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ADDITIVE AUTOMORPHIC FUNCTIONS AND BLOCH FUNCTIONS

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ABSTRACT. A function f analytic in the unit disk D is said to be *strongly uniformly continuous hyperbolically*, or SUCH, on a set $E \subset D$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - f(z')| < \varepsilon$ whenever z and z' are points in E and the hyperbolic distance between z and z' is less than δ. We show that f is a Bloch function in D if and only if $|f|$ is SUCH in D. A function f is said to be *additive automorphic in D* relative to a Fuchsian group Γ if, for each $\gamma \in \Gamma$, there exists a constant A_γ such that $f(\gamma(z)) = f(z) + A_\gamma$. We show that if an analytic function f is additive automorphic in D relative to a Fuchsian group Γ, where Γ is either finitely generated or if the fundamental region F of Γ has the right kind of structure, and if $|f|$ is SUCH in F, then f is a Bloch function. We show by example that some restrictions on Γ are needed.

1. **Introduction and preliminaries.** Let $D = \{z : |z| < 1\}$ denote the unit disk in the complex plane. For a pair of points z and z' in D, the hyperbolic distance between z and z' is denoted by $\sigma(z, z') = \frac{1}{2} \log \frac{1+h(z,z')}{1-h(z,z')} = \tanh^{-1}\big(h(z, z')\big)$, where $h(z, z') = \left|\frac{z-z'}{1-\bar{z}'z}\right|$. It is easy to verify that the differentials $d\sigma(z)$ and dz are related by the equation

$$d\sigma(z) = \frac{|dz|}{(1 - |z|^2)}.$$

A function f analytic in D is said to be *strongly uniformly continuous hyperbolically* (which we will indicate throughout by SUCH) on a set $E \subset D$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - f(z')| < \varepsilon$ whenever z and z' are points of E and $\sigma(z, z') < \delta$. Throughout, we will denote the closure of a set E by $\bar{E}$. It is easily seen that if f is SUCH on a set $E \subset D$ then f is SUCH on the set $\bar{E}$.

A function f analytic in D is said to be a *Bloch function* if

$$||f||_B = \sup\{|f'(z)|(1 - |z|^2) : z \in D\} < \infty.$$

A function f meromorphic in D is said to be a *normal function* if $C_f = \sup\{f^\#(z)(1-|z|^2) : z \in D\} < \infty$, where $f^\#(z) = \frac{|f'(z)|}{1+|f(z)|^2}$ is the spherical derivative of f (see [5]). It is easily seen that each Bloch function is also a normal function. We say that a disk Δ in the complex plane is a *schlicht disk* in the image of the function f analytic in D if there exists an open connected subset $E_\Delta \subset D$ such that $f\colon E_\Delta \longrightarrow \Delta$ is one-to-one and $f(E_\Delta) = \Delta$. If $z_0 \in E_\Delta$ is the point for which $f(z_0)$ is the center of the disk Δ, it is an easy consequence of Schwarz's Lemma that $|f'(z_0)|(1 - |z_0|^2)$ is at least as large as the radius of Δ. It is known

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that a function f analytic in D is a Bloch function if and only if the image of f contains no arbitrarily large schlicht disks. (For more details about Bloch functions, see [1].)

Let Γ be a Fuchsian group acting on D. We say that the function f analytic in D is *automorphic* (relative to Γ) if $f\big(\gamma(z)\big) = f(z)$ for each $\gamma \in \Gamma$ and each $z \in D$. We say that the function f analytic in D is *additive automorphic* (relative to Γ) if for each $\gamma \in \Gamma$ there exists a complex constant A_γ such that $f\big(\gamma(z)\big) = f(z) + A_\gamma$ for each $z \in D$. Finally, we say that the function f analytic in D is *rotation automorphic* (relative to Γ) if for each $\gamma \in \Gamma$ there exists a rotation R_γ of the Riemann sphere such that $f\big(\gamma(z)\big) = R_\gamma\big(f(z)\big)$ for each $z \in D$. It is clear that an automorphic function is both additive automorphic (with $A_\gamma = 0$ for each $\gamma \in \Gamma$), and rotation automorphic (with R_γ = identity for each $\gamma \in \Gamma$). For a rotation automorphic function f, we use the notation $\Sigma = \{R_\gamma : \gamma \in \Gamma\}$, and we refer to Σ as the *rotation group* for the function f.

For a Fuchsian group Γ acting on D, let F denote a *fundamental region* of Γ, that is, F is an open connected subset of D with the following two properties:

(a) for each $z \in D$, there exists an element $\gamma \in \Gamma$ such that $\gamma(z)$ is in $\bar{F}$, and

(b) if z_1 and z_2 are two points in F and if $\gamma \in \Gamma$ is such that $\gamma(z_1) = z_2$, then $z_1 = z_2$ and γ is the identity mapping. Many choices of a fundamental region are possible, but we will always deal with a hyperbolically convex fundamental region, that is, one for which any pair of points in the region can be connected by a hyperbolic line in the region. One such possibility is the "Ford fundamental region", that is, the interior of the set

$$\{z \in D : |z| \leq |\gamma(z)| \text{ for each } \gamma \in \Gamma\}.$$

The purpose of this paper is to investigate the relationships between Bloch functions and strongly uniformly continuous functions, and to give applications to automorphic functions, additive automorphic functions, and rotation automorphic functions. This continues the investigations reported in [2] and [3].

