Canad. Math. Bull. Vol. 19 (1), 1976

SOME RESULTS ON MATRICES WITH PRESCRIBED DIAGONAL ELEMENTS AND SINGULAR VALUES

BY
FUK-YUM SING†

1. **Introduction.** Let A be an $n\times n$ complex matrix. The singular values of A are the non-negative square-roots of the eigenvalues of A^*A. G. N. De Oliviera [4] gave a necessary condition for the existence of a matrix A with $a_1,\ldots,a_n$ as diagonal elements and $\alpha_1,\ldots,\alpha_n$ as singular values. We shall give another necessary condition which implies the above author's condition and we show that this is also a sufficient condition for the case $n=2$.

2. **Definitions and notations.** The notations adopted here can be found in [3].

Let $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$ be real n-vectors. We shall write $x\ll y$ if $\max_\sigma(\sum_{i=1}^k x_{\sigma(i)})\le\max_\sigma(\sum_{i=1}^k y_{\sigma(i)})$ for any permutation σ of $\{1,\ldots,n\}$ for any $k=1,\ldots,n$. $x\prec y$ if $x\ll y$ and in addition $\sum_{i=1}^n x_i=\sum_{i=1}^n y_i$. Note that if $x_1\ge x_2\ge\cdots\ge x_n$ and $y_1\ge y_2\ge\cdots\ge y_n$, then $x\ll y$ is equivalent to $\sum_{i=1}^k x_i\le\sum_{i=1}^k y_i$ for $k=1,\ldots,n$, and $x\prec y$ is equivalent to $\sum_{i=1}^k x_i\le\sum_{i=1}^k y_i$, $\sum_{i=k}^n x_i\ge\sum_{i=k}^n y_i$ for $k=1,\ldots,n$.

For any complex matrix $A=(a_{ij})$, denote by $|A|$ the matrix $(|a_{ij}|)$ where $|a_{ij}|$ is the modulus of a_{ij}.

3. **Diagonal and singular values of a matrix.**

Theorem 1. *Suppose there exists an $n\times n$ complex matrix A with $a=(a_1,\ldots,a_n$ as diagonal and $\alpha=(\alpha_1,\ldots,\alpha_n)$ as singular values, then $|a|\ll\alpha$.*

In order to prove Theorem 1, we state a lemma which can be found in [2, Theorem 2].

Lemma 1. *If $y=(y_1,\ldots,y_n)$ is a real n-vector such that $y_i\ge 0$ for $i=1,\ldots,n$, then $yP\ll y$ for any substochastic matrix P.*

Proof of Theorem 1. It is known that an $n\times n$ complex matrix A has $\alpha_1,\ldots,\alpha_n$ as singular values iff $A=U\operatorname{diag}(\alpha_1,\ldots,\alpha_n)V$ for some $n\times n$ unitary matrices U, V (see [1]). Let $U=(u_{ij})$ and $V=(v_{ij})$, then by direct computation $a=\alpha W$ where $W=(u_{ij}v_{ji})$. Clearly $|W|$ is a substochastic matrix. Furthermore, $|a|\ll\alpha|W|$ and so $|a|\ll\alpha$ by Lemma 1.

Received by the editors March 3, 1975.

†This paper formed part of the author's M. Phil. thesis.

We now prove that G. N. De Oliviera's results follow from Theorem 1.

COROLLARY. *Under the same assumptions as in Theorem* 1, *if* $|a_1|\ge|a_2|\ge\cdots\ge|a_n|$ *and* $\alpha_1\ge\alpha_2\ge\cdots\ge\alpha_n\ge 0$, *then*

$$\sum_{i=1}^{k}|a_i|^2\le\sum_{i=1}^{k}\alpha_i^2 \quad\text{for}\quad k=1,\ldots,n.$$

Proof. This follows from [2, Theorem 2] the fact that if x, y are non-negative n-vectors, then $x\ll y$ iff $\sum_{i=1}^n f(x_i)\le\sum_{i=1}^n f(y_i)$ for any convex increasing function f.

