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ON POSITIVITY OF FOURIER TRANSFORMS

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This note concerns Fourier transforms on the real positive line. In particular, we seek conditions on a real function $u(x)$ in $x > 0$, that ensure that its Fourier-cosine transform $v(t) = \int_0^\infty u(x) \cos xt \, dx$ is positive. We prove first that this is so for all $t > 0$, if $u''(x) > 0$ for all $x > 0$, that is, that everywhere-convex functions have everywhere-positive Fourier-cosine transforms. We then obtain a complex-plane criterion for some types of non-convex $u(x)$. Finally we consider criteria on $u(x)$ that imply positivity of $v(t)$ for $t > t_0$, for some $t_0 > 0$.

INTRODUCTION

Define for $t > 0$ the ordinary Fourier-cosine transform

$$v(t) = \int_0^\infty u(x) \cos xt \, dx \tag{1}$$

with inverse

$$u(x) = \frac{2}{\pi} \int_0^\infty v(t) \cos xt \, dt \, , \tag{2}$$

with a similar definition for the Fourier-sine transform.

Generally we shall assume here that $u(x)$ is real and smooth in $x > 0$ and that the Fourier integral (1) converges. In particular, $u(x)$ and all of its derivatives are bounded everywhere in $x > 0$ and tend to zero as $x \to +\infty$. Meanwhile, $u(x)$ could be bounded at the origin, but more generally could have a weak singularity, with $xu(x) \to 0$ as $x \to 0_+$, that is, $u(x)$ grows at a rate less than x^{-1}. For Fourier-sine transforms, we can allow a stronger singularity at $x = 0_+$, with any growth rate less than x^{-2}. We shall also generalise the results later, to allow even stronger singularities at the origin.

One important class of functions $u(x)$ is "convex", that is, such that $u''(x) > 0$ for all $x > 0$, which implies (since $u'(+\infty) = 0$) that $u'(x) < 0$ and (since $u(+\infty) = 0$) that $u(x) > 0$. That is, convex functions possessing Fourier transforms are also decreasing and positive. Such convex functions need not be smooth at $x = 0$, indeed not even bounded so long as they are integrable. In particular, they need not (indeed cannot) have all of their

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odd-order derivatives zero at $x = 0_+$, and hence do not extend smoothly as even functions into $x < 0$. We shall show that convex functions have everywhere-positive Fourier-cosine transforms. An elementary convex example is $u(x) = e^{-x}$ with $v(t) = 1/(1+t^2) > 0$.

However, we are more interested here in non-convex functions $u(x)$ which are bounded, positive and decreasing in $x > 0$, which extend smoothly as an even function to the whole real line, that is, all odd-order derivatives vanish at $x = 0_+$, and which usually have a single inflexion point in $x > 0$. Let us call such functions "bell-shaped" functions.

Some bell-shaped functions have positive Fourier transforms, and some don't. Thus compare $u(x) = 1/(1+x^2)$, which has transform $v(t) = (\pi/2)e^{-t}$, with $u(x) = 1/(1+x^4/4)$, which has transform $v(t) = (\pi/2)e^{-t}(\cos t + \sin t)$. One $v(t)$ is positive, the other oscillates between positive and negative values, but both $u(x)$ are bell-shaped and have quite similar graphs. A criterion for discriminating between such bell-shaped functions would be of some value.

Proof of positivity for convex functions

Positivity of Fourier-sine transforms is somewhat easier to prove than that of Fourier-cosine transforms. But by integration by parts we have

$$v(t) = -\frac{1}{t}\int_0^\infty u'(x)\sin xt\,dx\,, \tag{3}$$

given that the assumed convergence requirements ($u \to 0$ as $x \to +\infty$ and $xu(x) \to 0$ as $x \to 0_+$) eliminate the integrated part. That is, the Fourier-cosine transform of $u(x)$ is $-1/t$ times the Fourier-sine transform of its derivative $u'(x)$.

Now let us prove that the Fourier-sine transform of a decreasing function $w(x)$ is positive. That is,

$$\begin{aligned}\int_0^\infty w(x)\sin xt\,dx &= \sum_{j=0}^\infty \int_{2\pi j/y}^{2\pi(j+1)/y} w(x)\sin xt\,dx\\ &= \frac{1}{t}\sum_{j=0}^\infty \int_0^{2\pi} w\left(\frac{2\pi j+\theta}{t}\right)\sin\theta\,d\theta\\ &= \frac{1}{t}\sum_{j=0}^\infty \int_0^{\pi}\left[w\left(\frac{2\pi j+\theta}{t}\right) - w\left(\frac{2\pi j+\theta}{t}+\frac{\pi}{t}\right)\right]\sin\theta\,d\theta\end{aligned} \tag{4}$$

If $w(x)$ is a decreasing function for all x, the quantity in square brackets is positive for all t and all j, and so is $\sin\theta$ in $(0,\pi)$; hence the Fourier-sine transform is positive. This is essentially a simple geometrical result, each negative half-period loop of the sine function contributing less to the sum than the positive half-period loop preceeding it.