We have previously proved in [2, Theorem 3, p. 75] that an additive automorphic function is a Bloch function if and only if it is SUCH in the fundamental region F. Here, we investigate whether an additive automorphic function is a Bloch function under the weaker condition that the function $h(z) = |f(z)|$ is SUCH in the fundamental region. We note that the condition that f is SUCH in a set E implies that the function $h(z) = |f(z)|$ is SUCH in E, but not necessarily conversely. Thus, in general, we need some additional conditions either on f or on the Fuchsian group Γ. In Section 2, we first show, in Theorem 1, that f is SUCH in D if and only if $h(z) = |f(z)|$ is SUCH in D. Then, in Theorem 2, we give a new proof of the fact that an additive automorphic function f is a Bloch function if and only if f is SUCH in the fundamental region. Our proof enables us to extend this result to some rotation automorphic functions as well. Next we show in Theorem 3 that if f is an additive automorphic function relative to a finitely generated Fuchsian group, and if $h(z) = |f(z)|$ is SUCH in the fundamental region, then f is a Bloch function.

Finally, in Section 3, we give an example of a function f which is automorphic (and therefore both additive automorphic and rotation automorphic as well) for which $h(z) = |f(z)|$ is SUCH in the fundamental region but f is not a Bloch function.

2. **Results about Bloch functions.** We begin with a result about analytic functions in D which are not necessarily additive automorphic functions.

THEOREM 1. *Let f be a function analytic in D. The following statements are equivalent:*

(i) f is a Bloch function,

(ii) f is SUCH *in D, and*

(iii) $h(z) = |f(z)|$ is SUCH *in D.*

PROOF. Suppose (i). Then for each $z \in D$, $|f'(z)|(1-|z|^2) \leq \|f\|_B < \infty$, so that

$$\begin{aligned}|f(z)-f(z')| &= \left|\int_{z'}^{z} f'(\zeta)\,d\zeta\right| \leq \int_{z'}^{z} |f'(\zeta)|\,|d\zeta| \\ &= \int_{z'}^{z} |f'(\zeta)|(1-|\zeta|^2)\,d\sigma(\zeta) \leq \|f\|_B\sigma(z,z').\end{aligned}$$

This shows (ii). Thus (i) implies (ii).

That (ii) implies (iii) follows from the inequality

$$\big|\,|f(z)|-|f(z')|\,\big| \leq |f(z)-f(z')|.$$

Now assume that f is not a Bloch function. Then there exists a sequence of points $\{z_n\}$ in D such that, for each n, $f(z_n)$ is the center of a schlicht disk Δ_n in the image of f with radius n, where E_n is an open connected subset of D such that $z_n \in E_n$, $f(E_n) = \Delta_n$, and $f\colon E_n \longrightarrow \Delta_n$ is one-to-one. Let $C_{1,n}$ denote the circle with center $f(z_n)$ and radius 1. Let $g_n\colon \Delta_n \longrightarrow E_n$ be the inverse function of f restricted to E_n and let $E_{1,n} = g_n(C_{1,n})$. Since $|f'(z)|(1-|z|^2)$ is at least as large as the radius of the largest schlicht disk in the image of f with center at $f(z)$, we have $|f'(z)|(1-|z|^2) \geq n-1$ for each z inside or on the Jordan curve $E_{1,n}$. If L_n is a line segment from $f(z_n)$ to a point $W_n \in C_{1,n}$, then setting $z'_n = g_n(W_n)$ and $L'_n = g_n(L_n)$, we have

$$\begin{aligned}1 &= \int_{L'_n} |f'(z)|\,|dz| = \int_{L'_n} |f'(z)|(1-|z|^2)\,d\sigma(z) \\ &\geq (n-1)\sigma(z_n, z'_n).\end{aligned}$$

This means that $\sigma(z_n, z'_n) \leq 1/(n-1)$. But we may choose $w_n \in C_{1,n}$ such that $|w_n| = 1 + |f(z_n)|$, so that $|f(z'_n)| - |f(z_n)| = 1$. Thus, we have shown that if f is not a Bloch function then $h(z) = |f(z)|$ is not SUCH in D, and this means that (iii) implies (i), and completes the proof of the theorem.

In order to connect Theorem 1 to automorphic, additive automorphic, and rotation automorphic functions, we need some lemmas.