LEMMA 2. *Let H be an $n\times n$ Hermitian matrix with eigenvalues $\lambda=(\lambda_1,\ldots,\lambda_n)$ and diagonal elements $a=(a_1,\ldots,a_n)$. (The singular values of H are then $|\lambda_1|,\ldots,|\lambda_n|$.) Suppose $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n$ and $a_1\ge a_2\ge\cdots\ge a_n$, then we have*

$$\max_{1\le i\le n}\left(\sum_{j\ne i}|a_j|-|a_i|\right)\le\max_{1\le i\le n}\left(\sum_{j\ne i}|\lambda_j|-|\lambda_i|\right)$$

Proof. It is known that [2, Theorem 4] $a\prec\lambda$ and which is equivalent to

$$\sum_{i=1}^{k}a_i\le\sum_{i=1}^{k}\lambda_i \quad\text{and}\quad \sum_{i=k}^{n}a_i\ge\sum_{i=k}^{n}\lambda_i \quad\text{for}\quad k=1,\ldots,n.$$

Suppose $a_1\ge a_2\ge\cdots\ge a_j\ge 0\ge a_{j+1}\ge\cdots\ge a_n$. We have $\min_k|a_k|$ is a_j or $-a_{j+1}$. If a_j is the minimum, consider

$$\begin{aligned}&a_1+\cdots+a_{j-1}\pm a_j-(a_{j+1}+\cdots+a_n)\\&\quad\le\lambda_1+\cdots+\lambda_{j-1}\pm\lambda_j-(\lambda_{j+1}+\cdots+\lambda_n)\\&\quad\le|\lambda_1|+\cdots+|\lambda_{j-1}|\pm\lambda_j+|\lambda_{j+1}|+\cdots+|\lambda_n|.\end{aligned}$$

Therefore

$$\begin{aligned}&|a_1|+\cdots+|a_{j-1}|-|a_j|+|a_{j+1}|+\cdots+|a_n|\\&\quad\le|\lambda_1|+\cdots+|\lambda_{j-1}|-|\lambda_j|+|\lambda_{j+1}|+\cdots+|\lambda_n|\\&\quad\le\max_i\left(\sum_{k\ne i}|\lambda_k|-|\lambda_i|\right).\end{aligned}$$

The case for $-a_{j+1}$ is similar.

THEOREM 2. *Let M be the set of all $n\times n$ complex matrices of the form $U\operatorname{diag}(\alpha_1,\ldots,\alpha_n)V$ where U, V are $n\times n$ unitary matrices and $\alpha_1\ge\alpha_2\ge\cdots\ge\alpha_n\ge 0$, and let $f:M\to R$ be a mapping defined by $f(A)=|a_1|+\cdots+|a_{n-1}|-|a_n|$ where $a=(a_1,\ldots,a_n)$ is the diagonal of $A\in M$. Then $\max f=\alpha_1+\cdots+\alpha_{n-1}-\alpha_n$. (Hence $|a_1|+\cdots+|a_{n-1}|-|a_n|\le\alpha_1+\cdots+\alpha_{n-1}-\alpha_n$ for any matrix A with singular values $\alpha_1\ge\cdots\ge\alpha_n\ge 0$ and diagonal $a_1,\ldots,a_n$.)*

Proof. If the maximum is attained at certain matrix with non-zero diagonal elements, then using similar technique as in Horn's paper [2, Theorem 11] we can find another Hermitian matrix also attaining the maximum.

