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HIGGS BUNDLES OVER ELLIPTIC CURVES FOR COMPLEX REDUCTIVE GROUPS

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Abstract. We study Higgs bundles over an elliptic curve with complex reductive structure group, describing the (normalisation of) its moduli spaces and the associated Hitchin fibration. The case of trivial degree is covered by the work of Thaddeus in 2001. Our arguments are different from those of Thaddeus and cover arbitrary degree.

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1. Introduction. An *elliptic curve* is a pair (X, x_0), where X is a smooth complex projective curve of genus 1 and x_0 is a distinguished point on it. By abuse of notation, we usually refer to an elliptic curve simply as X. Let G be a connected complex reductive Lie group. A *G-Higgs bundle* over X is a pair (E, Φ), where E is a principal G-bundle over X and Φ, called the *Higgs field*, is a section of the adjoint bundle twisted by the canonical bundle of the curve. The canonical bundle of an elliptic curve is trivial, $\Omega^1_X \cong \mathcal{O}_X$, so $\Phi \in H^0(X, E(\mathfrak{g}))$. These objects were defined by Hitchin [**19**] over a smooth projective curve of any genus and the existence of their moduli spaces $\mathcal{M}(G)_d$ (here $d \in \pi_1(G)$ is a topological invariant known as the degree) follows from Simpson [**31**, **32**] (the existence of $\mathcal{M}(\mathrm{SL}(2, \mathbb{C}))$ was first given in [**19**] and the case of $\mathrm{GL}(n, \mathbb{C})$ was also given by Nitsure [**26**]).

A major result of the theory of G-Higgs bundles is the non-abelian Hodge correspondence which was proved by Hitchin [**19**], Donaldson [**10**], Simpson [**30**, **31**, **32**] and Corlette [**7**]. It is a generalisation of the Narasimhan–Seshadri–Ramanathan Theorem [**25**, **27**] to the non-unitary case and states the existence of a chain of homeomorphisms between the moduli space of G-Higgs bundles, the moduli space of G-bundles with projectively flat connections $\mathcal{C}(G)_d$ and the moduli space of representations $\mathcal{R}(G)_d$ of the curve

$$\mathcal{M}(G)_d \overset{\text{homeo}}{\cong} \mathcal{C}(G)_d \overset{\text{homeo}}{\cong} \mathcal{R}(G)_d.$$

The Hitchin fibration was defined by Hitchin [**20**] using a basis $p_1, \ldots, p_\ell$ of the invariant polynomials of the Lie algebra $\mathfrak{g}$

$$\begin{array}{rcl} \mathcal{M}(G)_d & \longrightarrow & B_G \cong \bigoplus H^0(X, (\Omega^1_X)^{\otimes \deg(p_i)}) \\ (E, \Phi) & \longmapsto & (p_1(\Phi), \ldots, p_\ell(\Phi)). \end{array}$$

A more canonical definition of the Hitchin fibration was provided by Donagi [**8**] redefining the Hitchin base B_G as the space of cameral covers $H^0(X, (\mathfrak{g} \otimes \Omega^1_X) /\!\!/ G)$. Another ground-breaking result of the theory of Higgs bundles says that, under this fibration, the space of G-Higgs bundles is an algebraically completely integrable system [**20**, **11**, **8**].

In 1957, Atiyah [**1**] studied vector bundles over an elliptic curve X leading to an identification of the moduli space of vector bundles $M(\mathrm{GL}(n, \mathbb{C}))_d$ with $\mathrm{Sym}^h X$, where h is the greatest common divisor of n and d. Some 40 years later, Laszlo [**22**] and Friedman, Morgan and Witten [**14**, **16**], gave a description of the moduli space of G-bundles $M(G)_d$ ([**22**] only deals with $M(G)_0$) as the quotient

$$M(G)_d \cong (X \otimes_{\mathbb{Z}} \Lambda_{G,d}) \big/ W_{G,d}, \tag{1}$$

where $\Lambda_{G,d}$ is a certain lattice, $W_{G,d}$ is a finite group acting on $\Lambda_{G,d}$ and $X \otimes_{\mathbb{Z}} \Lambda_{G,d}$ is the tensor product over $\mathbb{Z}$ (recall that X is an abelian variety and therefore has a natural $\mathbb{Z}$-module structure). When G is simply connected (and therefore $d = 0$), $\Lambda_{G,0} = \Lambda$ is the coroot lattice and $W_{G,0} = W$ is the Weyl group of G. In this case, by a result of Looijenga [**24**] (see also [**3**]), $M(G)_0$ is isomorphic to a weighted projective space. This isomorphism was obtained directly by Friedman and Morgan [**15**] working with deformations of unstable G-bundles (see also [**18**]).

The construction of the isomorphism (1) relies on two facts. The first one is the description of the moduli space of unitary representations $R(G)_d$ achieved by Schweigert [**29**] and more generally by Borel, Friedman and Morgan [**5**]. By the Narasimhan–Seshadri–Ramanathan theorem, $R(G)_d$ is homeomorphic to $M(G)_d$. This shows that an appropriate morphism from $(X \otimes_{\mathbb{Z}} \Lambda_{G,d})/W_{G,d}$ to $M(G)_d$ is bijective. The other key result is the fact that $M(G)_d$ is a normal projective variety, which allows us to apply Zariski's main theorem, proving that the previous bijective morphism is indeed an isomorphism.

In this paper, we describe $\mathcal{M}(G)_d$ for any complex reductive group G, thus, generalising [**13**], where the authors studied these objects when G is a classical group.

The results of this paper are structured as follows. After reviewing in Section 2, the theory of unitary representations and G-bundles over an elliptic curve, we prove in Section 3 that a G-Higgs bundle is (semi)stable if and only if the underlying G-bundle is (semi)stable [Propositions 3.1 and 3.3]. This fact shows the existence of a projection [Corollary 3.2]

$$\mathcal{M}(G)_d \longrightarrow M(G)_d \tag{2}$$

and, combined with the results of [**5**], implies that every polystable G-Higgs bundle of degree d reduces to a unique (up to conjugation) Jordan–Hölder Levi subgroup $L_{G,d}$

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[Proposition 3.7]. This allows us to give a complete description of the polystable G-Higgs bundles [Corollaries 3.8 and 3.9]. Using this description, we construct a family $\mathcal{H}_{G,d}$ of polystable G-Higgs bundles of degree d parametrised by $T^*X \otimes_{\mathbb{Z}} \Lambda_{G,d}$. Every polystable G-Higgs bundle can be constructed starting from a Higgs bundle for an abelian group [Remark 3.11], which shows that the non-abelian Hodge correspondence is not entirely non-abelian in the elliptic case. Next, we show that the morphism associated to the family $\mathcal{H}_{G,d}$ factors through a bijective morphism and, using Zariski's main theorem, this gives us a description of the normalisation of the moduli space [Theorem 3.14]

$$\overline{\mathcal{M}(G)_d} \cong (T^*X \otimes_{\mathbb{Z}} \Lambda_{G,d}) \big/ W_{G,d}. \tag{3}$$

It is not known whether $\mathcal{M}(G)_d$ is a normal quasiprojective variety (see [**13**, Section 3.4] for a detailed discussion), so we can not apply the method used to prove (1) since the hypothesis of Zariski's main theorem requires the normality of the target. By means of this bijection and the quotient (2), we define a natural orbifold structure on $M(G)_d$ and the projection (2) corresponds with the projection of the associated cotangent orbifold bundle [Remark 3.18].

In Section 4, we study the Hitchin fibration and we obtain that it corresponds to the projection [Proposition 4.1]

$$(T^*X \otimes_{\mathbb{Z}} \Lambda_{G,d}) \big/ W_{G,d} \longrightarrow (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{G,d}) \big/ W_{G,d},$$

induced by the obvious projection from $T^*X \cong X \times \mathbb{C}$ to $\mathbb{C}$. This gives us an explicit description of (the normalisation of) all the fibres of the Hitchin fibration and, more concretely, the generic ones [Corollary 4.2].

In Section 5, we use the non-abelian Hodge correspondence and our description of G-Higgs bundles to extend the results of [**5**] about unitary representations of surface groups of an elliptic curve to reductive representations of this surface group [Corollaries 5.1 and 5.2]. This allows us to construct a bijective morphism to the moduli space $\mathcal{R}(G)_d$ of representations and then the normalisation of the moduli space is [Corollary 5.4]

$$\overline{\mathcal{R}(G)_d} \cong (\mathbb{C}^* \times \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_{G,d} \big/ W_{G,d}. \tag{4}$$

In Section 6, we study the moduli space $\mathcal{C}(G)_d$ of G-bundles with projectively flat connections. Using only the Narasimhan–Seshadri–Ramanathan theorem and the fact that the underlying G-bundle of a polystable G-Higgs bundle is also polystable, we observe a splitting of the Hitchin equations [Proposition 6.1] that simplifies the proof of the Hitchin–Kobayashi correspondence over elliptic curves [Corollary 6.2, Remark 6.3]. We obtain that the normalisation of the moduli space is [Theorem 6.8]

$$\overline{\mathcal{C}(G)_d} \cong (X^\sharp \otimes_{\mathbb{Z}} \Lambda_{G,d}) \big/ W_{G,d}, \tag{5}$$

where we recall that $X^\sharp$ is the moduli space of rank 1 local systems on X.

In the trivial degree case, (3)–(5) become

$$\overline{\mathcal{M}(G)_0} \cong (T^*X \otimes_{\mathbb{Z}} \Lambda) \big/ W,$$

$$\overline{\mathcal{R}(G)_0} \cong ((\mathbb{C}^* \times \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda) \big/ W$$

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and

$$\overline{\mathcal{C}(G)_0} \cong (X^\sharp \otimes_{\mathbb{Z}} \Lambda) \big/ W,$$

where W is the Weyl group of G and Λ is the lattice given by the kernel of the exponential restricted to the Cartan subalgebra (i.e., the fundamental group of the Cartan subgroup). This was obtained by Thaddeus [**33**] in 2001. Our arguments are different from those of Thaddeus and work for arbitrary d.

When $G = \mathrm{GL}(n, \mathbb{C})$ or $\mathrm{SL}(n, \mathbb{C})$ (for any n, not only for $n \leq 4$ as stated in [**13**]) one actually obtains an isomorphism since the target is normal. In these cases,

$$\mathcal{R}(G)_0 := \mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z}, G) /\!\!/ G \subset \{x, y \in \mathfrak{g} : [x, y] = 0\} /\!\!/ G$$

is normal due to [**21**, Section 0.2] (although the hypothesis of [**21**] requires G to be semisimple, the proof can be extended to $\mathrm{GL}(n, \mathbb{C})$ as in [**23**, Corollary 7.4]). Normality of $\mathcal{M}(G)_0$ and $\mathcal{C}(G)_0$ follow from the Isosingularity theorem [**32**, Theorem 10.6] and normality of $\mathcal{R}(G)_0$. The corresponding results for $\mathcal{R}(G)_0$ for general reductive G constitute a long-standing open problem and the case of $\mathcal{R}(G)_d$ is still more uncertain. Indeed, it is not even clear whether the moduli spaces are reduced.