LEMMA 1. *Let f be a function analytic on D such that $f(0) = 0$, and suppose that there exist a region Ω_0 in D and a constant $K > 0$ such that $0 \in \Omega_0$ and $f\colon \Omega_0 \longrightarrow \{w \in \mathbb{C} : |w| < K+1\}$ is both one-to-one and onto. Then there exists a simple closed curve $J_K \subset \Omega_0$ such that 0 is in the interior of J_K, $f(J_K) = \{w \in \mathbb{C} : |w| = \frac{1}{2}\}$ and $J_K \subset \{z \in \mathbb{C} : |z| \leq \frac{1}{2K}\}$.*

PROOF. Letting $D_{1/2} = \{w \in \mathbb{C} : |w| < 1/2\}$, and if $w' \in \bar{D}_{1/2}$, there exists a unique $z' \in \Omega_0$ such that $f(z') = w'$. Since $f(\Omega_0)$ contains a disk with center at w' and radius at least K, it follows that $|f'(z')|(1-|z'|^2) \geq K$. Let g: $\{w \in \mathbb{C} : |w| < K+1\} \longrightarrow \Omega_0$ be the local inverse function for f. Then $f'(z') \cdot g'(w') = 1$, which means that

$$|g'(w')| \leq \frac{1-|z|^2}{K} \leq \frac{1}{K}.$$

Now, let $D_{1/2} = \{w : |w| < 1/2\}$, let $J_K = g(\partial D_{1/2})$, and suppose that $w' \in J_K$. Then

$$|z'| = \left|\int_0^{w'} g'(w)\,dw\right| \leq \int_0^{w'} |g'(w)|\,|dw| \leq |w'| \cdot \frac{1}{K} = \frac{1}{2K}.$$

Further, it follows from the conformal mapping properties of f that J_K is a simple closed curve with 0 in its interior. This completes the proof of the lemma.

LEMMA 2. *Let f be a function analytic in D such that f is not a Bloch function. Then there exists a sequence of points $\{z_n\}$ and a sequence of simple closed curves $\{J_n\}$ in D such that*

(1) z_n is in the interior of J_n for each positive integer n,

(2) if $\mu_n = \sup\{\sigma(z_1, z_2) : z_1, z_2 \in J_n\}$, then $\mu_n \longrightarrow 0$, and

(3) for each fixed positive integer n, if $z' \in J_n$, then

$$|f(z') - f(z_n)| = \frac{1}{2}.$$

PROOF. Since f is not a Bloch function, for each positive integer n there exists a point z_n and an open subset Ω_n of D such that $z_n \in \Omega_n$ and f maps Ω_n in a one-to-one manner onto a disk with center at $f(z_n)$ and radius $n + 1$. Define $\gamma_n(z) = \frac{z+z_n}{1+\bar{z}_n z}$, and let $f_n(z) = f\big(\gamma_n(z)\big) - f(z_n)$. If we let $\Omega'_n = \gamma_n^{-1}(\Omega_n)$, then $0 \in \Omega'_n$, f_n is analytic in D, $f_n(0) = 0$ and f_n maps Ω'_n in a one-to-one manner onto the disk $\{w \in \mathbb{C} : |w| < n+1\}$. Applying Lemma 1 to the function f_n, there exists a simple closed curve $J'_n \subset \Omega'_n \bigcap \{z \in \mathbb{C} : |z| \leq \frac{1}{2n}\}$ such that 0 is in the interior of J'_n and, if $z''_n \in J'_n$, then $|f_n(z''_n) - f_n(0)| = \frac{1}{2}$. Letting $J_n = \gamma_n(J'_n)$, we have that z_n is in the interior of J_n and, since

$$\sigma(z''_n, 0) = \sigma\big(\gamma_n(z''_n), \gamma_n(0)\big) = \sigma\big(\gamma_n(z''_n), z_n\big) \leq \tanh^{-1}\Big(\frac{1}{2n}\Big),$$

we have that $\mu_n \leq 2\tanh^{-1}(\frac{1}{2n})$ and $|f(z') - f(z_n)| = \frac{1}{2}$ whenever $z' \in J_n$. Since $\tanh^{-1}(x) \longrightarrow 0$ as $x \longrightarrow 0$, the proof is complete.

THEOREM 2. *Let f be a function analytic in D such that f is either an additive automorphic function or a rotation automorphic function whose rotation group Σ consists of rotations around the origin. Then f is a Bloch function if and only if f is* SUCH *in the fundamental region F.*

Note that, in the case of an additive automorphic function, this result appears in [2, Theorem 3, p. 75]. The proof given here is different and more elementary.

PROOF. If f is a Bloch function, then by Theorem 1, f is strongly uniformly continuous hyperbolically in all of D, and hence in F. This proves the "only if" part.

We prove the "if" part by contradiction. Suppose that f is not a Bloch function. By Lemma 2, there exists a sequence $\{z_n\}$ of points in D and a sequence $\{J_n\}$ of simple closed curves in D such that, for each n, the point z_n is in the interior of J_n and, for each point $z' \in J_n$ we have $|f(z') - f(z_n)| = \frac{1}{2}$, and also, if μ_n is the hyperbolic diameter of J_n, then $\mu_n \longrightarrow 0$. Let γ_n be the element of Γ (the Fuchsian group for f) for which $z_n'' = \gamma_n(z_n) \in \bar{F}$, let $J_n' = \gamma_n(J_n)$, let L_n be the line segment from the origin to the point z_n'', and let ζ_n denote a point of intersection of L_n with J_n'. Since F is starshaped, we have that L_n lies in the closure of F, so that both ζ_n and z_n'' are in the closure of F. Further, the assumptions on f mean that $f\big(\gamma_n(z)\big)$ results from $f(z)$ by a rigid motion, which means that

$$|f(\zeta_n) - f(z_n'')| = \left|f\big(\gamma_n^{-1}(\zeta_n)\big) - f(z_n)\right| = \frac{1}{2},$$

while $\sigma(\zeta_n, z_n'') \leq \mu_n \longrightarrow 0$. This means that f is not SUCH in F. This completes the proof of the theorem.

