Can. J. Math., Vol. XXVIII, No. 4, 1976, pp. 885–888

HOLOMORPHIC FUNCTIONALS ON OPEN RIEMANN SURFACES

P. M. GAUTHIER AND L. A. RUBEL

Let $\mathcal{O}(R)$ denote the space of functions holomorphic on an open Riemann surface R, where $\mathcal{O}(R)$ has the topology of uniform convergence on compact sets. In this note, we characterize the dual space $\mathcal{O}(R)^*$. The result is not new, for it is implicitly contained in the more general results of Tillmann [**5**] and, in case R is planar, in those of Köthe [**4**]. However, the paper of Gunning and Narasimhan [**3**], which appeared subsequently, allows us to give a short proof of this important result. Actually, our characterization is a natural one in terms of differentials, while the Köthe-Tillmann characterization is in terms of functions, but we show that these two characterizations are isomorphic. We end our paper by using our characterization to prove an interpolation result. The second author gratefully acknowledges a helpful discussion with Professor George Szekeres.

We denote by $D(R)$ the space of holomorphic differentials on R. Thus $\omega \in D(R)$ has the form $f(z)dz$ in local coordinates, where f is holomorphic. We denote by $D(\infty)$ the space of germs of holomorphic differentials "at ∞". That is, $D(\infty)$ consists of equivalence classes of differentials holomorphic outside of compact subsets of R. Two such differentials are equivalent if they agree outside some compact subset of R. Further let $\mathcal{O}(\infty)$ denote the space of germs of holomorphic functions at ∞.

THEOREM. $\mathcal{O}(R)^* \simeq D(\infty)/D(R) \simeq \mathcal{O}(\infty)/\mathcal{O}(R)$.

Thus, the elements of $\mathcal{O}(R)^*$, namely the holomorphic functionals, can be represented both by holomorphic differentials and by holomorphic functions. The second representation is the classical one. While it has the advantage of putting functionals in correspondence with tangible objects, namely functions, the correspondence itself is not very tangible and involves an arbitrary choice. The correspondence between functionals and differentials, on the other hand, is tangible and canonical.

Received January 13, 1976.

The research of the first author was partially supported by the N.R.C. of Canada, and that of the second author was partially supported by a grant from the National Science Foundation of the U.S.A.

Our discussion centers on the following diagram:

(1)
$$
\begin{tikzcd}
\mathcal{O}(\infty) \arrow[dr] \arrow[rrdd, "t"] \arrow[dddd, "r"'] & & \\
& \mathcal{O}(\infty)/\ker t \arrow[d, "\tilde{r}"'] \arrow[rd, "\tilde{t}"] & \\
& D(\infty)/\ker s \arrow[r, "\tilde{s}"] & \mathcal{O}(R)^* \\
& & \\
D(\infty) \arrow[ruu] \arrow[rruu, "s"'] & &
\end{tikzcd}
$$

The first step is to define r. Gunning and Narasimhan [**3**] have shown that every open Riemann surface R can be spread without ramification over the finite plane $\mathbf{C}$. That is, R admits a locally injective holomorphic function ρ. Since ρ yields an atlas for R in which change of charts is just the identity function, ρ thus induces an isomorphism r from the vector space of holomorphic functions onto the vector space of holomorphic differentials. By abuse of notation, we also denote by r the induced isomorphism of $\mathcal{O}(\infty)$ onto $D(\infty)$.

Referring to (1), we shall now define the canonical map s. Let $[\omega] = r[f]$ be in $D(\infty)$, where $\omega = f(z)dz$ and $f \in \mathcal{O}(R\backslash K)$, and K is compact in R. Let G be a relatively compact neighborhood of K, holomorphically convex and with smooth boundary. We call such a G a regular domain. The map s is defined by

$$[\omega] \rightarrow L_{[\omega]} \quad \text{where} \quad \langle L_{[\omega]}, g\rangle = \int_{\partial G} g\omega, \quad g \in \mathcal{O}(R).$$

It is easily seen that s is a well-defined morphism.

Again returning to (1), we set $t = s \circ r$. We shall show now that the morphism t is surjective. Suppose then $L \in \mathcal{O}(R)^*$ and let μ be a measure of compact support in R that represents L:

(2) $$\langle L, g\rangle = \int g d\mu, \quad g \in \mathcal{O}(R).$$

We digress briefly to introduce the *Cauchy kernel* of Behnke-Stein [**1**]. Recall that R is spread over the finite z-plane by ρ. Hence $\rho \times \rho$ spreads $R \times R$ over $\mathbf{C} \times \mathbf{C}$, and we write the local variable as the pair (ζ, z). Since $R \times R$ is Stein, we may solve the first Cousin problem. In particular, there is a function Φ, meromorphic on $R \times R$, such that

(3) $$\Phi(\zeta, z) - (\zeta - z)^{-1}$$

is locally holomorphic. The function Φ is called a *Cauchy kernel.*

Now referring back to (2), we set

$$(4)\quad f(z) = -(2\pi i)^{-1} \int \Phi(\zeta, z)d\mu(\zeta), \quad z \notin \operatorname{supp} \mu.$$

Let G be a regular neighborhood of supp μ. Then for $g \in \mathscr{O}(R)$, we have

$$\langle L_f, g\rangle = \int_{\partial G} g(z)f(z)dz = -\int_{\partial G} g(z)(2\pi i)^{-1} \int \Phi(\zeta, z)d\mu(\zeta)dz$$
$$= -\int (2\pi i)^{-1} \int_{\partial G} g(z)\Phi(\zeta, z)dzd\mu(\zeta) = \int g(\zeta)d\mu(\zeta) = \langle L, g\rangle.$$

Thus t is surjective, and it follows that $\tilde{t}$ and $\tilde{s}$ are isomorphisms.