We work in the category of algebraic schemes over $\mathbb{C}$. Unless, otherwise stated, all the bundles considered are algebraic bundles.

2. Review on G-bundles and unitary representations over elliptic curves.

2.1. Review on the abelian case. If X is an elliptic curve, the Abel–Jacobi map gives an isomorphism $X \cong \mathrm{Pic}^1(X)$. Fixing a point $x_0 \in X$ and tensoring by $\mathcal{O}(x_0)^{-1}$ one obtains $\varsigma_{1,0} : X \xrightarrow{\cong} \mathrm{Pic}^0(X)$, which induces an abelian group structure on X. There is a unique Poincaré bundle $\mathcal{P} \to X \times \mathrm{Pic}^0(X)$ such that its restriction to the slice $\{x_0\} \times \mathrm{Pic}^0(X)$ is the trivial line bundle.

Let S be a compact connected abelian group and let $S^{\mathbb{C}}$ be its complexification. The universal cover of S (resp. $S^{\mathbb{C}}$) is its Lie algebra $\mathfrak{s}$ (resp. $\mathfrak{s}^{\mathbb{C}}$) and the covering map is the exponential $\exp : \mathfrak{s} \to S$ (resp. $\mathfrak{s}^{\mathbb{C}} \to S^{\mathbb{C}}$). By construction, the kernels of the two maps coincide and we write

$$\Lambda_S := \Lambda_{S^{\mathbb{C}}} := \ker \exp,$$

which is a lattice in $\mathfrak{s} \subset \mathfrak{s}^{\mathbb{C}}$. Note that the fundamental groups $\pi_1(S)$ and $\pi_1(S^{\mathbb{C}})$ coincide since both are identified with the kernel of the exponential map.

Every element $\gamma \in \Lambda_S$ defines a cocharacter $\theta : \mathbb{C}^* \to S^{\mathbb{C}}$ that restricts to $\theta : \mathrm{U}(1) \to S$. Let $\mathcal{B} = \{\gamma_1, \ldots, \gamma_k\}$ be a basis of Λ_S and let $\{\theta_1, \ldots, \theta_k\}$ be the associated cocharacters. These give isomorphisms

$$\begin{array}{rccc} \Theta_S & : & \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_S \xrightarrow{\cong} & S^{\mathbb{C}} \\ & & \mathrm{U}(1) \otimes_{\mathbb{Z}} \Lambda_S \xrightarrow{\cong} & S \\ & & \sum_{i=1}^k \ell_i \otimes_{\mathbb{Z}} \gamma_i \longmapsto & \Pi_{i=1}^k \theta_i(\ell_i), \end{array} \tag{6}$$

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where $\ell_i \in \mathbb{C}^*$ (resp. U(1)), and

$$\begin{array}{rcccc} d\Theta_S & : & \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_S & \xrightarrow{\cong} & \mathfrak{s}^{\mathbb{C}} \\ & & \mathbb{R} \otimes_{\mathbb{Z}} \Lambda_S & \xrightarrow{\cong} & \mathfrak{s} \\ & & \sum_{i=1}^{k}(\lambda_i \otimes_{\mathbb{Z}} \gamma_i) & \longmapsto & \sum_{i=1}^{k} \lambda_i \cdot \gamma_i, \end{array} \tag{7}$$

where $\lambda_i \in \mathbb{C}$ (resp. $\mathbb{R}$).

Using (6) and fibre products of the Poincaré bundle $(\mathrm{id} \times \varsigma_{1,0})^* \mathcal{P} \to X \times X$, one can construct a family of $S^{\mathbb{C}}$-bundles with trivial degree

$$\mathcal{P}_S \longrightarrow X \times (X \otimes_{\mathbb{Z}} \Lambda_S), \tag{8}$$

whose restriction to the slice $\{x_0\} \times (X \otimes_{\mathbb{Z}} \Lambda_S)$ is the trivial $S^{\mathbb{C}}$-bundle over $(X \otimes_{\mathbb{Z}} \Lambda_S)$.

Among other references, the following result is contained in [**32**, Theorem 9.6] (recall that for an elliptic curve $X \cong \mathrm{Pic}^0(X)$).

THEOREM 2.1. *Let $S^{\mathbb{C}}$ be an abelian, connected complex Lie group. Then, the moduli space of topologically trivial $S^{\mathbb{C}}$-bundles over the elliptic curve X is*

$$M(S^{\mathbb{C}})_0 \cong X \otimes_{\mathbb{Z}} \Lambda_S.$$

2.2. Notation and some results on Lie groups. We refer to [**12**] for an expanded version of this section. Let G denote a compact (resp. complex reductive) connected Lie group. We set some notations, which are as follows:

- Z_0 denotes the connected component of the identity of the centre $Z_G(G)$ of the group,
- $p : D \to [G, G]$ denotes the universal covering of the semisimple group $[G, G]$,
- $F := Z_0 \cap [G, G]$,
- $C := p^{-1}(F) \subset Z_D(D)$,
- $\tau : C \to Z_0$ denotes the homomorphism given by the inclusion $F \hookrightarrow Z_0$.
- $\overline{G} := G/F$,
- $\overline{Z} := Z_0/F$,
- $\overline{D} := D/C$ or equivalently $[G, G]/F$,
- $H \subset G$ denotes a maximal torus (resp. Cartan subgroup) with Lie algebra $\mathfrak{h}$,
- $H' \subset D$ denotes a maximal torus (resp. Cartan subgroup) with Lie algebra $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$,
- $W = N_G(H)/Z_G(H) = N_D(H')/Z_D(H')$ denotes the Weyl group.

Note that we have natural isomorphisms

$$G \cong Z_0 \times_\tau D \tag{9}$$

and

$$\overline{G} \cong \overline{Z} \times \overline{D}. \tag{10}$$

The finite covering $G \to \overline{G}$ induces an injection

$$\begin{array}{ccc} \pi_1(G) & \hookrightarrow & \pi_1(\overline{Z}) \times \pi_1(\overline{D}) \\ d & \longmapsto & (u, c). \end{array} \tag{11}$$

Since D is simply connected and C finite, we have

$$\pi_1(\overline{D}) = C.$$

Let us suppose for simplicity that D is a simple compact Lie group (resp. simple complex Lie group). Take an alcove $A \subset \mathfrak{h}'$ containing the origin. For $c \in Z_D(D)$, we know (see, for instance, [**6**]) that there is a vertex a_c of the alcove A such that $c = \exp(a_c)$. We see that $A - a_c$ is another alcove containing the origin. Hence, there is a unique element $\omega_c \in W$ such that

$$A - a_c = \omega_c(A).$$

In the trivial case, we obviously have $\omega_0 = \mathrm{id}$.

We denote the connected component of the fixed point set of the action of ω_c on H by

$$S_c := (H^{\omega_c})_0. \tag{12}$$

Let us take its normaliser $N_G(S_c)$ and define the quotient

$$W_c := {N_G(S_c)} \big/ {Z_G(S_c)} = {N_G(\mathfrak{h}^{\omega_c})} \big/ {Z_G(\mathfrak{h}^{\omega_c})}. \tag{13}$$

When c is the identity, one recovers the usual Weyl group W.

We define

$$L_c := Z_G(S_c). \tag{14}$$

Since L_c is the centraliser of a torus, we know that it is connected. One can easily check that $N_G(S_c) = N_G(L_c)$ and therefore $W_c = {N_G(L_c)} \big/ {L_c}$.

By [**5**, Lemma 2.1.1] and [**5**, Proposition 3.4.4], $D_c = [L_c, L_c]$ is simply connected. Define $F_c = S_c \cap D_c$ and note that S_c is the centre of L_c. By (9), we have $L_c \cong S_c \times_{F_c} D_c$. Note, by (11), that $\pi_1(L_c)$ injects into $\pi_1(\overline{S_c}) \times \pi_1(D_c/F_c)$, where

$$\overline{S_c} := S_c/F_c. \tag{15}$$

The inclusion $L_c \hookrightarrow G$ induces a morphism $\pi_1(L_c) \to \pi_1(G)$.

LEMMA 2.2. *Let $d = (u, c) \in \pi_1(G)$ and let L_c be associated to c. Then, there is a unique $\ell_d \in \pi_1(L_c)$ that maps to d and furthermore $\ell_d = (u, p(c))$.*

Proof. By construction, we have that $p(c) \in D_c = [L_c, L_c]$ and $p(c) \in S_c$, thus, $p(c) \in F_c \subset Z_{D_c}(D_c)$. If $\ell \in \pi_1(L_c)$ is given by $(v, f) \in \mathfrak{s} \times Z_{D_c}(D_c)$ and it maps to d, then $f = p(c)$ and $v = u$, since $v \in \exp^{-1}(p(c)) \subset \exp^{-1}(F) \subset \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$. The choice of d fixes (v, f), so its preimage $\ell \in \pi_1(L_c)$ is unique. □

Recall that $W_c = N_G(\mathfrak{s}_c)/Z_G(\mathfrak{s}_c)$, where $\mathfrak{s}_c = \mathfrak{h}^{\omega_c}$ is the Lie algebra of S_c, and note that W_c preserves $\Lambda_{S_c} \subset \mathfrak{s}_c$. This gives us an action of W_c on $\mathrm{U}(1) \otimes_{\mathbb{Z}} \Lambda_{S_c}$ (resp. on $\mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_{S_c}$) and this action commutes with the isomorphism Θ_{S_c} defined in (6).

In (15), we have defined $\overline{S}_c$ as S_c/F_c. We can check that W_c preserves F_c, so the action of W_c on S_c gives a well-defined action of W_c on $\overline{S}_c$. Note that $\Lambda_{\overline{S}_c} = \exp_S^{-1}(F_c)$, so W_c also preserves $\Lambda_{\overline{S}_c}$, inducing an action on $\mathrm{U}(1) \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$ (resp. on $\mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$). We can check that the action of W_c commutes with $\Theta_{\overline{S}_c}$ too.

2.3. Representations and c-pairs. In this section, we present some results from [**5**] (see also [**12**]). We say that two elements of a Lie group G *almost commute* if their commutator lies in the centre of the Lie group. Let c be an element of $C \subset Z_D(D)$. Suppose a and b are two almost commuting elements of the form $a = [(z_1, \delta_1)]_\tau$ and $b = [(z_2, \delta_2)]_\tau$, where $z_1, z_2 \in Z_0$ and $\delta_1, \delta_2 \in D$. We say that (a, b) is a *c-pair* if $[\delta_1, \delta_2] = c$. Let $C(G)_c$ denote the subset of $G \times G$ of c-pairs.