LEMMA 3. *Let f be a function analytic and additive automorphic in D relative to a Fuchsian group Γ generated by a parabolic transformation γ. Further, suppose that $h(z) = |f(z)|$ is SUCH in F, the fundamental region of Γ. Then f is a Bloch function.*

PROOF. Let A_γ be the complex number such that $f\big(\gamma(z)\big) = f(z) + A_\gamma$ for each $z \in D$. First assume that $A_\gamma \neq 0$. By a conformal mapping of the domain and by replacing the function f by the function $2f/A_\gamma$, we may assume that f is analytic in the upper half plane H, $\gamma(z) = z + b$, and $f\big(\gamma(z)\big) = f(z) + 2$. (We note that multiplying f by a constant does not change the SUCH property, nor does a conformal mapping of the domain, provided that we use the hyperbolic distance in H in place of the hyperbolic distance in D in our definition.)

We now may take $F = \{z = x + iy : -b/2 < x < b/2\}$. Further, if, for each n, the image of f contains a schlicht disk with center at $w_n = f(\zeta_n)$ and radius n, then for each integer k there is a schlicht disk with center at $f\big(\gamma^k(\zeta_n)\big)$ with radius n. Thus, if f is not a Bloch function, we may assume that there exists a sequence $\{z_n\}$ in F such that f has a schlicht disk with center at $f(z_n)$ and radius n. We note that we may assume that either $\operatorname{Im} z_n \longrightarrow \infty$ or $\operatorname{Im} z_n \longrightarrow 0$, for otherwise f would not be analytic at a limit point of the sequence $\{z_n\}$.

Now, assume that f is not a Bloch function. According to Lemma 2, for each positive integer n there exists a simple closed curve J_n in H with z_n in the interior of J_n such that $f(J_n) = C_n$, the circle with center at $f(z_n)$ and radius $1/2$, and the hyperbolic diameter μ_n of J_n goes to zero with n. If $\operatorname{Im} z_n \longrightarrow 0$, it is a consequence of the fact that $\mu_n \longrightarrow 0$ that J_n can meet F and at most one other copy F_1 of F. But then it is easy to see that if $\zeta_{1,n}$ and $\zeta_{2,n}$ are the pre-images of the points of intersection of $f(J_n)$ and the line L_n determined by the origin and $f(z_n)$ (if $f(z_n) = 0$ then let L_n be any line through the origin), then either F or F_1 must contain two of the three points $\zeta_{1,n}$, $\zeta_{2,n}$, and z_n, and this violates the assumption that $h(z) = |f(z)|$ is SUCH in F. Thus, we must have that $\operatorname{Im} z_n \longrightarrow \infty$.

For each n, let $E_n = \{z = x + iy \in F : \sigma(iy, i\operatorname{Im}(z_n)) < 2\mu_n\}$. (Here, we use σ to denote hyperbolic distance in H.) Since $h(z) = |f(z)|$ is SUCH in F, there exists a $\delta > 0$ such that $\big||f(z)| - |f(z')|\big| < 1/16$ whenever $z, z' \in F$ and $\sigma(z, z') < \delta$. Since $\mu_n \longrightarrow 0$, there exists an integer n_0 such that $E_n \subset \{z \in F : \sigma(z, z_n) < \delta\}$ for each $n > n_0$. Now let $B_{n,1} = \partial E_n \bigcap \{z = x + iy : x = -b/2\}$ and let $B_{n,2} = \gamma(B_{n,1})$ and let $B_{n,3} = \gamma(B_{n,2})$. Further, let $A_{n,1} = \{w \in \mathbb{C} : |f(z_n)| - 1/16 < |w| < |f(z_n)| + 1/16\}$ and let $A_{n,2} = \{w' = w + 2 : w \in A_{n,1}\}$.

Since $h(z) = |f(z)|$ is SUCH in F, it follows that $f(E_n) \subset A_{n,1}$, so that both $f(B_{n,1})$ and $f(B_{n,2})$ are contained in the closure of $A_{n,1}$. Since f is additive automorphic, it follows that $f(B_{n,2})$ is also contained in the closure of $A_{n,2}$, the translate of $A_{n,1}$ by 2. But this means that $f(B_{n,2})$ is contained in a component of the intersection of the closures of $A_{n,1}$ and $A_{n,2}$, and this intersection is a set of diameter less than $1/4$. Thus, the diameter of $f(B_{n,2})$ is less than $1/4$, and, since $f(B_{n,2})$ is a translate of $f(B_{n,1})$ through a distance 2, it follows that C_n, which has diameter 1, cannot intersect both sets $f(B_{n,1})$ and $f(B_{n,2})$, which are a distance at least $3/2$ apart. If we assume, for definiteness, that C_n intersects $f(B_{n,2})$, then a similar argument shows that C_n cannot intersect $f(B_{n,3})$. It follows that the interior of C_n is contained in $\overline{f(E_n)} \cup \overline{f(\gamma(E_n))}$ (or in $\overline{f(E_n)} \cup \overline{f(\gamma_n^{-1}(E_n))}$). This means that a disk with radius $1/2$ is contained in the union of two annuli, each with thickness $1/8$, which is impossible. It follows that A_γ must be 0, which means that f is automorphic relative to Γ.