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Now define $w(x) = -u'(x)$. Then $u''(x) > 0$ implies $w'(x) < 0$ so this $w(x)$ is a decreasing function. Therefore its Fourier-sine transform is positive, and hence so is the Fourier-cosine transform of $u(x)$. Thus we have proved that $u''(x) > 0$ for all $x > 0$ guarantees $v(t) > 0$ for all $t > 0$. That is, convex functions have everywhere-positive Fourier-cosine transforms.

However, bell-shaped functions are not convex, and it is doubtful if there is any criterion based solely on behaviour of $u(x)$ for positive real x, for positivity of the Fourier-cosine transform of bell-shaped functions. Somewhat reluctantly, we must move into the complex plane.

Complex detours

Suppose we can continue the function $u(z)$ into the upper half complex z plane, and that it is an even analytic function of z, real on the real axis, satisfying $\overline{u}(z) = u(\overline{z})$. Then we can write

$$v(t) = \frac{1}{2}\int_{-\infty}^{\infty} u(z)e^{izt}\,dz\,. \tag{5}$$

Now suppose that $|u(z)| \to 0$ as $\Re z \to \pm\infty$ for some range of positive values of the imaginary part of z, say for $\Im z < p$. Then we can shift the path of integration upward, writing $z = x + ip$ and giving

$$v(t) = \frac{1}{2}e^{-pt}\int_{-\infty}^{\infty} u(x+ip)e^{ixt}\,dx \tag{6}$$

$$= e^{-pt}\int_0^{\infty} \left[\Re\, u(x+ip)\cos xt - \Im\, u(x+ip)\sin xt\right] dx\,. \tag{7}$$

Equation (7) expresses $v(t)$ as the sum of a Fourier cosine and a Fourier sine transform, each multiplied by an exponential decay factor. Hence if $\Re\, u(x+ip)$ is convex (and decreasing and positive) and also $-\Im\, u(x+ip)$ is decreasing (and positive), then $v(t)$ is positive for all $t > 0$.

An example is $u(z) = 1/\sqrt{1+z^2}$ where we can take $p = 1$. Then $\Re\, u(x+i) = R\cos\theta$ and $-\Im\, u(x+i) = R\sin\theta$, where $R = x^{-1/2}(x^2+4)^{-1/4}$ and $\tan 2\theta = 2/x$. These functions have the required properties, which proves that $v(t)$ is positive for all $t > 0$. In fact, $v(t) = K_0(t)$ is a modified Bessel function [**1**], which is indeed positive and decays exponentially for large t.

Non-integrable singularities

The above analysis is valid as it stands if $u(z)$ is integrable along the whole line $z = x + ip$, including the case of bounded $u(z)$. However, it is of no use for the present purpose if $u(z)$ is bounded as $z \to ip$, because then evenness of $u(z)$ necessarily implies that $\Im u(ip) = 0$, so $-\Im u(x+ip)$ cannot be decreasing and positive for $x > 0$. Thus we are only interested in choices of p such that $u(z)$ has a singularity at $z = ip$ on the imaginary axis, and no other singularity closer to the origin. The above example $u(z) = 1/\sqrt{1+z^2}$ has an (integrable) inverse square root branch point at $z = i$.

But what if the nearest singularity is stronger than that? For example, $u(z) = 1/(1+z^2)$ is not integrable through the simple pole at $z = i$, nor is $u(z) = (1+z^2)^{-\alpha}$ for any $\alpha \geqslant 1$. Nevertheless these happen to be functions with positive Fourier-cosine transforms. We would like to be able to prove that statement using methods like those in the previous section. For the present, we shall only discuss the simple-pole case $\alpha = 1$; although a similar analysis can be performed for stronger singularities, it requires generalisation of the concept of a Fourier transform to non-integrable functions.

Thus we now assume that as $z \to ip$ we have

$$u(z) \to U_0\big[i(z-ip)\big]^{-1} \tag{8}$$

for some real constant U_0. The example $u(z) = 1/(1+z^2)$ has $U_0 = 1/2$. Note that when (8) holds, only the imaginary part of u is singular as $x \to 0_+$ on the line $z = x+ip$, with $-x\Im\, u(x+ip) \to U_0$, but $x\Re\, u(x+ip) \to 0$. Hence both Fourier integrals in (7) converge in spite of the non-integrable character of the singularity in $u(z)$. Nevertheless we must modify (7) to take account of the pole.