The fundamental group of an elliptic curve is $\pi_1(X) = \langle \alpha, \beta : [\alpha, \beta] = \mathrm{id}\rangle \cong \mathbb{Z}^2$. Take the universal central extension $\Gamma = \langle \alpha, \beta, \delta : [\alpha, \beta] = \delta, [\alpha, \delta] = \mathrm{id}, [\beta, \delta] = \mathrm{id}\rangle$ and define $\Gamma_{\mathbb{R}}$ as $\Gamma \times_{\mathbb{Z}} \mathbb{R}$. A representation $\rho : \Gamma_{\mathbb{R}} \to G$ is *central* if $\rho(\mathbb{R})$ is contained in $Z_G(G)$; since $\rho(\mathbb{R})$ is connected and contains the unit element, it is contained in $Z_0 = Z_G(G)_0$. From a central representation $\rho : \Gamma_{\mathbb{R}} \to G$, one obtains a pair (ν, u), where $\nu : \Gamma \to G$ is such that $\nu = \rho|_\Gamma$ and $u \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$ is given by $u = d\rho(1)$ and, thanks to the exponential map, u can be viewed as an element of the fundamental group of $\overline{Z}$. Conversely, (ν, u) determines uniquely a central representation $\rho : \Gamma_{\mathbb{R}} \to G$. We observe that $u \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$ is an invariant of the conjugacy class of the representation ρ. We denote by $\mathrm{Hom}^c(\Gamma_{\mathbb{R}}, G)_d$, the set of central representations with topological invariant d and we define the *moduli space of such representations* as the GIT quotient by the conjugation action of the group

$$\mathcal{R}(G)_d := \mathrm{Hom}^c(\Gamma_{\mathbb{R}}, G)_d /\!\!/ G.$$

Every central representation $\nu : \Gamma \to G$ is completely determined by two elements of G, $a = \nu(\alpha)$ and $b = \nu(\beta)$. Since ν is central, $\nu(\delta) = [a, b]$ is contained in Z_0 and therefore in $F = Z_0 \cap [G, G]$. Take $a = [(z_1, \delta_1)]_\tau$ and $b = [(z_2, \delta_2)]_\tau$, and write $c = [\delta_1, \delta_2]$, where $\nu(\delta) = \tau(c)$. Then, (a, b) completely determines the representation $\nu : \Gamma \to G$ and is a c-pair. Furthermore, $c \in C \subset Z_D(D)$ is an invariant of the conjugacy class of the representation ν.

Remark 2.3. Every central representation $\rho : \Gamma_{\mathbb{R}} \to G$ is determined by a c-pair $(a, b) \in C(G)_c$ and an element u of $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$ that satisfies $\tau(c) = \exp(u)$. The pair $d = (u, c) \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}) \times Z_D(D)$ is an invariant of the conjugacy class of ρ. Indeed, d is an element of $\pi_1(G)$ as indicated by (11).

For any $g \in G$, the representation $g\rho g^{-1}$ is determined by (gag^{-1}, gbg^{-1}, u), where (gag^{-1}, gbg^{-1}) is a c-pair.

By Remark 2.3, we see that $\mathrm{Hom}^c(\Gamma_{\mathbb{R}}, G)_{(u,c)}$ can be identified with $C(G)_c$. As a consequence, the moduli space of representations of $\Gamma_{\mathbb{R}}$ for an elliptic curve with invariant $d \in \pi_1(G)$ determined by $(u, c) \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}) \times Z_D(D)$ coincides with the moduli space of c-pairs

$$\mathcal{R}(G)_d \cong C(G)_c /\!\!/ G.$$

Suppose now that G is a connected complex reductive algebraic group and let K be its maximal compact subgroup. A representation ρ is *reductive* if and only if the Zariski closure of $\mathrm{im}\,\rho$ is a reductive group. It is proved in [**28**] that the orbit $[\rho]_G$ is closed if and only if ρ is a reductive representation. Denote by $\mathrm{Hom}^c(\Gamma_{\mathbb{R}}, G)_d^+$, the set of central reductive representations, and by $C(G)_c^+$, the set of reductive c-pairs (those associated to reductive representations). Then,

$$\mathcal{R}(G)_d \cong \mathrm{Hom}^c(\Gamma_{\mathbb{R}}, G)_d^+ / G \cong C(G)_c^+ / G. \tag{16}$$

Note that, for G compact, every representation of $\Gamma_{\mathbb{R}}$ is reductive. So, the moduli space of unitary representations is a categorical quotient

$$R(G)_d := \mathcal{R}(K)_d \cong C(K)_c \big/ K.$$

A representation ρ is *irreducible* if the centraliser of its image, $Z_G(\rho)$, is equal to $Z_G(G)$. Analogously, we say that a c-pair (a, b) is *irreducible* if the centraliser of its elements, $Z_G(a, b)$, is equal to $Z_G(G)$.

2.4. Review on unitary representations over elliptic curves. Following [**5**], in this section, we study the moduli space of central representations of $\Gamma_{\mathbb{R}}$ into a compact Lie group K. Let $C = p^{-1}(F) = \pi_1(\overline{D})$ as defined at the beginning of Section 2.2 and set $c \in C$.

PROPOSITION 2.4. **([5, Proposition 4.1.1]).** *Let K be a simply connected compact semisimple Lie group. Let (a, b) be an irreducible c-pair in K. Then,*

(1) *the group K is a product of simple factors K_i, where each K_i is isomorphic to* $\mathrm{SU}(n_i)$ *for some $n_i \geq 2$;*
(2) *$c = (c_1, \ldots, c_r)$, where each c_i generates the centre of K_i;*
(3) *conversely, if K is as in (1) and c as in (2), then there is an irreducible c-pair in K and all c-pairs in K are conjugate.*

Recall that $L_c \cong S_c \times_{\tau_c} D_c$, where S_c, L_c are defined in (12), (14) and $D_c = [L_c, L_c]$.

PROPOSITION 2.5. **([5, Proposition 4.2.1]).** *Let K be a compact Lie group. Let (a, b) be any c-pair. Any maximal torus of $Z_K(a, b)$ is conjugate in K to S_c, so (a, b) is contained in L_c after conjugation and, as a c-pair in L_c, is irreducible.*

Now, we have the ingredients to describe the moduli space of unitary representations. Fix $d \in \pi_1(G)$ determined by $(u, c) \in \pi_1(\overline{Z}) \times \pi_1(\overline{D})$ as described in (11). Let us take (δ_1, δ_2) to be one representative of the unique conjugation class of the irreducible c-pair in D_c. Consider the following continuous map:

$$\begin{aligned} (S_c \times S_c) &\longrightarrow \mathcal{R}(K)_d \\ (s_1, s_2) &\longmapsto ([s_1, \delta_1]_{\tau_c}, [s_2, \delta_2]_{\tau_c}). \end{aligned} \tag{17}$$

Using Proposition 2.5, one can check that (17) is surjective.

REMARK 2.6. By Proposition 2.4, we have $D_c = \mathrm{SU}(n_1) \times \cdots \times \mathrm{SU}(n_\ell)$. Let $\delta_{1,i}$ and $\delta_{2,i}$ be the projections of δ_1 and δ_2 to $\mathrm{SU}(n_i)$. The conjugation of the c-pair $([s_1, \delta_1]_{\tau_c}, [s_2, \delta_2]_{\tau_c})$ by $[\mathrm{id}, \delta_{1,i}]_{\tau_c}$ gives us $([s_1, \delta_1]_{\tau_c}, [s_2, c_i\delta_2]_{\tau_c})$ and similarly, conjugating by $[\mathrm{id}, \delta_{2,i}]_{\tau_c}$ gives $([s_1, c_i^{-1}\delta_1]_{\tau_c}, [s_2, \delta_2]_{\tau_c})$. By Proposition 2.4(2), the c_i generate $Z_{D_c}(D_c)$, so it is obvious that (17) factors through

$$\overline{S}_c \times \overline{S}_c \longrightarrow \mathcal{R}(K)_d. \tag{18}$$

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One can further prove that (18) factors through the quotient by the finite group W_c, defined in (13).

THEOREM 2.7. **([5, Corollary 4.2.2]).** *Let K be a compact connected Lie group. There is a homeomorphism*

$$(\overline{S_c} \times \overline{S_c}) \Big/ W_c \xrightarrow{homeo} \mathcal{R}(K)_d.$$

REMARK 2.8. Since (6) gives us the isomorphism $\Theta_{\overline{S}_c} : \mathrm{U}(1) \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c} \xrightarrow{\cong} \overline{S}_c$ and the action of W_c commutes with $\Theta_{\overline{S}_c}$, we have a natural homeomorphism

$$\big((\mathrm{U}(1) \times \mathrm{U}(1)) \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}\big) \Big/ W_c \xrightarrow{homeo} \mathcal{R}(K)_d.$$

2.5. Review on G-bundles over an elliptic curve. Let G be a connected complex reductive Lie group with maximal compact K. The notions of stability, semistability, polystability and S-equivalence for G-bundles are well known (see, for example, [**27**]).

Given a unitary representation $\rho : \Gamma_{\mathbb{R}} \to K \subset G$, after [**2**], we can construct the G-bundle E^ρ as follows (see also [**27**] for a similar construction). Consider the line bundle $\mathcal{O}(x_0)$ associated with the divisor given by the fixed point x_0 of X and let $Q'_{x_0} \to X$ be the fixed U(1)-bundle obtained from reduction of structure group of $\mathcal{O}(x_0)$. The universal covering $\widetilde{X} \to X$ is a $\pi_1(X)$-bundle. Consider the fibre product $\widetilde{X} \times_X Q'_{x_0}$ and denote by Q_{x_0} its lifting to $\Gamma_{\mathbb{R}}$. We set E^ρ as the extension of structure group associated to ρ of Q_{x_0}, i.e.,

$$E^\rho = \rho_* Q_{x_0}. \tag{19}$$

As shown in [**2**] (see also [**27**]),

- the bundles E^ρ are polystable,
- two bundles E^{ρ_1} and E^{ρ_2} are isomorphic if and only if ρ_1 and ρ_2 are conjugate,
- every polystable G-bundle E is isomorphic to some E^ρ, and
- the bundle E^ρ is stable if and only if the representation ρ is irreducible.

We can interpret as follows the results of [**5**] given in Section 2.4.