Now if f is automorphic in D and we assume that f is not a Bloch function, we can repeat the argument above, with the same notation, to obtain that $C_n = f(J_n) \subset \overline{f(E_n)} \subset \bar{A}_{n,1}$ for $n > n_0$. But this means that a circle of radius $1/2$ is contained in a single annulus with thickness $1/4$, which is impossible. Thus, we conclude that f is a Bloch function, and the lemma is proved.

We need one more lemma.

LEMMA 4. *Let f be a function analytic in D such that f is an additive automorphic function such that $h(z) = |f(z)|$ is* SUCH *in F. If p is a parabolic vertex of the fundamental region F and if $\{z_n\}$ is a sequence of points in F which converges to the point p, then the sequence $\{|f'(z_n)|(1 - |z_n|^2)\}$ is a bounded sequence.*

PROOF. Using the same notation as in the proof of Lemma 3, we may assume, without loss of generality, that f is defined in the upper half plane H, that ∞ is a parabolic vertex of F, that $\gamma_0(\zeta) = \zeta + b$ is the generator of the parabolic elements of Γ which fix ∞, and that F is a subset of $\{\zeta = \xi + i\eta \in H : -b/2 < \xi < b/2\}$. In this context, the condition equivalent to the conclusion of the lemma is that the sequence $\{|f'(\zeta_n)|/\operatorname{Im}(\zeta_n)\}$ is a bounded sequence, where $\{\zeta_n\} \in F$ and $\operatorname{Im}(\zeta_n) \longrightarrow \infty$. Because ∞ is a parabolic vertex of F, there exists a positive number y_0 such that the set $\{\zeta = \xi + i\eta : -b/2 < \xi < b/2, \eta > y_0\}$ is a subset of F. Now define a function $g(\zeta) = f(\zeta + iy_0)$ in H. It is easily verified that g is additive automorphic relative to the group generated by γ_0, and hence g is a Bloch function by Lemma 3. Thus, $|g'(\zeta)|/\operatorname{Im}(\zeta) \leq ||g||_B$. Since $g'(\zeta) = f'(\zeta + iy_0)$, it follows that, for $\mathrm{Im}(\zeta) > y_0$, $|f'(\zeta)| / \mathrm{Im}(\zeta) = |g'(\zeta - iy_0)| / \mathrm{Im}(\zeta) < |g'(\zeta - iy_0)| / \mathrm{Im}(\zeta - iy_0) \leq \|g\|_B$. This proves the lemma.

We now consider some cases in which $h(z) = |f(z)|$ is SUCH in the fundamental region.

THEOREM 3. *Let f be a function analytic in D such that f is additive automorphic relative to a finitely generated Fuchsian group Γ. Then f is a Bloch function function if and only if $h(z) = |f(z)|$ is* SUCH *on the fundamental region F.*

PROOF. The "only if" part follows directly from Theorem 1.

Now suppose that f is not a Bloch function. Then there exists a sequence of points $\{\zeta_n\}$ in D such that there is a schlicht disk with center at $f(\zeta_n)$ and radius n in the image of f. Since f is additive automorphic, there exists a point z_n in $\bar{F}$ and an element γ of Γ such that $\gamma(z_n) = \zeta_n$ and $f\big(\gamma(z)\big) = f(z) + A_\gamma$ for each $z \in D$. It follows that there is a schlicht disk with center at $f(z_n)$ and radius n, where this schlicht disk is simply a translate of the schlicht disk with center at $f(\zeta_n)$. Since $|f'(z_n)|(1 - |z_n|^2) \geq n$ for each n, it follows from Lemma 4 that the sequence $\{z_n\}$ cannot converge to a parabolic vertex of F. Also, because Γ is finitely generated, F can have at most a finite number of parabolic vertices, so it follows that the sequence $\{z_n\}$ is bounded away from the parabolic vertices of F (see [4, pp. 143–146]).

Since we must have that $|z_n| \longrightarrow 1$, it follows that any convergent subsequence of $\{z_n\}$ must converge to a point on a free arc of $\partial D \cap \partial F$. Then the geometry of F for a finitely generated group Γ (see, for example, [4, Theorem, p. 75]) provides that there exists a number $\delta > 0$ and a number r, $0 < r < 1$, such that any disk Δ in D with hyperbolic radius δ and center $\zeta \in \bar{F}$ and $|\zeta| > r$ has the property that there exists a single element $\gamma_\Delta \in \Gamma$ such that $\Delta \subset \bar{F} \cup \gamma_\Delta(F)$. (We note that it is possible that γ_Δ is the identity, so that it may happen that $\gamma_\Delta(F) = F$.)

