Glasgow Math. J. **60** (2018) 123–134.
doi:10.1017/S0017089516000604.

COMPACT FACTORIZATION OF OPERATORS WITH λ-COMPACT ADJOINTS

ANTARA BHAR and ANIL K. KARN
School of Mathematical Sciences, National Institute of Science Education and Research, Bhimpur, Padanpur via Jatni, Khurda-752050, India
e-mails: antara.music@gmail.com, anilkarn@niser.ac.in

(Received September 11 2015; revised October 9 2016; accepted 20 October 2016; first published online 20 March 2017)

Abstract. Let λ be a symmetric, normal sequence space equipped with a k-symmetric, monotone norm $\|.\|_\lambda$. Also, assume that $(\lambda, \|.\|_\lambda)$ is AK-BK. Corresponding to this sequence space λ, we study compactness of the operator ideal K_λ. We proved compactness, completeness and injectivity of the dual operator ideal K_λ^d. We also investigate the factorization of these operators.

2010 *Mathematics Subject Classification.* 46B50, 46B20, 46B28, 47B07.

1. Introduction. Compactness is a significant topological notion of Banach space theory. This notion, in general, has been used to bridge the gap while passing from finite dimensional spaces to infinite dimensional spaces. Grothendieck's [**5**] development of various norms on the tensor product of two Banach (locally convex) spaces is an important milestone in this direction. It may be noted that this process also developed certain classes of operators which are known as operator ideals. Some of these operator ideals are closely related to the operator ideal K of compact operators. Let N $(= N_1)$ denote the operator ideal of nuclear operators, I $(= I_1)$ denote the operator ideal of integral operators and Π $(= \Pi_1)$ denote the operator ideal of absolutely summing operators. (These ideals were introduced by A. Grothendieck.) Then,

$$N^{\max} = I, \;\; K^{\max} = \Pi$$

$$I^{\text{inj}} = \Pi, \;\; N^{\text{inj}} = K.$$

Pietsch [**12**] and others generalized the notions of nuclear to p-nuclear (N_p), integral to p-integral (I_p) and absolutely summing to p-absolutely summing (Π_p) operators for $1 < p < \infty$. In particular, they established the following relations:

$$N_p^{\max} = I_p, \;\; (N_p^d)^{\max} = I_p^d,$$

and

$$I_p^{\text{inj}} = \Pi_p, \;\; (I_p^d)^{\text{sur}} = \Pi_p^d.$$

However, a suitable extension of compact operators to a p-case $(1 \le p < \infty)$ remained missing till 2002. Motivated by Grothendieck's characterization of a relatively compact set as one sitting inside convex hull of a vector-valued null sequence, Sinha and Karn [**16**] in 2002 introduced the notion of p-compactness $(1 \le p < \infty)$ in Banach spaces.

They defined a set K in a Banach space X as relatively p-compact if there exists a sequence $\overline{x} = \{x_n\} \in l_p^s(X)$ such that $K \subseteq \{\sum_{n\geq 1} \alpha_n x_n : \overline{\alpha} \in B_{l_q}, 1/p + 1/q = 1\}$. An operator $T \in \mathcal{L}(X, Y)$, X and Y are Banach spaces, is said to be relatively p-compact operator if T maps bounded sets of X to relatively p-compact sets in Y. They proved the factorization of adjoint of a relatively p-compact operator through a subspace of l_p. They defined a norm on the class of relatively p-compact operators using this factorization and proved that K_p, the class of relatively p-compact operators, is a Banach operator ideal when equipped with this norm. It is also proved that $K_p^d = N_p^{\text{inj}}$ and $(K_p^d)^{\max} = \Pi_p$, cf.[**16,17**]. See also [**1,13**] for a different approach to these equalities, which expresses K_p as a surjective hull of a certain well-known operator ideal.