PROPOSITION 2.9. *Let G be a connected, complex semisimple Lie group and denote by $\widetilde{G}$ its universal cover. Let E^ρ be a stable G-bundle of degree $d \in \pi_1(G) \subset Z_{\widetilde{G}}(\widetilde{G})$. Then,*

(1) *the group $\widetilde{G}$ is a product of simple factors G_i, where each G_i is isomorphic to* $\mathrm{SL}(n_i, \mathbb{C})$ *for some $n_i \geq 2$;*
(2) *$d = (d_1, \dots, d_r)$, where each d_i generates $\pi_1(G_i) \cong \mathbb{Z}_{n_i}$;*
(3) *conversely, if G is as in (1) and d as in (2), then there is a stable G-bundle of degree d and all G-bundles of degree d are isomorphic, i.e.,*

$$M(G)_d = M^{st}(G)_d = \{pt\}.$$

Proof. Since, by Remark 2.3, a representation ρ is determined by a c-pair, and the c-pair is irreducible if and only if the representation is irreducible, the proof follows from Proposition 2.4 ([**5**, Proposition 4.1.1]) and the existence of a bijective correspondence between irreducible representations and stable G-bundles. □

REMARK 2.10. Note that Proposition 2.9 implies that, for G simple, the only stable bundles occur when $G = \mathrm{PGL}(n, \mathbb{C})$ and d generating $\mathbb{Z}_n$ (*i.e.*, n and d coprime).

Let G be a complex reductive Lie group, and let F be as defined at the beginning of Section 2.2. Since $F \subset Z_G(G)$, the extension of structure group given by the multiplication map $\mu : F \times G \to G$ is well defined. Given an F-bundle J and a G-bundle E, we denote by $J \otimes E$ the G-bundle $\mu_*(J \times_X E)$.

COROLLARY 2.11. *Let E be a stable G-bundle of topological class d and let J be any element of $H^1(X, F)$. Then,*

$$E \cong J \otimes E,$$

so $J \otimes E$ has the same topological invariant as E.

Proof. This follows from Remark 2.6. □

By [**27**, Proposition 7.1], a G-bundle is stable if and only if the induced (G/Z_0)-bundle is stable. Let $\overline{Z}$ and $\overline{D}$ be as defined at the beginning of Section 2.2.

THEOREM 2.12. *Let G be a connected complex reductive Lie group and let $d \in \pi_1(G)$. Then,*

$$M^{st}(G)_d = \emptyset,$$

unless G/Z_0 decomposes into $\mathrm{PGL}(n_1, \mathbb{C}) \times \ldots \times \mathrm{PGL}(n_s, \mathbb{C})$ *and $d \in \pi_1(G_i)$ projects to* $(d_1, \ldots, d_s) \in \pi_1(\mathrm{PGL}(n_1, \mathbb{C})) \times \pi_1(\mathrm{PGL}(n_s, \mathbb{C}))$, *where* $\gcd(n_i, d_i) = 1$. *In that case, there is a natural isomorphism*

$$M^{st}(G)_d = M(G)_d \cong X \otimes_{\mathbb{Z}} \Lambda_{\overline{Z}}.$$

Proof. The first statement follows from Proposition 2.9.

The extension of structure group associated to $G \to \overline{G} \cong \overline{Z} \times \overline{D}$ (see (10)) induces a morphism

$$M^{st}(G)_d \longrightarrow M^{st}(\overline{G})_{(u,c)} \cong M^{st}(\overline{Z})_u \times M^{st}(\overline{D})_c. \tag{20}$$

This morphism is injective by Corollary 2.11. For any stable G-bundle E, the morphism

$$\begin{array}{ccc} M^{st}(Z)_u & \longrightarrow & M^{st}(\overline{Z})_u \\ J & \longmapsto & \overline{J} := (J \otimes E)/D \cong J/F \end{array}$$

is surjective, as $\overline{J}$ is the extension of structure group of J associated to $Z \to \overline{Z}$. Then, the morphism (20) is bijective, and, therefore, it is an isomorphism. By Proposition 2.9, $M^{st}(\overline{D})_c = \{pt\}$, so the second statement follows from Theorem 2.1. □

REMARK 2.13. Note that the point $x_0 \in X$ defines an origin in $M^{st}(\overline{Z})_u$. For G and d of the form given in Theorem 2.12, we write $E^{x_0}_{G,d}$ for the stable G-bundle of degree d associated to this point of $M^{st}(\overline{Z})_u$. Let Z_0 be the connected component of the centre of G and consider the universal family of Z_0-bundles $\mathcal{P}_{Z_0}$ parametrised by $X \otimes_{\mathbb{Z}} \Lambda_{Z_0}$, which is defined in (8). We define the family $(\mathcal{E}')_{G,d} = \mathcal{P}_{Z_0} \otimes E^{x_0}_{G,d}$ of G-Higgs bundles with degree d. By Corollary 2.11, this family descends to a family parametrised by the quotient of $X \otimes_{\mathbb{Z}} \Lambda_{Z_0}$ by the image of $H^1(X, F)$. Recalling that $\exp^{-1}(F) = \Lambda_{\overline{Z}} \subset \Lambda_{Z_0}$, one can check that this quotient is isomorphic to $X \otimes_{\mathbb{Z}} \Lambda_{\overline{Z}}$. Then, we have a family

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$\mathcal{E}_{G,d} \to X \times (X \otimes_{\mathbb{Z}} \Lambda_{\overline{Z}})$ such that

$$\begin{array}{ccc} X \otimes_{\mathbb{Z}} \Lambda_{\overline{Z}} & \xrightarrow{\cong} & M^{st}(G)_d \\ t & \longmapsto & [\mathcal{E}_{G,d}|_{X\times\{t\}}]_{\cong}. \end{array}$$

PROPOSITION 2.14. *Every polystable G-bundle of topological type $d = (u, c)$ admits a reduction of structure group to L_c, giving a stable L_c-bundle of topological class $\ell_d = (u, p(c))$.*

Proof. Every polystable G-bundle is isomorphic to some E^ρ. By Remark 2.3, ρ is determined by u and a c-pair $(a, b) \in K \times K$. By Proposition 2.5 ([**5**, Proposition 4.2.1]), (a, b) is contained (after conjugation) in the maximal compact subgroup of L_c and is irreducible as a c-pair in that group. Then, $\operatorname{im}\rho \subset L_c$ and ρ is irreducible in L_c, so E^ρ reduces to a stable L_c-bundle. □

By Proposition 2.14, it makes sense to define the following family parametrising all polystable G-bundles of degree d,

$$\mathcal{E}_{G,d} := i_*(\mathcal{E}_{L_c,\ell_d}), \tag{21}$$

where $i : L_c \hookrightarrow G$ is the natural inclusion. Note that this family is parametrised by $X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$, where S_c is the centre of L_c. This family induces a morphism to the moduli space

$$X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c} \longrightarrow M(G)_d, \tag{22}$$

which is surjective by Proposition 2.14.

THEOREM 2.15. *Let G be a connected complex reductive Lie group and let $d \in \pi_1(G)$. Then,*

$$M(G)_d \cong \left(X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}\right) \big/ W_c.$$

Proof. It is clear that (22) descends to a surjective morphism

$$\zeta_{G,d} : \left(X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}\right) \big/ W_c \longrightarrow M(G)_d.$$

Injectivity follows from Corollary 2.11 and the fact that the reduction of structure group to L_c is unique up to conjugation. Now $\varsigma_{G,d}$ is an isomorphism by Zariski's main theorem. □

COROLLARY 2.16. *Let E_1 and E_2 be two polystable G-bundles of topological class d parametrised by $\mathcal{E}_{G,d}$ at the points t_1 and $t_2 \in X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$. Then, E_1 and E_2 are isomorphic G-bundles if and only if there exists $\omega \in W_c$ such that $t_2 = \omega \cdot t_1$.*

3. *G*-Higgs bundles over an elliptic curve.

Let G be a connected complex reductive Lie group. Recall that a G-Higgs bundle over an elliptic curve X is a pair (E, Φ), where E is an algebraic G-bundle over X and $\Phi \in H^0(X, E(\mathfrak{g}))$. We say that (E, Φ) is *stable* (resp. *semistable*) if, for every proper parabolic subgroup P with Lie algebra $\mathfrak{p}$, any non-trivial antidominant character $\chi : P \to \mathbb{C}^*$, and any reduction of structure group σ to the parabolic subgroup P giving the P-bundle E_σ such that $\Phi \in H^0(X, E_\sigma(\mathfrak{p}))$, we

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have

$$\deg \chi_* E_\sigma > 0 \quad (\text{resp. } \geq 0).$$

Let (E_1, Φ_1) and (E_2, Φ_2) be two semistable G-Higgs bundles and suppose that there exists a family $\mathcal{H}$ parametrised by $\mathbb{C}$ such that $\mathcal{H}|_{X\times\{\lambda\}} \cong (E_1, \Phi_1)$ if $\lambda \neq 0$ and $\mathcal{H}|_{X\times\{0\}} \cong (E_2, \Phi_2)$. We say that these two G-Higgs bundles are *S-equivalent* and we call the induced equivalence relation *S-equivalence*, writing $(E_1, \Phi_1) \sim_S (E_2, \Phi_2)$. Two families of semistable G-Higgs bundles parametrised by Y are *S-equivalent*, $\mathcal{H}_1 \sim_S \mathcal{H}_2$, if for every point $y \in Y$, one has $\mathcal{H}_1|_{X\times\{y\}} \sim_S \mathcal{H}_2|_{X\times\{y\}}$.

We denote by $\mathcal{M}(G)_d$ the moduli space of S-equivalence classes of semistable G-Higgs bundles of degree d and by $\mathcal{M}^{st}(G)_d$, the corresponding moduli space for stable G-Higgs bundles.

The G-Higgs bundle E is *polystable* if it is semistable and, when there exists a parabolic subgroup $P \subsetneq G$, a strictly antidominant character $\chi : P \to \mathbb{C}^*$ and a reduction of structure group σ giving the P bundle E_σ such that

$$\Phi \in H^0(X, E_\sigma(\mathfrak{p}))$$

and

$$\deg \chi_* E_\sigma = 0,$$

there exists a reduction ς of the structure group of E_σ to the Levi subgroup $L \subset P$ such that $\Phi \in H^0(X, E_\varsigma(\mathfrak{l}))$, where E_ς denotes the principal L-bundle obtained from the reduction of structure group ς and $\mathfrak{l}$ is the Lie algebra of L. There is a unique (up to isomorphism) polystable G-Higgs bundle in each S-equivalence class. Let us recall that every polystable G-Higgs bundle has a reduction of structure group to some Levi subgroup $L \subset G$ giving a stable L-Higgs bundle. Such a reduction is called a *Jordan–Hölder reduction* and is unique in a certain sense (see, for example, [**17**]).

The triviality of the canonical bundle Ω^1_X in the case of an elliptic curve leads us to the following well-known results.

PROPOSITION 3.1. *Let (E, Φ) be a semistable G-Higgs bundle. Then, E is a semistable G-bundle.*

Proof. If E is unstable, then E reduces to the Harder–Narasimhan parabolic subgroup P, giving E_σ, and there exists a character $\chi : P \to \mathbb{C}^*$ such that $\deg \chi^* E_\sigma < 0$. Moreover, $H^0(X, E(\mathfrak{g})) = H^0(X, E_\sigma(\mathfrak{p}))$. So, $\Phi \in H^0(X, E_\sigma(\mathfrak{p}))$ and hence the Higgs bundle (E, Φ) is unstable. □

We have the following consequence.