It remains only to identify the kernels of these isomorphisms. Suppose that $f \in \mathscr{O}(R\backslash K)$ and $L_f = 0$. We shall show that $f \in \mathscr{O}(R)$. Now we have already associated to f a domain G for which

$$\langle L_f, g\rangle = \int_{\partial G} g(z)f(z)dz.$$

Let $\tilde{G}$ be a regular neighborhood of $\bar{G}$ in R. These neighborhoods exhaust R. We set

$$\tilde{f}(z) = (2\pi i)^{-1} \int_{\partial \tilde{G}} f(\zeta)\Phi(\zeta, z)d\zeta, \quad z \in \tilde{G}.$$

Then, for $z \in \tilde{G}\backslash\bar{G}$,

$$(5)\quad \tilde{f}(z) = (2\pi i)^{-1} \int_{\partial(\tilde{G}/G)} f(\zeta)\Phi(\zeta, z)d\zeta + (2\pi i)^{-1} \int_{\partial G} f(\zeta)\Phi(\zeta, z)d\zeta$$
$$= f(z) + (2\pi i)^{-1} \int_{\partial G} f(\zeta)\Phi(\zeta, z)d\zeta.$$

Now by assumption,

$$\int_{\partial G} f(\zeta)g(\zeta)d\zeta = 0, \quad g \in \mathscr{O}(R).$$

Hence by the Runge-Behnke-Stein Theorem [**1**], we have

$$(6)\quad \int_{\partial G} f(\zeta)\Phi(\zeta, z)d\zeta = 0, \quad z \notin G.$$

From (5) and (6), $f = \tilde{f}$ on $\tilde{G}\backslash\bar{G}$. Thus $f \in \mathscr{O}(R)$ and so ker $t = \mathscr{O}(R)$. From (1), it follows that ker $s = D(R)$ and our theorem is proved.

We conclude with an application of our theorem. The result is known, but our proof is new. Let $\{z_n\}$ be any sequence of points in R, with no limit points, and let $\{w_n\}$ be any sequence of complex numbers. Then there exists an $f \in$

$\mathcal{O}(R)$ such that $f(z_n) = w_n$ for all n. To prove this, let $L_n \in \mathcal{O}(R)^*$ be defined by $L_n(f) = f(z_n)$. In the language of [**2**], we must prove that $\{L_n\}$ is an interpolating sequence, and it is enough by [**2**] to prove that $\{L_n\}$ is totally linearly independent. But L_n is associated with the unit point mass $d\epsilon_{z_n}$ at z_n, and $-(2\pi i)^{-1} \int \Phi(\zeta, z)d\epsilon_{z_n}(\zeta)$ has a simple pole at z_n. If we denote by $[\omega_n]$ the element of $D(\infty)/D(R)$ associated with L_n, then $[\omega_n]$ has such a pole at z_n and it follows easily that $\{[\omega_n]\}$ and hence $\{L_n\}$ is totally linearly independent. Of course, one must check the expected result that if $[\omega_n] \to 0$ in the weak-star topology, then the ω_n are all holomorphic outside a fixed compact set. Consider the correspondence $[\omega_n] \leftrightarrow f_n \leftrightarrow \mu_n$ where $\omega_n = f_n dz$ in the ρ-atlas and $T_n g = \int g d\mu_n = \int g\omega_n$ where μ_n is a suitable measure of compact support. By the uniform boundedness principle, since T_n converges weak-*, we have $|T_n(f)| \leqq \sigma ||f||_K$ for some constant σ and some compact $K \subseteq R$. It follows that we may choose μ_n so that supp $\mu_n \subseteq K$. But $f_n(z) = -(2\pi i)^{-1} \int \Phi(\zeta, z)d\mu_n(\zeta)$ is holomorphic outside K, and hence so is ω_n. This completes the proof.

References

1. H. Behnke and K. Stein, *Entwicklungen analytischer Funktionen auf Riemannschen Flächen*, Math. Ann. *120* (1949), 430–461.
2. P. M. Gauthier and L. A. Rubel, *Interpolation in separable Fréchet spaces with applications to spaces of analytic functions*, Can. J. Math. *27* (1975), 1110–1113.
3. R. C. Gunning and R. Narasimhan, *Immersion of open Riemann surfaces*, Math. Ann. *174* (1967), 103–108.
4. G. Köthe, *Dualität in der Funktionentheorie*, J. reine u. angew. Math. *191* (1953), 30–49.
5. H. G. Tillmann, *Dualität in der Funktionentheorie auf Riemannschen Flächen*, J. reine u. angew. Math. *195* (1956), 76–101.

Université de Montréal,
Montréal, Québec;
University of Illinois at Urbana-Champaign,
Champaign, Illinois