We may then assume there is a matrix A with diagonal $a_1 \geq a_2 \geq \cdots \geq a_{n-1} > a_n = 0$ that attains the maximum. As in Horn's paper, A must be of the form $\begin{pmatrix} H & \times \\ \times & 0 \end{pmatrix}$ where H is an $(n-1)\times(n-1)$ Hermitian matrix. A suitable unitary matrix U can be found so that $B = UAU^*$ is of the form

$$\begin{pmatrix} \beta_1 \cdots & 0 & x_1 \\ \cdot\cdot & \cdot & \cdot \\ \cdot \quad \cdot & \cdot & \cdot \\ \cdot \quad & \cdot\,\cdot & \cdot \\ 0 \cdots & \beta_{n-1} & x_{n-1} \\ y_1 \cdots & y_{n-1} & 0 \end{pmatrix} \quad \text{with} \quad \beta_1 + \cdots + \beta_{n-1} = a_1 + \cdots + a_{n-1}.$$

By the maximality of A, we must have

$$\beta_i \geq 0 \quad \text{for} \quad i = 1, \ldots, n-1.$$

Without loss of generality we may assume $x_1 < 0$ and $y_1 \leq 0$. Consider a unit vector $u = (u_1, 0, \ldots, 0, u_n)$ with $u_1, u_n > 0$. We have

$$uBu' = \beta_1 u_1^2 + x_1 u_1 u_n + y_1 u_1 u_n = (\beta_1 u_1 + x_1 u_n + y_1 u_n) u_1$$

which is negative if u_1 is sufficiently small. Find a unitary matrix W with u as its last row. Then $WBW^* = C$ has diagonal $(c_1, \ldots, c_n)$ with $c_n < 0$. Since $c_1 + \cdots + c_{n-1} + c_n = a_1 + \cdots + a_{n-1}$, by the maximality of A, none of the c_i is zero for $i = 1, \ldots, n$.

Using the same argument as above, we can find a Hermitian matrix which attains the maximum. The Theorem now follows from Lemma 2.

We now give a complete answer for $n=2$.

THEOREM 3. *For $n=2$, a necessary and sufficient condition for the existence of a matrix A with diagonal a_1, a_2 and singular values α_1, α_2 such that $|a_1| \geq |a_2|$ and $\alpha_1 \geq \alpha_2$ is*

$$(*) \qquad \begin{aligned} \alpha_1 + \alpha_2 &\geq |a_1| + |a_2| \\ \alpha_1 - \alpha_2 &\geq |a_1| - |a_2|. \end{aligned}$$

Proof. Necessity follows from Theorem 1 and 2.

Sufficiency: We may assume $a_1 \geq a_2 \geq 0$. It is sufficient to find

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

so that $A = U\,\mathrm{diag}(\alpha_1, \alpha_2)V$ has diagonal (a_1, a_2). This is equivalent to

$$\begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ -\sin\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

https://doi.org/10.4153/CMB-1976-012-5 Published online by Cambridge University Press

which is equivalent to

$$\cos(\theta+\phi)(\alpha_1+\alpha_2) = a_1+a_2$$

$$\cos(\theta-\phi)(\alpha_1-\alpha_2) = a_1-a_2.$$

This system is solvable in θ and φ when $(*)$ is satisfied.

Added in proof. After this note was completed, the author proved that the necessary condition is also sufficient for $n \geq 2$. But he found that this result had been obtained by R. C. Thompson in the paper "Singular Values Diagonal Elements, and Convexity." To be published in SIAM Journal of Mathematical Analysis.

ACKNOWLEDGEMENT. The author wishes to express his gratitude to Dr. Y. H. Au-Yeung for his encouragement.

REFERENCES

1. A. Horn, *On the eigenvalues of a matrix with prescribed singular values*, Proc. Amer. Math. Soc. **5** (1954), 4–7.

2. A. Horn, *Doubly stochastic matrices and diagonal of a rotation matrix*, Amer. J. of Math. **76** (1954) 620–630.

3. L. Mirsky, *On a convex set of matrices*, Arch. Math. Vol. **10** (1959), 88–92.

4. G. N. De Oliviera, *Matrices with prescribed principal elements and singular values*, Canad. Math. Bull. **14** (2), 1971, 247–249.

UNIVERSITY OF HONG KONG,
HONG KONG