COROLLARY 3.2. *The moduli space of G-Higgs bundles projects onto the moduli space of G-bundles*

$$\begin{aligned} \mathcal{M}(G)_d &\longrightarrow M(G)_d \\ [(E, \Phi)]_S &\longmapsto [E]_S. \end{aligned}$$

PROPOSITION 3.3. *Let (E, Φ) be a stable G-Higgs bundle. Then, E is stable.*

Proof. We first note that $\Phi \in H^0(X, E(\mathfrak{g}))$ is contained in $\mathrm{aut}(E, \Phi)$.

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If (E, Φ) is stable, then, by [**17**, Proposition 2.14], $\mathrm{aut}(E, \Phi) \subset H^0(X, E(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})))$ and it follows easily that $(E, 0)$ is stable too. □

COROLLARY 3.4. *Let (E, Φ) be a polystable G-Higgs bundle. Then, E is a polystable G-bundle.*

Proof. The polystable G-Higgs bundle (E, Φ) reduces to the Jordan–Hölder Levi subgroup L giving the stable L-Higgs bundle (E_L, Φ_L). By Proposition 3.3, E_L is a stable L-bundle and therefore E is a polystable G-bundle. □

With the results above, we are able to describe stable and polystable G-Higgs bundles. Recall the bundle E^ρ defined in (19).

PROPOSITION 3.5. *A stable G-Higgs bundle (E, Φ) is isomorphic to $(E^\rho, z \otimes 1_{\mathcal{O}})$ where $\rho : \Gamma_{\mathbb{R}} \to K$ is some representation such that $\mathfrak{z}_{\mathfrak{g}}(\rho) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$, $1_{\mathcal{O}}$ is the constant section of the trivial bundle $\mathcal{O}$ equal to 1 and $z \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$.*

Proof. By Proposition 3.3, E is stable and therefore polystable. Then, $E \cong E^\rho$ for some ρ. By [**27**, Proposition 3.2], we have $H^0(X, E(\mathfrak{g})) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$, so $\Phi = z \otimes 1_{\mathcal{O}}$ for some $z \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$. Note that $\mathfrak{z}_{\mathfrak{g}}(\rho) \subseteq H^0(X, E^\rho(\mathfrak{g}))$, and then $\mathfrak{z}_{\mathfrak{g}}(\rho)$ is contained in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$ so they are equal. □

We recall the isomorphism (7) and note that $T^*X \cong X \times \mathbb{C}$. With all this in mind, we provide a result for G-Higgs bundles analogous to Theorem 2.1.

THEOREM 3.6. *Let $S^{\mathbb{C}}$ be an abelian, connected complex Lie group. Then, the moduli space of topologically trivial $S^{\mathbb{C}}$-Higgs bundles over the elliptic curve X is*

$$\mathcal{M}(S^{\mathbb{C}})_0 \cong T^*X \otimes_{\mathbb{Z}} \Lambda_S.$$

Proof. The description follows from the construction of a family of $S^{\mathbb{C}}$-Higgs bundles using $\mathcal{P}_S$ defined in (8) and $d\Theta_S$ from (7). □

Recall the definition of L_c given in (14).

PROPOSITION 3.7. *Every polystable G-Higgs bundle of topological type $d = (u, c)$ admits a reduction of structure group to L_c giving a stable L_c-Higgs bundle of topological class $\ell_d = (u, p(c))$.*

Proof. Take a polystable G-Higgs bundle (E, Φ) of type $d = (u, c)$, and suppose that L is a Jordan–Hölder Levi subgroup of (E, Φ). Since (E, Φ) reduces to L giving a stable L-Higgs bundle, it follows from Proposition 3.5 that there exists (ρ, z) such that $(E, \Phi) \cong (E^\rho, z \otimes 1_{\mathcal{O}})$. Here, $z \in \mathfrak{z}_{\mathfrak{g}}(\rho)$, which is a reductive Lie algebra since

$$Z_G(\rho) = Z_G(a, b) = Z_K(a, b)^{\mathbb{C}}$$

and $Z_K(a, b)$ is a compact subgroup. Then, we can conjugate $z \in \mathfrak{z}_{\mathfrak{g}}(\rho)$ to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{l}_c$. As a consequence of the above and Proposition 2.14, $(E^\rho, z \otimes 1_{\mathcal{O}})$ reduces to a stable L_c-Higgs bundle and so does (E, Φ). □

Recall that $\mathfrak{h}^{\omega_c}$ is the centre of $\mathfrak{l}_c$. Propositions 3.5 and 3.7 imply the following.

COROLLARY 3.8. *Let K_{L_c} be a maximal compact subgroup of L_c. A polystable G-Higgs bundle (E, Φ) of type $d \in \pi_1(G)$ is isomorphic to $(E^\rho, z \otimes 1_{\mathcal{O}})$, where $\rho : \Gamma_{\mathbb{R}} \to K_{L_c} \subset L_c$ is some representation, $1_{\mathcal{O}}$ is the constant section of the trivial bundle $\mathcal{O}$ equal to 1 and $z \in \mathfrak{h}^{\omega_c}$.*

Recall that $E^{\rho_1} \cong E^{\rho_2}$ if and only if ρ_1 and ρ_2 are conjugate. This fact, together with Corollary 3.8, implies the following.

COROLLARY 3.9. *In the notation of Corollary 3.8, two pairs* (ρ, z) *and* (ρ', z') *determine isomorphic polystable G-Higgs bundles if and only if there exists an element* $k \in K$ *such that* $(\rho', z') = (k\rho k^{-1}, \mathrm{ad}_k(z))$.

The automorphism group of the polystable G-Higgs bundle $(E^\rho, z \otimes 1_{\mathcal{O}})$ *is* $Z_G(\rho, z)$ *and its Lie algebra is* $\mathfrak{z}_{\mathfrak{g}}(\rho, z)$.

Recall the family of polystable G-bundles $\mathcal{E}_{G,d} \to X \times (X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c})$ defined in Remark 2.13 and in (21). Recalling the isomorphism

$$d\Theta_{\overline{S}_c} : \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c} \to \mathfrak{s}_c = \mathfrak{h}^{\omega_c},$$

defined in (7), as well as the discussion immediately before Theorem 3.6, we define a family of G-Higgs bundles $\mathcal{H}_{G,d}$ parametrised by $T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$, setting, for each point $(t, s) \in T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$,

$$\mathcal{H}_{G,d}|_{X \times \{(t,s)\}} = \left(\mathcal{E}_{G,d}|_{X \times \{t\}}\, ,\, d\Theta_{\overline{S}_c}(s) \otimes 1_{\mathcal{O}}\right),$$

where $1_{\mathcal{O}}$ is the section of the trivial bundle $\mathcal{O}$ equal to 1.

REMARK 3.10. By Corollary 3.8, every polystable G-Higgs bundle of degree d is parametrised by $\mathcal{H}_{G,d}$.

REMARK 3.11. The family $\mathcal{H}_{G,d}$ can be constructed starting from $\mathcal{H}_{S_c,0} \otimes (E^{x_0}_{L_c,\ell_d}, 0)$, quotienting by $H^1(X, F)$ as described in Corollary 2.11 and taking the extension of structure group associated to $L_c \hookrightarrow G$. This shows that all polystable G-Higgs bundles are described by Higgs bundles for the abelian group S_c.

THEOREM 3.12. *Let G be a connected complex reductive Lie group and let* $d \in \pi_1(G)$. *Then,*

$$\mathcal{M}^{st}(G)_d = \emptyset,$$

unless G/Z_0 *decomposes into* $\mathrm{PGL}(n_1, \mathbb{C}) \times \ldots \times \mathrm{PGL}(n_s, \mathbb{C})$ *and* $d \in \pi_1(G)$ *projects to* $(d_1, \ldots, d_s) \in \pi_1(\mathrm{PGL}(n_1, \mathbb{C})) \times \pi_1(\mathrm{PGL}(n_s, \mathbb{C}))$, *where* $\gcd(n_i, d_i) = 1$. *In that case, there is a natural isomorphism*

$$\mathcal{M}^{st}(G)_d = \mathcal{M}(G)_d \cong T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{Z}}.$$

Proof. The first statement is a consequence of Propositions 3.3 and 3.5 and Theorem 2.12.

As in Theorem 2.12, the extension of structure group associated to $G \to \overline{G} \cong \overline{Z} \times \overline{D}$ induces a morphism

$$\mathcal{M}^{st}(G)_d \longrightarrow \mathcal{M}^{st}(\overline{G})_{(u,c)} \cong \mathcal{M}^{st}(\overline{Z})_u \times \mathcal{M}^{st}(\overline{D})_c, \tag{23}$$

which, as in the case of G-bundles, can be proved to be bijective. By Proposition 3.5, Corollary 3.9 and Theorem 2.12, $\mathcal{M}^{st}(\overline{D})_c = \{pt\}$. Noting also that $\mathcal{M}^{st}(\overline{Z})_u$ is smooth as $\overline{Z}$ is abelian, we have that $\mathcal{M}^{st}(G)_d \cong \mathcal{M}^{st}(\overline{Z})_u$ and the second statement follows from Theorem 3.6. □

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Recall W_c defined in (13). Note that W_c acts on $\overline{S}_c$ and therefore it acts on $T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$.

PROPOSITION 3.13. *Let* (E_1, Φ_1) *and* (E_2, Φ_2) *be two polystable G-Higgs bundles of topological class d parametrised by* $\mathcal{H}_{G,d}$ *at the points* (t_1, s_1) *and* $(t_2, s_2) \in T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$. *Then,* (E_1, Φ_1) *and* (E_2, Φ_2) *are isomorphic G-Higgs bundles if and only if there exists* $\omega' \in W_c$ *such that* $(t_2, s_2) = \omega' \cdot (t_1, s_1)$.

Proof. It is clear that, if $(t_2, s_2) = \omega' \cdot (t_1, s_1)$, then $(E_1, \Phi_1) \cong (E_2, \Phi_2)$. Suppose conversely that $(E_1, \Phi_1) \cong (E_2, \Phi_2)$ and that (E_1, Φ_1) and (E_2, Φ_2) are associated to (ρ_1, z_1) and (ρ_2, z_2) in the sense of Corollary 3.8. Then, by Corollary 3.9, there exists $k \in K$ such that (ρ_2, z_2) is equal to $(k\rho_1 k^{-1}, \mathrm{ad}_k z)$.