Using the duality theory of sequence spaces, we introduced the notion of λ-compactness in [**6**], corresponding to an arbitrary Banach sequence space λ. In this paper, we prove that every λ-compact operator is compact for a symmetric, normal, AK-BK sequence space λ. For a pair of Banach spaces X and Y, the space $K_\lambda(X, Y)$ is equipped with the quasi-norm $\|.\|_{k_\lambda}$ which is defined as $\|T\|_{k_\lambda} = \inf \|\overline{y}\|_\lambda^s$, where $\overline{y} = \{y_n\} \in \lambda^s(Y)$ appears in the definition of λ-compact operators. In this paper, we prove that when λ is a symmetric, normal, AK-BK reflexive sequence space, then for $T \in K_\lambda(X, Y)$, X and Y Banach spaces, T^* factors compactly through a subspace of λ. We define another quasi-norm on $K_\lambda(X, Y)$ through this factorization and establish the equality of the two quasi-norms. We also show that $K_\lambda^d(X, Y)$ is topologically isomorphic to $N_\lambda^{\text{inj}}(X, Y)$ for a symmetric, normal, AK-BK sequence space λ which is equipped with k-symmetric, monotone norm $\|.\|_\lambda$. In [**6**], we proved that K_λ, the collection of all λ-compact operators, form a quasi-normed operator ideal. In this paper, we prove completeness of K_λ^d under the operator ideal norm. This follows from the fact that $K_\lambda^d = N_\lambda^{\text{inj}}$ as quasi-Banach operator ideals. This paper is in sequel of [**6**].

2. Preliminaries. For the rudimentary results and notions of sequence spaces, we essentially follow [**6, 7**]. Our references for operator ideals, λ-summing operators, λ-nuclear, quasi-λ-nuclear, λ-compact operators are [**6, 12, 14, 15**].

Let λ be an arbitrary sequence space. Then, λ is called (i) **symmetric** if $\bar{\alpha_\sigma} = \{\alpha_{\sigma(i)}\} \in \lambda$ whenever $\bar{\alpha} = \{\alpha_i\} \in \lambda$ and $\sigma \in \Pi$, where Π is the collection of all permutations of the set of natural numbers $\mathbb{N}$, (ii) **normal or solid** if $\bar{\beta} = \{\beta_i\} \in \lambda$ whenever $|\beta_i| \leq |\alpha_i|$, $i \geq 1$ for some $\bar{\alpha} = \{\alpha_i\} \in \lambda$ and (iii) **monotone** provided λ contains canonical preimages of all its stepspaces.

A sequence space λ is said to be **perfect** if $\lambda = \lambda^{\times\times} = (\lambda^\times)^\times$, where $\lambda^\times$ is the Köthe-dual of λ.

A Banach sequence space $(\lambda, \|.\|_\lambda)$ is called a **BK-space** provided each of the projection maps $P_i : \lambda \to \mathbb{K}$, $P_i(\bar{\alpha}) = \alpha_i$ is continuous, for $i \geq 1$, where $\mathbb{K}$ is the field of scalars and $\bar{\alpha} = \{\alpha_1, \alpha_2, \ldots\}$. A BK-space $(\lambda, \|.\|_\lambda)$ is called an **AK-space** if $\bar{\alpha}^{(n)} \to \bar{\alpha}$, for each $\bar{\alpha} \in \lambda$.

For a BK-space $(\lambda, \|.\|_\lambda)$, the dual-norm on $\lambda^\times$ is defined as follows:

$$\|\bar{\beta}\|_{\lambda^\times} = \sup\{\sum_{i\geq 1} |\alpha_i||\beta_i| : \bar{\alpha} \in \lambda, \|\bar{\alpha}\|_\lambda \leq 1\}.$$

https://doi.org/10.1017/S0017089516000604 Published online by Cambridge University Press

The space $(\lambda^\times, \|.\|_{\lambda^\times})$ becomes a BK-space provided $0 < \sup_n \|e^n\|_\lambda < \infty$. If $(\lambda, \|.\|_\lambda)$ is also an AK-space, then $(\lambda^\times, \|.\|_{\lambda^\times})$ is topologically isomorphic to its topological dual $(\lambda^*, \|.\|)$, where $\|f\| = \sup\{|f(\bar{\alpha})| : \bar{\alpha} \in \lambda, \|\bar{\alpha}\|_\lambda \leq 1\}$.

The norm $\|.\|_\lambda$ is said to be (i) **k-symmetric** if $\|\bar{\alpha}\|_\lambda = \|\bar{\alpha}_\sigma\|_\lambda$, for all $\sigma \in \Pi$ and (ii) **monotone** if $\|\bar{\alpha}\|_\lambda \leq \|\bar{\beta}\|_\lambda$ for $\bar{\alpha}, \bar{\beta}$ in λ with $|\alpha_i| \leq |\beta_i|, \forall\, i \geq 1$.