By Lemma 2, for each integer $n > 1$, there exists a simple closed curve J_n in D such that z_n is in the interior of J_n and $f(J_n)$ is the circle with center at $f(z_n)$ and radius $1/2$, and, in addition, the hyperbolic diameter μ_n of J_n goes to zero with n. Let R_n denote the union of J_n and its interior. Since $\mu_n \longrightarrow 0$, there exists an integer n_0 such that, for $n > n_0$ we have that

$$R_n \subset D_n\{z \in D : \sigma(z, z_n) < \delta\}.$$

Recalling that $h(z) = |f(z)|$ is SUCH in F, there exists a number $\alpha > 0$ such that $\big| |f(z)| - |f(z')| \big| < 1/16$ whenever $z, z' \in F$ and $\sigma(z, z') < \alpha$. Since $\mu_n \longrightarrow 0$, there exists n_1 such that for $n > n_1$ we have that $\big| |f(z)| - |f(z')| \big| \leq 1/16$ whenever z and z' are in $R_n \cap \bar{F}$. Similarly, for $n > n_1$, we have $\big| |f(z)| - |f(z')| \big| \leq 1/16$ whenever z and z' are in $\gamma_{D_n}^{-1}(R_n) \cap \bar{F}$. (Recall that γ_{D_n} is the element of Γ such that $D_n \subset \bar{F} \cup \gamma_{D_n}(F)$.)

Define the two annuli $A_{n,1} = \{w \in \mathbb{C} : |f(z_n)| - 1/16 \leq |w| \leq |f(z_n)| + 1/16\}$ and $A_{n,2} = \{w \in \mathbb{C} : |f(z_n')| - 1/16 \leq |w| \leq |f(z_n')| + 1/16\}$, where z_n' is an arbitrary point in $\gamma_{D_n}^{-1}(R_n) \cap \bar{F}$. (If γ_{D_n} is the identity, we can take $A_{n,2}$ to be the empty set in what follows.) Let $A_{n,3}$ be the translation of $A_{n,2}$ by the constant $A_{\gamma_{D_n}}$. Then for $n > n_1$, we have that

$f(R_n)$ is contained in the union of the two annuli $A_{n,1}$ and $A_{n,3}$, which means that a disk of radius $1/2$ is contained in the union of two closed annuli each having thickness $1/8$. This is impossible, so we have arrived at a contradiction. It follows that f is a Bloch function, and the proof is complete.

We note that the proof of Theorem 3 does not make full use of the assumption that the Fuchsian group Γ is finitely generated. Thus, we can use virtually the same proof to prove the following.

COROLLARY 1. *Let f be an additive automorphic function in D for which the fundamental region F satisfies the following two conditions:*

(α) F has a finite number of parabolic vertices, and

(β) there exists a $\delta > 0$ and a positive integer N such that, if $U(z,\delta) = \{\zeta \in D : \sigma(z,\zeta) < \delta\}$, then for each point $z \in \bar{F}$ if $U(z,\delta)$ meets more than N copies of $\bar{F}$, then there exists a parabolic transformation $\gamma_0 \in \Gamma$ such that $U(z,\delta) \subset \bigcup_{n=-\infty}^{\infty}(\gamma_0)^n(\bar{F})$. Then f is a Bloch function if and only if $h(z) = |f(z)|$ is SUCH *in the fundamental region F.*

For a Fuchsian group Γ, we say that the fundamental region F is *thick* if there exist positive constants r and r' such that for each sequence $\{z_n\}$ of points in F there is a sequence of points $\{z_n'\}$ such that $\sigma(z_n, z_n') < r$ and, for each positive integer n, $\{z : \sigma(z_n', z) < r\} \subset F$. It is easy to see that if F is thick then F has no parabolic vertices and that for each $\delta > 0$, there exists a positive integer N such that for each $z \in F$ the set $U(z,\delta) = \{\zeta \in D : \sigma(z,\zeta) < \delta\}$ meets at most N copies of F. Thus, if the fundamental region F is thick, then both conditions (α) and (β) of Corollary 1 are satisfied. Thus, we have the following result.

COROLLARY 2. *Let f be an additive automorphic function with respect to a Fuchsian group Γ for which the fundamental region F is thick and for which the function $h(z) = |f(z)|$ is* SUCH *in the fundamental region F. Then f is a Bloch function.*

3. **An example.** In this section, we show that the condition that $h(z) = |f(z)|$ is SUCH in the fundamental region F is not a sufficient condition for an automorphic function to be a Bloch function.

THEOREM 4. *There exists an automorphic function f for which $h(z) = |f(z)|$ is* SUCH *in the fundamental region F but f is not a Bloch function.*

PROOF. Let $a_0 = 0$, $a_1 = 1$, and, for $n > 1$, $a_{n+1} = a_n + (a_n)^{-1/2}$. Further, for n negative, define $a_n = -a_{|n|}$. Define $p(x) = \max\{2, |x|^{1/4}\}$ for $-\infty < x < \infty$. Let R be the region $R = \{z = x+iy : 0 < y < p(x)\}$, and, for each integer n, let $T_n = \{z = x+iy : a_n < x < a_{n+1}, 1 < y < p(x)\}$ and let $P_n = \{z = a_n + iy : 1 < y < p(a_n)\}$. Now define R_0 to be the region R with the set of points $\{(a_n, 1) : -\infty < n < \infty\}$ removed. We wish to form a Riemann surface W from a countable number of copies of R_0.