By Corollary 2.16, there exists $\omega \in W_c = N_K(S_c)/Z_K(S_c)$ such that $t_2 = \omega \cdot t_1$. Then, there exists $n \in N_K(L_c) = N_K(S_c)$ projecting to ω and such that $\rho_2 = n\rho_1 n^{-1}$. Let us set $z' = \mathrm{ad}_{n^{-1}}(z_2)$ in $\mathfrak{s}_c = \mathfrak{h}^{\omega_c}$ and note that

$$(\rho_2, z_2) = \left(n\rho_1 n^{-1}, \mathrm{ad}_n(z')\right).$$

Then, $(\rho_1, z') = \left((n^{-1}k)\rho_1(n^{-1}k)^{-1}, \mathrm{ad}_{n^{-1}k}\, z_1\right)$, so $n^{-1}k$ belongs to $Z_K(\rho_1)$ and conjugates z_1 to z', both elements of $\mathfrak{s}_c = \mathfrak{h}^{\omega_c}$.

Let T be the maximal torus of $Z_K(\rho_1, z')$ such that its complexification is S_c. Note that $T' = n^{-1}kT(n^{-1}k)^{-1}$ is another maximal torus of $Z_K(\rho_1, z')$. Since $Z_K(\rho_1, z')$ is compact there exists an element h' that conjugates T to T'. Then, there exists $h = n^{-1}kh' \in Z_K(\rho_1) \cap N_K(S_c)$ with $z' = \mathrm{ad}_h(z_1)$. Setting $n' = nh = kh'$, we obtain an element of $N_K(S_c)$ such that

$$(\rho_2, z_2) = \left(n'\rho_1(n')^{-1}, \mathrm{ad}_{n'}(z_1)\right).$$

Finally, let $\omega' \in W_c$ be given by the projection of n'. It is clear that it sends (t_1, s_1) to (t_2, s_2). □

THEOREM 3.14. *There exists a bijective morphism*

$$\left(T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}\right) \big/ W_c \xrightarrow{1:1} \mathcal{M}(G)_d. \tag{24}$$

Hence, the normalisation $\overline{\mathcal{M}(G)_d}$ *of* $\mathcal{M}(G)_d$ *is isomorphic to* $\left(T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}\right) \big/ W_c$.

Proof. By moduli theory, the family $\mathcal{H}_{G,d} \to X \times (T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c})$ induces a morphism

$$\begin{array}{ccc} T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c} & \longrightarrow & \mathcal{M}(G)_d \\ (t, s) & \longmapsto & [\mathcal{H}_{G,d}|_{X\times\{(t,s)\}}]_S. \end{array}$$

As we have seen in Remark 3.10, this morphism is surjective. It descends to a surjective morphism (24). By Proposition 3.13, (24) is also injective.

The quasiprojective variety $\left(T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}\right) \big/ W_c$ is normal since it is the quotient of a smooth (and therefore normal) variety by a finite (and therefore reductive) group. Zariski's main theorem and (24) give us the description of the normalisation of $\mathcal{M}(G)_d$. □

REMARK 3.15. This is proved in [**33**] for the trivial degree case. For $G = \mathrm{GL}(n, \mathbb{C})$ or $\mathrm{SL}(n, \mathbb{C})$ and $d = 0$, (24) is indeed an isomorphism since the target is normal (see the discussion at the end of Section 1).

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The irreducibility of the quotient $\left(T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}\right) / W_c$ implies the following.

COROLLARY 3.16. *The moduli space of G-Higgs bundles $\mathcal{M}(G)_d$ is irreducible.*

A G-Higgs bundle is *infinitesimally regular* if the dimension of $\operatorname{aut}(E, \Phi)$ is the minimal possible one.

PROPOSITION 3.17. *The Zariski open subset of points represented by polystable G-Higgs bundles which are infinitesimally regular lies in the smooth locus of $\mathcal{M}(G)_d$.*

Proof. Consider the infinitesimal deformation space T of (E, Φ). By [**4**], one has the exact sequence

$$H^0(X, E(\mathfrak{g})) \xrightarrow{e_0(\Phi)} H^0(X, E(\mathfrak{g})) \longrightarrow T \longrightarrow H^1(X, E(\mathfrak{g})) \xrightarrow{e_1(\Phi)} H^1(X, E(\mathfrak{g})),$$

where $e_i(\Phi)(\psi) = [\psi, \Phi]$ and $e_1(\Phi)$ is the Serre dual of $e_0(\Phi)$ (recall that the canonical bundle is trivial in our case). Hence, $\operatorname{codim}(\operatorname{im} e_0(\Phi)) = \dim(\ker e_1(\Phi))$, so $\dim(T) = 2\dim(\ker e_1(\Phi))$.

Suppose that $(E, \Phi) \cong (E^\rho, z \otimes 1_{\mathcal{O}})$. Recall that dx is a generator of $H^1(X, \mathcal{O})$, so $H^1(X, E(\mathfrak{g})) = \{z' \otimes dx : z' \in \mathfrak{z}_{\mathfrak{g}}(\rho)\}$. We observe that the kernel of $e_1(\Phi)$ corresponds to $\mathfrak{z}_{\mathfrak{g}}(\rho, z)$ and therefore

$$\dim(T) = 2\dim(\mathfrak{z}_{\mathfrak{g}}(\rho, z)) = 2\dim(\operatorname{aut}(E, \Phi)),$$

where the last step in the equality follows from Corollary 3.9.

Suppose that ρ is associated to the c-pair (a, b) with (up to conjugation) $a \in H$. Recall that Proposition 2.5 implies that $\mathfrak{h}^{\omega_c}$ is a Cartan subalgebra of $\mathfrak{z}_{\mathfrak{g}}(\rho)$ and therefore a Cartan subalgebra of $\mathfrak{z}_{\mathfrak{g}}(\rho, z)$ since $z \in \mathfrak{h}^{\omega_c}$. Then, for every polystable G-Higgs bundle (E, Φ),

$$\dim(\mathcal{M}(G)_d) = 2\dim(\mathfrak{h}^{\omega_c}) \leq 2\dim(\mathfrak{z}_{\mathfrak{g}}(\rho, z)) = 2\dim(\operatorname{aut}(E, \Phi)).$$

Recalling [**14**, Corollary 5.18], we observe that, if a is a regular element of H, then $\mathfrak{z}_{\mathfrak{g}}(\rho, z) = \mathfrak{h}^{\omega_c}$, so $\dim(T) = \dim(\mathcal{M}(G)_d)$ is achieved in a Zariski open subset and the statement follows. □

We define the projection

$$\begin{array}{cccc} p_{G,d} & : \left(T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}\right) / W_c & \longrightarrow & \left(X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}\right) / W_c \\ & [(t, s)]_{W_c} & \longmapsto & [t]_{W_c}. \end{array}$$

Recalling the projection of Corollary 3.2, we have the commutative diagram

$$\begin{array}{ccc} \left(T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}\right) / W_c & \xrightarrow{p_{G,d}} & \left(X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}\right) / W_c \\ {\scriptstyle 1:1}\downarrow & & \downarrow{\scriptstyle \cong} \\ \mathcal{M}(G)_d & \longrightarrow & M(G)_d. \end{array}$$

REMARK 3.18. We can give an interpretation of the projection $p_{G,d}$ in terms of a certain orbifold bundle. Given an orbifold defined as a global quotient Z/Γ, one can define its cotangent orbifold bundle as the orbifold given by T^*Z/Γ, where the action of Γ on T^*Z is the action induced by the action of Γ on Z. Denote by $\widetilde{M}(G)_d$ and $\widetilde{\mathcal{M}}(G)_d$

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the orbifolds given, respectively, by the quotients of $(X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c})$ and $(T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c})$ by the finite group W_c. Since $T^*(X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c})$ is $(T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c})$, we have that $\widetilde{\mathcal{M}}(G)_d$ is the cotangent orbifold bundle of $\widetilde{M}(G)_d$, i.e.,

$$\widetilde{\mathcal{M}}(G)_d \cong \mathcal{T}^* \widetilde{M}(G)_d.$$

4. The Hitchin fibration. We describe the Hitchin map in the spirit of [**9**]. Consider the adjoint action of the group G on the Lie algebra $\mathfrak{g}$ and take the quotient map

$$q : \mathfrak{g} \longrightarrow \mathfrak{g} /\!\!/ G .$$

Let E be any holomorphic G-bundle. Since the adjoint action of G on $\mathfrak{g}/\!\!/G$ is obviously trivial, we note that the fibre bundle induced by E is trivial

$$E(\mathfrak{g}/\!\!/G) = \mathcal{O} \otimes (\mathfrak{g}/\!\!/G).$$

The projection q induces a surjective morphism of fibre bundles

$$q_E : E(\mathfrak{g}) \longrightarrow E(\mathfrak{g}/\!\!/G),$$

and q_E induces a morphism on the set of holomorphic global sections

$$\begin{array}{rccc} (q_E)_* & : H^0(X, E(\mathfrak{g})) & \longrightarrow & H^0(X, \mathcal{O} \otimes (\mathfrak{g}/\!\!/G)) \\ & \Phi & \longmapsto & \Phi/\!\!/G \end{array}.$$

If (E_1, Φ_1) and (E_2, Φ_2) are two S-equivalent semistable G-Higgs bundles, one can check that $(q_{E_1})_* \Phi_1 = (q_{E_2})_* \Phi_2$. Hence, we can define the Hitchin map

$$\begin{array}{rccc} b_G & : \mathcal{M}(G) & \longrightarrow & H^0(X, \mathcal{O} \otimes (\mathfrak{g}/\!\!/G)) \\ & [(E, \Phi)]_S & \longmapsto & (q_E)_* \Phi. \end{array} \qquad (25)$$

When the base variety is a Riemann surface of genus greater than or equal to 2, the restriction of b_G to every component $\mathcal{M}(G)_d$ is surjective. This is not the case for genus $g = 1$ and, to preserve the fact that the Hitchin map is a fibration, we set

$$B(G, d) := b_G(\mathcal{M}(G)_d),$$

and we denote by $b_{G,d}$ the restriction of (25) to $\mathcal{M}(G)_d$.

If H is a Cartan subgroup with Cartan subalgebra $\mathfrak{h}$ and Weyl group W, Chevalley's theorem says that

$$\mathfrak{g}/\!\!/G \cong \mathfrak{h}/W.$$

So, $H^0(X, \mathcal{O} \otimes (\mathfrak{g}/\!\!/G)) \cong H^0(X, \mathcal{O} \otimes \mathfrak{h}/W)$ and, since X is a compact holomorphic variety, we have $H^0(X, \mathcal{O} \otimes \mathfrak{h}/W) \cong \mathfrak{h}/W \cong \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_H/W$. There is a natural isomorphism

$$\beta_{G,0} : \left(\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_H\right) / \, W \xrightarrow{\cong} B(G, 0).$$

Now we take $d \in \pi_1(G)$ non-trivial associated to $(u, c) \in \pi_1(\overline{Z}) \times \pi_1(\overline{D})$. By Corollary 3.8, we see that every polystable G-Higgs bundle of topological class d

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is isomorphic to $(E^\rho, z \otimes 1_{\mathcal{O}})$, where $z \in \mathfrak{h}^{\omega_c}$. We can check that the quotient map q induces a bijective morphism

$$\beta_{G,d} : (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) \big/ W_c \xrightarrow{1:1} B(G, d).$$

Let $\mathcal{B}(\Lambda_{\overline{S}_c}) = \{\gamma_1, \dots, \gamma_\ell\}$ be a basis of $\Lambda_{\overline{S}_c}$. Recalling that $T^*X \cong X \times \mathbb{C}$, we see that the projection $\pi : T^*X \to \mathbb{C}$ induces

$$\begin{array}{rccc} \pi_{G,c} & : (T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) \big/ W_c & \longrightarrow & (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) \big/ W_c \\ & [(t,s)]_{W_c} & \longmapsto & [s]_{W_c}. \end{array} \tag{26}$$

We use this morphism to better understand the Hitchin map.