The necessary modification is simply to allow the path of integration to pass below the pole, on a semicircle of vanishingly small radius. The net effect is to add a term proportional to the residue at the pole, so (7) becomes

$$v(t) = e^{-pt}\left[\int_0^\infty \left[\Re\, u(x+ip)\cos xt - \Im\, u(x+ip)\sin xt\, dx\right] + U_0\frac{\pi}{2}\right]. \tag{9}$$

For example, suppose $u(z) = 1/(1+z^2)$ and $p = 1$. Then

$$v(t) = e^{-t}\left[\int_0^\infty \frac{1}{x^2+4}\cos xt\, dx + \int_0^\infty \frac{2}{x(x^2+4)}\sin xt\, dx + \frac{\pi}{4}\right]. \tag{10}$$

Since the coefficient of $\sin xt$ is positive and decreasing, the Fourier-sine integral in (10) is positive. Although the coefficient of $\cos xt$ is not convex, we no longer need the Fourier-cosine integral to be positive (though it is!), so long as it is overwhelmed by the positive correction term $\pi/4$. This is clearly so, since (replacing $\cos xt$ by -1), the Fourier-cosine integral can be seen to be greater than $-\pi/4$. Hence $v(t) > 0$. Of course, given that we can actually evaluate this $v(t) = (\pi/2)e^{-t}$ and the other Fourier integrals in (10), this appears a clumsy way to prove something obvious, but is important in principle, in that it does not depend on a knowledge of the exact integrals, so generalises to more complicated functions.

POSITIVITY ONLY FOR $t > t_0$

In fact, in some applications it is neither necessary nor desirable to insist that $v(t) > 0$ for all $t > 0$, and it may be enough to show that there is a finite $t_0 > 0$ such that $v(t) > 0$ for all $t > t_0$. Can we find criteria on $u(x)$ for this to be true, and if so, can we estimate t_0? Only preliminary discussions of this generalised task are given here.

Assuming validity of (7), that is, ruling out for the time being non-integrable singularities, on integration of the first term of (7) by parts, $v(t)$ can be expressed as a single Fourier-sine integral

$$v(t) = e^{-pt} \int_0^\infty F(x;t) \sin xt \, dx \tag{11}$$

where

$$F(x;t) = -\Im u(x+ip) - \frac{1}{t}\frac{d}{dx}\Re u(x+ip) \tag{12}$$

$$= \Re\left[iu(x+ip) - \frac{1}{t}u'(x+ip)\right]. \tag{13}$$

Now if (in any range of t values) the function $F(x;t)$ is a decreasing (and positive) function of x for all $x > 0$, then $v(t)$ is positive for that range of t. This is true for all t when the two terms of (12) are both decreasing and positive for all $x > 0$, as in the examples already given.

However, suppose it is not true for all t, but only for $t > t_0$, for some $t_0 > 0$. Then in particular it must be true for large t, when the second term of (12) tends to zero, so the first term $F(x;\infty) = -\Im u(x+ip)$ of (12) must be decreasing and positive for all $x > 0$. If the second term was also decreasing and positive for all $x > 0$, we would have $t_0 = 0$ as above, so let us assume that this is not so for some x values. Then there is still a chance of finding a finite t_0 such that the sum of the two terms of (12) is decreasing and positive for all $x > 0$. This will be possible if the second term of (12) is bounded (together with its derivative) in $x > 0$, and does not become asymptotically large relative to the first term, either as $x \to 0_+$ or as $x \to \infty$.

For example, consider

$$\int_0^\infty \frac{\sin xt - x\cos xt}{1+x^2}\, dx = e^{-t}\mathrm{Ei}(t) \tag{14}$$

where Ei is the exponential integral ([**1**, p. 230]). Now

$$\int_0^\infty \frac{\sin xt - x\cos xt}{1+x^2}\, dx = \int_0^\infty F(x;t)\sin xt\, dx \tag{15}$$

where

$$F(x;t) = \frac{1}{1+x^2} + \frac{1}{t}\frac{1-x^2}{(1+x^2)^2} \tag{16}$$

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is positive and decreasing for all x if $t > t_0 = 1$. This is a conservative estimate of t_0, since in fact $\mathrm{Ei}(t) > 0$ for all $t > 0.37253$.

There is a potential application to the celebrated Riemann hypothesis [2]. This hypothesis might well be true if $v(t) = V'(t)^2 - V(t)V''(t)$ could be proved positive for all $t > t_0$, where $V(t) = \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$ is a real-valued scaling of the Riemann zeta function ([1, p. 807]) on its critical line $s = 1/2 + it$. Numerical evidence [4] is that this is so with $t_0 \approx 5.9009$, but a proof is elusive. The inverse Fourier transform of this $v(t)$ is the bell-shaped function

$$u(x) = \frac{1}{4}\int_0^\infty y^2\, U\Big(\frac{x+y}{2}\Big)\, U\Big(\frac{x-y}{2}\Big)\,dy\,, \tag{17}$$

where

$$U(x) = -2e^{-x/2} + 4e^{x/2}\sum_{n=1}^{\infty} e^{-n^2\pi e^{2x}} \tag{18}$$

is the (also bell-shaped) inverse Fourier transform of the (sign-oscillatory) Riemann function $V(t)$ [3]. The nearest singularity of $u(z)$ is at $z = i\pi/2$, so we could try $p = \pi/2$ in the above. However, there also appear to be many other singularities along the line $z = x + i\pi/2$, which may or may not be integrable. Further study of $u(z)$ near that line would seem to be of value.

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