To begin our construction, we slit R_0 along each of the segments P_n, leaving each of the sets T_n as "tabs" for the region R_0. If w is a point of R_0 and if R' is a copy of R_0, we will denote by $w(R')$ the point of R' with the same coordinates (in R') as those of w (in

R_0). Similarly, we will use the notations $T_n(R')$ and $P_n(R')$ to denote the region on R' and the slit on R' corresponding to T_n and P_n, respectively, on R_0. We will connect the copies of R_0 to each other in stages.

At the first stage, for each integer n, we join a copy $R_{n,1}$ to R_0 by indentifying the left edge of the slit $P_{n+1}(R_0)$ with the right edge of the slit $P_{n+1}(R_{n,1})$. Thus, we have that $R_0 \cup R_{n,1}$ is a connected set (as a surface) where paths which join a point of R_0 to a point of $R_{n,1}$ must cross the segment representing the identified left edge of the slit $P_{n+1}(R_0)$ and the right edge of the slit $P_{n+1}(R_{n,1})$. There are no other connections made between R_0 and $R_{n,1}$, and there are no direct connections between $R_{n,1}$ and $R_{j,1}$ for $n \neq j$. Similarly, for each integer n, we connect R_0 to a copy $R_{n,-1}$ by identifying the right edge of the slit $P_n(R_0)$ with the left edge of the slit $P_n(R_{n,-1})$. There are no other connections made between R_0 and $R_{n,-1}$, and there are no direct connections between $R_{n,c}$ and $R_{j,d}$, where n and j are integers, c and d are in the set $\{-1, 1\}$, and $(n, c) \neq (j, d)$. Of course, each of the sets $R_{n,1}$ and $R_{n,-1}$ is connected to R_0, so the set

$$R_0 \cup \bigcup_{n=-\infty}^{\infty} R_{n,1} \cup \bigcup_{n=-\infty}^{\infty} R_{n,-1}$$

is a connected set. (For example, for each integer n, there is a path from $T_{n-1}(R_{n,-1})$ to $T_n(R_0)$ and continuing from $T_n(R_0)$ to $T_{n+1}(R_{n,1})$, going across identified slits.)

We now complete the construction of the surface W inductively. If R' is a copy of R_0 introduced at the k-th stage, for each integer n, we introduce sets $R'_{n,1}$ and $R'_{n,-1}$ at the $(k+1)$-st stage by making connections analoguous to those at the first stage, where the role of R_0 at the first stage is played by R', and the roles of $R_{n,1}$ and $R_{n,-1}$ are played by $R'_{n,1}$ and $R'_{n,-1}$ at the current stage. Taking the union of all the sets at all stages results in the surface W. We note that the only direct connection between "sheets" of W are those explicitly described in the construction. This ensures that W is simply connected.

Let π denote the projection mapping from W onto R_0 (here considered as a subset of the complex plane), where, for each sheet R' of W, $\pi\big(w(R')\big) = w$. Further, the collection of continuous automorphisms γ^* of W which satisfy the condition $\pi(w) = \pi\big(\gamma^*(w)\big)$ for each $w \in W$ form a group Γ^*. This group is generated by mappings which send $w(R')$ to $w(R'_{n,1})$ and those which send $w(R')$ to $w(R'_{n,-1})$, $-\infty < n < \infty$.

Since W is a simply connected Riemann surface, there exists a one-to-one continuous function $\tilde{f}\colon D \longrightarrow W$ such that the function $f = \pi \circ \tilde{f}$ is an analytic function. Further, it is easily seen that the group Γ^* of automorphisms of W described above induces a Fuchsian group Γ on D such that $f\big(\gamma(z)\big) = f(z)$ for each $\gamma \in \Gamma$ and each $z \in D$. Thus, f is an automorphic function relative to Γ which maps D onto R_0. Let F denote the fundamental region in D relative to Γ. Then the function f is a univalent function from F onto $R_0 - \bigcup_{n=-\infty}^{\infty} P_n$ (considered as a region in the complex plane). We claim that $|f|$ is SUCH in F, and we proceed to show this.

Let M be a positive number, and let

$$G_M = \{z \in D : \operatorname{Im} f(z) < M\}.$$

We claim that $|f|$ is SUCH in G_M. If not, then there exist a pair of sequences $\{z_n\}$ and $\{z_n'\}$ in G_M and a number $\varepsilon_0 > 0$ such that $\sigma(z_n, z_n') \longrightarrow 0$ and $\big|\, |f(z_n)| - |f(z_n')| \,\big| \geq \varepsilon_0$. Assuming that such sequences can be found, for each n, let

$$f_n(z) = f\big((z + z_n)/(1 + \bar{z}_n z)\big) - f(z_n).$$

Then $f_n(0) = 0$, and $\mathrm{Im}\big(f_n(z)\big) > -M$ for each n, which implies that the collection $\{f_n\}$ is a normal family in D. Letting $z_n'' = (z_n' - z_n)/(1 - \bar{z}_n z_n')$, we have that $\sigma(0, z_n'') = \sigma(z_n, z_n') \longrightarrow 0$, and the fact that $\{f_n\}$ is a normal family means that $f_n(z_n'') = f(z_n') - f(z_n) \longrightarrow 0 = \lim_{n\to\infty} f_n(0)$, contradicting the property assumed for the sequences $\{z_n\}$ and $\{z_n'\}$. This shows that $|f|$ is SUCH in G_M (which contains points outside of F).