Proposition 4.1. *Recall the bijective morphism (24). The following diagram is commutative:*

$$\begin{array}{ccc} (T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) \big/ W_c & \xrightarrow{1:1} & \mathcal{M}(G)_d \\ \downarrow{\scriptstyle \pi_{G,c}} & & \downarrow{\scriptstyle b_{G,d}} \\ (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) \big/ W_c & \xrightarrow[1:1]{\beta_{G,d}} & B(G,d). \end{array}$$

The normalisation of the Hitchin fibre corresponding to $s \in \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{S_c}$ is isomorphic to

$$\pi^{-1}_{G,c}([s]_{W_c}) \cong (X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) \big/ Z_{W_c}(s). \tag{27}$$

Proof. Take $(t, s) \in (T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c})$, and consider

$$b_G(\mathcal{H}_{G,d}|_{X \times \{(t,s)\}}) = [s]_G.$$

Clearly, this equality is W_c-invariant. On the other hand, note that

$$\beta_{G,d} \circ \pi_{G,c} ([(t,s)]_{W_c}) = \beta_{G,d}([s]_{W_c}) = [s]_G$$

and the first statement follows.

Next, consider the following projection

$$\widetilde{\pi}_{G,c} : T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c} \longrightarrow \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}.$$

We observe that

$$\pi^{-1}_{G,c}([s]_{W_c}) \cong \left(\bigcup_{\omega \in W_c} \widetilde{\pi}^{-1}_{G,c}(\omega \cdot s) \right) \big/ W_c.$$

Since, for $\omega \cdot s \neq \omega' \cdot s$ the sets $\widetilde{\pi}^{-1}_{G,c}(\omega \cdot s)$ and $\widetilde{\pi}^{-1}_{G,c}(\omega' \cdot s)$ are disjoint, it follows that

$$\pi^{-1}_{G,c}([s]_{W_c}) \cong \widetilde{\pi}^{-1}_{G,c}(s) \big/ Z_{W_c}(s)$$

and therefore we obtain the isomorphism (27). Finally, we observe that the bijection (24) sends $\pi^{-1}_{G,c}([s]_{W_c})$ to the Hitchin fibre corresponding with the Higgs field $\Phi = z \otimes 1_{\mathcal{O}}$. Hence, by Zariski's main theorem, it describes an isomorphism with the normalisation of this subset. □

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We denote by $U_{G,c}$ the subset of $\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c} / W_c$ given by the points $[s]_{W_c}$ such that there exists a non-trivial $\omega \in W_c$ with $s = \omega \cdot s$. Since the only element of W_c that acts trivially on $\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$ is the identity, $U_{G,c}$ is a finite union of closed subsets of codimension at least equal to 1. By construction, for any $s \notin U_{G,c}$, we have $Z_{W_c}(s) = \{\mathrm{id}\}$.

The *generic Hitchin fibre* is the fibre over any element of the complement of $U_{G,c}$.

COROLLARY 4.2. *The normalisation of the generic Hitchin fibre is isomorphic to the abelian variety* $X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$.

5. The moduli space of representations $\mathcal{R}(G)_d$. From the non-abelian Hodge correspondence on a compact Riemann surface [**19**, **32**, **10**, **7**], it follows that a polystable G-Higgs bundle is associated to a reductive representation $\rho : \Gamma_{\mathbb{R}} \to G$ and two representations are conjugate if and only if they are associated to isomorphic polystable G-Higgs bundles. Furthermore, irreducible representations correspond to stable G-Higgs bundles.

Using this correspondence and Remark 2.3, we can use the results on G-Higgs bundles obtained in Section 3, to generalise the description given in Section 2.3 of c-pairs on compact groups, to complex reductive Lie groups.

PROPOSITION 5.1. *Let G be a simply connected complex semisimple Lie group. Let $C = p^{-1}(F) = \pi_1(\overline{D})$ as defined at the begining of Section 2.2 and set $c \in C$. Let (a, b) be an irreducible c-pair in G. Then,*

(1) *the group G is a product of simple factors G_i, where each G_i is isomorphic to* $\mathrm{SL}(n_i, \mathbb{C})$ *for some $n_i \geq 2$;*
(2) *$c = (c_1, \ldots, c_r)$, where each c_i generates the centre of G_i;*
(3) *conversely, if G is as in (1) and c as in (2), then there is an irreducible c-pair in G and all c-pairs in G are conjugate.*

Proof. This follows from Theorem 3.12 and the fact that the universal cover of $\mathrm{PGL}(n, \mathbb{C})$ is $\mathrm{SL}(n, \mathbb{C})$. □

PROPOSITION 5.2. *Let G be a connected complex reductive Lie group. Let (a, b) be a reductive c-pair; then (a, b) is contained in L_c after conjugation and, as a c-pair in L_c, is irreducible.*

Proof. This follows from Proposition 3.7. □

Recall the notation introduced in Section 2.2.

THEOREM 5.3. *Let G be a connected complex reductive Lie group and let $d \in \pi_1(G)$, corresponding under the injection (11) to $(u, c) \in \pi_1(\overline{Z}) \times \pi_1(\overline{D})$. Then, there is a bijective morphism*

$$\zeta_{G,d} : (\overline{S_c} \times \overline{S_c}) \big/ W_c \xrightarrow{1:1} \mathcal{R}(G)_d.$$

Proof. Take a representative (δ_1, δ_2) of the unique conjugation class of c-pairs in D_c. Recall that $C(G)^+_c$ denotes the space of reductive c-pairs in G and consider the following morphisms:

$$\begin{array}{ccccc} (S_c \times S_c) & \longrightarrow & C(G)^+_c & \longrightarrow & \mathcal{R}(G)_d \\ (s_1, s_2) & \longmapsto & ([s_1, \delta_1]_{\tau_c}, [s_2, \delta_2]_{\tau_c}) & \longmapsto & \left[([s_1, \delta_1]_{\tau_c}, [s_2, \delta_2]_{\tau_c})\right]_G . \end{array}$$

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By an argument analogous to that of Remark 2.6, the composition morphism factors through

$$\overline{S}_c \times \overline{S}_c \longrightarrow \mathcal{R}(G)_d.$$

By (16) and Proposition 5.2, it is clear that this morphism is surjective. The group W_c acts on $\overline{S}_c \times \overline{S}_c$ via conjugation by $N_G(S_c)$. Since the points of $\mathcal{R}(G)_d$ are the conjugation classes of c-pairs, the morphism factors through this quotient, giving the morphism $\zeta_{G,d}$ of the statement. We only need to prove that it is injective.

Take two reductive c-pairs of the form $([s_1, \delta_1]_{\tau_c}, [s_2, \delta_2]_{\tau_c})$ and $([s_1', \delta_1]_{\tau_c}, [s_2', \delta_2]_{\tau_c})$. Write $Z' = Z_G([s_1', \delta_1]_{\tau_c}, [s_2', \delta_2]_{\tau_c})$ which is a complex reductive group since the c-pair is reductive. Suppose that there is $g \in G$ such that

$$([s_1, \delta_1]_{\tau_c}, [s_2, \delta_2]_{\tau_c}) = g([s_1', \delta_1]_{\tau_c}, [s_2', \delta_2]_{\tau_c})g^{-1}.$$

Then, S_c and gS_cg^{-1} are Cartan subgroups of Z', so there is an element $h \in Z'$ such that $hS_ch^{-1} = gS_cg^{-1}$ and then $g' = h^{-1}g$ is contained in $N_G(S_c)$. We have

$$g'([\mathrm{id}, \delta_1]_{\tau_c}, [\mathrm{id}, \delta_2]_{\tau_c})(g')^{-1} = ([\mathrm{id}, \delta_1']_{\tau_c}, [\mathrm{id}, \delta_2']_{\tau_c}),$$

where (δ_1', δ_2') is an irreducible c-pair in D_c and therefore, by Proposition 5.1, there exists $\delta \in D_c$ such that $\delta(\delta_1', \delta_2')\delta^{-1} = (\delta_1, \delta_2)$. Noting that $[\mathrm{id}, \delta]_{\tau_c}$ conmutes with S_c since S_c is the centre of $Z_G(S_c)$, it follows that $g'' = [\mathrm{id}, \delta]_{\tau_c} \cdot g' \in N_G(S_c)$ and

$$([s_1, \delta_1]_{\tau_c}, [s_2, \delta_2]_{\tau_c}) = ([g''s_1'(g'')^{-1}, \delta_1]_{\tau_c}, [g''s_2'(g'')^{-1}, \delta_2]_{\tau_c}).$$

Thus, (s_1, s_2) and (s_1', s_2') define the same point of $(\overline{S}_c \times \overline{S}_c)/W_c$. □

COROLLARY 5.4. *There is a bijective morphism*

$$\left((\mathbb{C}^* \times \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}\right) \big/ W_c \xrightarrow{1:1} \mathcal{R}(G)_d. \tag{28}$$

and $\overline{\mathcal{R}(G)}_d = \left((\mathbb{C}^* \times \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}\right) \big/ W_c$ *is the normalisation of* $\mathcal{R}(G)_d$.

Proof. Due to the isomorphism $\Theta_{\overline{S}_c} : \overline{S}_c \xrightarrow{\cong} \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$ defined in (6) and knowing that the action of W_c commutes with it, the first statement follows from Theorem 5.3.

The second statement follows from (28) and Zariski's main theorem. □

REMARK 5.5. This is proved in [**33**] for the case $d = 0$. When the degree is trivial and $G = \mathrm{GL}(n, \mathbb{C})$ or $\mathrm{SL}(n, \mathbb{C})$, one obtains an isomorphism due to the normality of the target (see the discussion at the end of Section 1).

6. Hitchin equation and projectively flat bundles. Fix a maximal compact subgroup K of G and denote its Lie algebra by $\mathfrak{k}$. Take $\tau : \mathfrak{g} \to \mathfrak{g}$ to be the Cartan involution associated to the compact real form $\mathfrak{k} \subset \mathfrak{g}$. Then, $\tau(k) = k$ and $\tau(ik) = -ik$ for every $k \in \mathfrak{k}$.

