2$ where q_3 is the smallest positive integer such that

$$r_2/(q_3 - 1) < r_1/(q_2 + 1).$$

We continue this process n times to obtain the desired n annuli about 0. We name these annuli $A_1, A_2, \dots, A_n$.

Finally we choose our circles $C_1, C_2, \dots, C_n$ such that C_1 lies outside A_1, C_2 lies between A_1 and A_2, C_3 lies between A_2 and A_3, etc., until C_n lies between A_{n-1} and A_n. We then have concentric circles $C_1, \dots, C_n$ of decreasing radii such that $F_1(\text{sp } A)$ lies between C_1 and C_2, $F_2(\text{sp } A)$ lies between C_2 and C_3, etc., until $F_n(\text{sp } A)$ lies inside C_n. Making an analysis of integrals similar to that in the proof of Theorem 1, we arrive at the listed integral formulas for $J_1, J_2, \dots, J_n$ in terms of integrals around the circles $C_1, C_2, \dots, C_n$ separating the Q-spectrum of A. We omit the analogous details.

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