Now suppose that $\{z_n\}$ and $\{z_n'\}$ are two sequences of points in F and $\varepsilon_0 > 0$ is a real number such that $\big|\, |f(z_n)| - |f(z_n')| \,\big| > \varepsilon_0$. If there exists a number M such that $\mathrm{Im} f(z_n) < M$ and $\mathrm{Im} f(z_n') < M$ for each n, then the fact that f is SUCH in G_M requires that there exists a $\delta > 0$ such that $\sigma(z_n, z_n') > \delta$ for each n sufficiently large. Thus, if we assume that $\sigma(z_n, z_n') \longrightarrow 0$, we must have that at least one of the sequences $\{\mathrm{Im} f(z_n)\}$ and $\{\mathrm{Im} f(z_n')\}$ is unbounded. Thus, we may assume, without loss of generality, that $\mathrm{Im} f(z_n) \longrightarrow \infty$.

Recalling the fact that the sequence $\{a_n\}$ converges to ∞, let n_0 be such that $|a_n|^{-1/2} < \varepsilon_0/6$ whenever $n > n_0$, and let $M = a_{n_0+1}$. Let T_n^* denote the extended tab

$$T_n^* = \{z = x + iy : a_n < x < a_{n+1}, 0 < y < P(x)\} \cap R_0.$$

If both $f(z_n)$ and $f(z_n')$ are in the same extended tab T_n with $|n| > n_0$, then

$$\big|\, |f(z_n)| - |f(z_n')| \,\big| \leq |a_{n+1} + i(a_{n+1})^{1/4}| - a_n.$$

Recalling that $a_{n+1} = a_n + (a_n)^{-1/2}$, we have that

$$\begin{aligned} |a_{n+1} + i(a_{n+1})^{1/4}| - a_n &= \frac{|a_{n+1} + i(a_{n+1})^{1/4}|^2 - (a_n)^2}{|a_{n+1} + i(a_{n+1})^{1/4}| + a_n} \\ &\leq \frac{(a_{n+1})^2 - (a_n)^2 + (a_{n+1})^{1/2}}{a_{n+1} + a_n} \\ &\leq a_{n+1} - a_n + (a_{n+1})^{-1/2} \\ &\leq 2(a_n)^{-1/2} < \varepsilon_0/3. \end{aligned}$$

Now suppose that $f(z_n)$ and $f(z_n')$ are in different tabs, say, $f(z_n) \in T_j$ and $f(z_n') \in T_k$, where $j \neq k$. Let L_n be the hyperbolic line segment in D between z_n and z_n'. Since the Ford fundamental region is convex in the hyperbolic metric, we have that $L_n \subset F$. Further, $K_n = f(L_n)$ is a curve in R_0 which does not cross any of the segments P_n. By the calculation above, we have that if j is sufficiently large and w_1 and w_2 are points in $K_n \cap T_j^*$, then $\big|\, |w_1| - |w_2| \,\big| < \varepsilon_0/3$. Similarly, if k is sufficiently large and w_3 and w_4 are points in $K_n \cap T_k^*$ then $\big|\, |w_3| - |w_4| \,\big| < \varepsilon_0/3$. We note, using $M = 1$, that K_n must meet both $f(G_1) \cap T_j^*$ and $f(G_1) \cap T_k^*$. If we choose $w_2 \in K_n \cap f(G_1) \cap T_j^*$ and $w_3 \in K_n \cap f(G_1) \cap T_k^*$, where $w_2 = f(\zeta_2)$, $w_3 = f(\zeta_3)$, and ζ_2 and ζ_3 are points of $L_n \cap G_1$, then because $|f|$ is

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SUCH in G_1 we have that there exists a constant $\beta > 0$ such that $\sigma(\zeta, \zeta') > \beta$ whenever $\big| |f(\zeta)| - |f(\zeta')| \big| \geq \varepsilon_0/3$, where the constant β depends only on ε_0 and the function f. Using $\zeta = \zeta_2$ and $\zeta' = \zeta_3$, we conclude that either $\big| |f(z_n)| - |f(z'_n)| \big| < \varepsilon_0$ or else $\sigma(z_n, z'_n) > \beta$, and this shows that $|f|$ is SUCH in F.

Next, we note that f is not a Bloch function, because the Riemann surface W contains arbitrarily large schlicht disks. For example, we might take the "corridor"

$$\cdots \cup T_{-1}(R_{-1,-1}) \cup T_0(R_0) \cup T_1(R_{1,1}) \cup \cdots$$

which contains a copy of each of the tabs. (Simply start with $T_0(R_0)$ and move through $T_1(R_{1,1})$ to the copy of T_2 joined to $T_1(R_{1,1})$, *etc.*) This "corridor" contains arbitrarily large disks, and f maps a subset of D in a one-to-one manner onto the "corridor".

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