Let (E, Φ) be a G-Higgs bundle and let h be a metric on E, i.e., a C^∞ reduction of E to the maximal compact subgroup K giving the K-bundle E_h. We define the involution on the adjoint bundle $\tau_h : E_h(\mathfrak{g}) \to E_h(\mathfrak{g})$ using τ fibrewise.

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Let $\overline{\partial}_E$ denote the Dolbeault operator of E and set $A_h := \overline{\partial}_E + \tau_h(\overline{\partial}_E)$, which is the unique K-connection on E_h compatible with $\overline{\partial}_E$, also known as the *Chern connection*. We denote by F_h the curvature of A_h.

Take the C^∞ (1, 0)-form $dx \in \mathcal{A}^{1,0}(X, \mathcal{O})$ and $d\overline{x} \in \mathcal{A}^{0,1}(X, \mathcal{O})$. Given a G-Higgs bundle (E, Φ), Hitchin introduced in [**19**], the following equation for a metric h on E:

$$F_h + [\Phi\, dx, \tau_h(\Phi)\, d\overline{x}] = u \otimes \omega, \tag{29}$$

where $u \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$ and $\omega \in \mathcal{A}^2(X)$ is the volume form of the curve normalised to $2\pi i$. Recall that u is determined by $d \in \pi_1(G)$.

In the elliptic case, we have a splitting of the Hitchin equation.

PROPOSITION 6.1. *If the G-Higgs bundle (E, Φ) is polystable, then there exists a metric h on E that satisfies*

$$F_h = u \otimes \omega \qquad \text{and} \qquad [\Phi\, dx, \tau_h(\Phi)\, d\overline{x}] = 0.$$

Proof. By Corollary 3.4, if the G-Higgs bundle (E, Φ) is polystable, then E is polystable and by the Narasimhan–Seshadri–Ramanathan theorem there exists a metric for which $F_h = u \otimes \omega$.

By Corollary 3.8, (E, Φ) is isomorphic to $(E^\rho, z \otimes \mathrm{id}_E)$ where $z \in \mathfrak{h}^{\omega_c}$. Then,

$$[\Phi\, dx, \tau_h(\Phi)\, d\overline{x}] = [z, \tau(z)] \otimes \mathrm{id}_E \otimes (dx \wedge d\overline{x}) = 0,$$

since both z and $\tau(z)$ belong to the abelian subalgebra $\mathfrak{h}$. □

One can easily show that a G-Higgs bundle (E, Φ) admitting a metric that satisfies (29) is always polystable. Thus, we see that Proposition 6.1 completes the proof of the Hitchin–Kobayashi correspondence in the elliptic case.

COROLLARY 6.2. *A G-Higgs bundle (E, Φ) is polystable if and only if it admits a metric h that satisfies the Hitchin equation (29).*

REMARK 6.3. Note that to prove the Hitchin–Kobayashi correspondence in the elliptic case we only make use of the Narasimhan–Seshadri–Ramanathan theorem, the Jordan–Hölder reduction and Propositions 3.1 and 3.3.

Let $\mathbf{E}_{G,d}$ be the (unique up to isomorphism) differentiable G-bundle of degree $d \in \pi_1(G)$ over the elliptic curve X. A G-connection A on $\mathbf{E}_{G,d}$ is *flat* if the curvature vanishes, $F_A = 0$ (note that this forces $d = 0$). A G-connection A on $\mathbf{E}_{G,d}$ is *projectively flat* or equivalently *A has constant central curvature* if $F_A = a \otimes \omega$ for some $a \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$. Due to topological considerations $a = u$, where $u \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$ is determined by $d \in \pi_1(G)$. Let us denote by $\mathcal{C}(G)_d$ the moduli space of projectively flat connections on $\mathbf{E}_{G,d}$ and consequently, $\mathcal{C}(G)_0$ is the moduli space of flat connections on $\mathbf{E}_{G,0}$.

We denote by $X^\sharp$ the moduli space of line bundles with flat connections over the elliptic curve X. Recalling that $T^*X \cong \mathrm{Pic}^0(X) \times H^0(X, \Omega^1_X)$, we have a homeomorpshism

$$X^\sharp \xrightarrow{\;homeo\;} T^*X, \tag{30}$$

given by Hodge theory.

Let S be a connected complex reductive abelian group. Recalling the isomorphism Θ_S given in (6), one can give a description of the moduli space of flat S-connections,

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denoted by $\mathcal{C}(S)_0$. Write $\mathbf{E}_{S,0}$ for the differentiable S-bundle with trivial topological class and recall that it is unique up to isomorphism.

Recall the isomorphism (6). For instance, the following result is contained in [**32**, Theorem 9.10].

THEOREM 6.4. Let $S^{\mathbb{C}}$ be an abelian, connected complex Lie group. Then, the moduli space of flat $S^{\mathbb{C}}$-connections over the elliptic curve X is

$$\mathcal{C}(S)_0 \cong X^\sharp \otimes_{\mathbb{Z}} \Lambda_S.$$

Let $L \subset G$ be a reductive subgroup. We say that the G-connection A *reduces* to the L-connection A' when A is gauge equivalent to the extension of structure group of A' given by the natural injection $i : L \hookrightarrow G$.

Recall from (11) that $d \in \pi_1(G)$ is determined by $(u, c) \in \pi_1(\overline{Z}) \times \pi_1(\overline{D})$, where $\pi_1(\overline{Z}) \subset \mathfrak{z}_\mathfrak{g}(\mathfrak{g})$ and $\pi_1(\overline{D}) = C$ as described in Section 2.2. Take L_c as defined in (14) and denote by K_c its maximal compact subgroup.

PROPOSITION 6.5. *Every projectively flat connection A on $\mathbf{E}_{G,d}$ reduces to a projectively flat L_c-connection. Furthermore, A is gauge equivalent to*

$$A_{(\rho,z)} = A_\rho + z\, dx + \tau(z)\, d\overline{x},$$

where A_ρ is the Chern connection of E^ρ given by $\rho : \Gamma_{\mathbb{R}} \to K_c$ and $z \in \mathfrak{h}^{\omega_c}$.

The projectively flat connections $A_{(\rho,z)}$ and $A_{(\rho',z')}$ are gauge equivalent if and only if there exists $g \in K$ such that $(\rho', z') = (g\rho g^{-1}, \mathrm{ad}_g\, z)$.

Proof. From a polystable G-Higgs bundle (E, Φ), we can construct a G-connection on $\mathbf{E}_{G,d}$ as follows:

$$A = A_h + \Phi\, dx + \tau_h(\Phi)\, d\overline{x}.$$

Two isomorphic polystable G-Higgs bundles give rise to gauge equivalent flat G-connections. By Corollary 6.2, the above G-connection is projectively flat if and only if (E, Φ) is polystable. The description of polystable G-Higgs bundles in Corollary 3.8 implies the proposition. □

Denote by $\mathbf{E}_{L_c,\ell_d}$ the differentiable bundle underlying $E^{x_0}_{L_c,\ell_d}$, the L_c-bundle with degree ℓ_d defined in Remark 2.13, and let $A^{x_0}_{L_c,\ell_d}$ be its Chern connection. Setting $p : X \times (X^\sharp \otimes \Lambda_{S_c}) \to X$, we define the family

$$\left(\mathbf{F}'_{L_c,\ell_d}, (\mathcal{A}')_{L_c,\ell_d}\right) = \left(\mathbf{P}_{S,0} \otimes p^*\mathbf{E}_{L_c,\ell_d}, \mathcal{A}_{S_d,0} \otimes p^* A^{x_0}_{L_c,\ell_d}\right),$$

noting that S_c is the centre of L_c. This family is parametrised by $X^\sharp \otimes_{\mathbb{Z}} \Lambda_{S_c}$.

Recall F_c and $\overline{S}_c$ as defined in (15). Let $J \in H^1(X, F_c)$ be a F_c-bundle and A_J its Chern connection. By Corollary 2.11, one has the following.

PROPOSITION 6.6. *Let A be any L_c-connection on $\mathbf{E}_{L_c,\ell_d}$, then $A_J \otimes A$ is gauge equivalent to A.*

As a consequence of Proposition 6.6, it follows that $(\mathbf{F}'_{L_c,\ell_d}, (\mathcal{A}')_{L_c,\ell_d})$ induces a family of L_c-connections parametrised by the quotient of Λ_{S_c} by the subgroup associated to $H^1(X, F_c)$. This quotient is $\Lambda_{\overline{S}_c}$, and therefore we obtain a family parametrised by $X^\sharp \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$ that we denote by $(\mathbf{F}_{L_c,\ell_d}, \mathcal{A}_{L_c,\ell_d})$.

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Using the natural injection $i : L_c \hookrightarrow G$, we construct, by extension of structure group,

$$(\mathbf{F}_{G,d}, \mathcal{A}_{G,d}) = i_*(\mathbf{F}_{L_c,\ell_d}, \mathcal{A}_{L_c,\ell_d}),$$

a family of projectively flat G-connections, which is also parametrised by $X^\sharp \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$.

REMARK 6.7. The flat G-connection parametrised by $(\mathbf{F}_{G,d}, \mathcal{A}_{G,d})$ at the point $f \in X^\sharp \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$ is of the form $A_{(\rho,z)}$. It is therefore associated to the polystable G-Higgs bundle $(E^\rho, z \otimes 1_{\mathcal{O}})$ parametrised by $\mathcal{H}_{G,d}$ at the point $(t, s) \in T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$, where (t, s) is the image of f under the homeomorphism (30). Therefore, by Proposition 3.13, two points $f_1, f_2 \in X^\sharp \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$ parametrise gauge equivalent connections if $f_2 = \omega \cdot f_1$ for some $\omega \in W_c$.

THEOREM 6.8. *There exists a bijective morphism*

$$(X^\sharp \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) \big/ W_c \xrightarrow{1:1} \mathcal{C}(G)_d \tag{31}$$

and $\overline{\mathcal{C}(G)}_d = \left(X^\sharp \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}\right) \big/ W_c$ *is the normalisation of* $\mathcal{C}(G)_d$.

Proof. The family $(\mathbf{F}_{G,d}, \mathcal{A}_{G,d})$ induces a morphism from the parametrising space to the moduli space

$$X^\sharp \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c} \longrightarrow \mathcal{C}(G)_d,$$

which is surjective by Proposition 6.5. By Remark 6.7, this surjection factors through (31) giving an injection.

The second statement follows from (31) and Zariski's main theorem. □

REMARK 6.9. This is proved in [**33**] for the trivial degree case. In the case of $G = \mathrm{GL}(n, \mathbb{C})$ or $\mathrm{SL}(n, \mathbb{C})$ and $d = 0$, (31) is an isomorphism since the target is normal. Normality of $\mathcal{C}(G)_0$ follows from the isosingularity theorem [**32**, Theorem 10.6] and normality of $\mathcal{R}(G)_0$.

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