The space $(\lambda, \|.\|_\lambda)$ is said to have the **norm iteration property** if for each sequence $\{\bar{\alpha}^n\}$ in λ, $\bar{\alpha}_i = \{\alpha_i^1, \alpha_i^2, \alpha_i^3, \ldots.\} \in \lambda$ for each $i \geq 1$ and $\|\{\|\bar{\alpha}^n\|_\lambda\}_n\|_\lambda = \|\{\|\bar{\alpha}_i\|_\lambda\}_i\|_\lambda$, cf.[**14**].

Corresponding to a sequence space λ and a Banach space X with its topological dual X^* equipped with the operator norm topology generated by $\|.\|$, the vector-valued sequence spaces $\lambda^s(X)$ and $\lambda^w(X)$ are defined as

$$\lambda^s(X) = \{\bar{x} = \{x_n\} \subset X : \{\|x_n\|\} \in \lambda\}$$

and

$$\lambda^w(X) = \{\bar{x} = \{x_n\} \subset X : \{f(x_n)\} \in \lambda, \ \forall f \in X^*\}.$$

If λ is equipped with a monotone norm $\|.\|_\lambda$, the space $\lambda^s(X)$ becomes a normed linear space with respect to the norm defined as

$$\|\bar{x}\|_\lambda^s = \|\{x_n\}\|_\lambda^s = \|\{\|x_n\|\}\|_\lambda, \ \bar{x} = \{x_n\} \in \lambda^s(X).$$

However, for $\bar{x} \in \lambda^w(X)$, the norm on $\lambda^w(X)$ is defined as

$$\|\bar{x}\|_\lambda^w = \|\{x_n\}\|_\lambda^w = \sup\{\|\{f(x_n)\}\|_\lambda, \ f \in X^*, \|f\| \leq 1\}.$$

The symbol $\mathcal{L}(X, Y)$ is used for the set of bounded linear operators between any two Banach spaces X and Y; whereas $\mathcal{L}$ denotes the collection of all bounded operators between any pair of Banach spaces.

An operator $T \in \mathcal{L}(X, Y)$ is said to be

(i) **absolutely λ-summing** [**15**] if for each $\bar{x} = \{x_i\} \in \lambda^w(X)$, the sequence $\{Tx_i\}$ is in $\lambda^s(Y)$;

(ii) **λ-nuclear** [**14**] if T has the representation

$$Tx = \sum_{n\geq 1} f_n(x) y_n,$$

where $\{f_n\} \subseteq X^*$ with $\{f_n\} \in \lambda^s(X^*)$, $\bar{y} = \{y_n\} \in (\lambda^\times)^w(Y)$;

(iii) **quasi-λ-nuclear** [**14**] if there exists $\{f_n\} \subseteq X^*$ such that $\bar{f} = \{f_n\} \in \lambda^s(X^*)$ and $\|Tx\| \leq \|\{f_n(x)\}\|_\lambda$, for each $x \in X$;

(iv) **λ-compact** [**6**] if there exists $\bar{y} = \{y_n\} \in \lambda^s(Y)$ such that $T(B_X) \subseteq \lambda - co\{y_n\} = \{\sum_{n\geq 1} \alpha_n y_n : \bar{\alpha} \in B_{\lambda^\times}\}$.

The symbols $\Pi_\lambda(X, Y)$, $N_\lambda(X, Y)$, $QN_\lambda(X, Y)$ and $K_\lambda(X, Y)$ denote respectively the collection of all λ-summing, λ-nuclear, quasi-λ-nuclear and λ-compact operators from X to Y. Quasi-norms on $\Pi_\lambda(X, Y)$, $N_\lambda(X, Y)$, $QN_\lambda(X, Y)$ and $K_\lambda(X, Y)$ are defined as

(i) $\|T\|_{\Pi_\lambda} = \inf\{C > 0 : \|\{Tx_i\}\|_\lambda^s \leq C\|\{x_i\}\|_\lambda^w$, for $\{x_i\} \in \lambda^w(X)\}$, $T \in \Pi_\lambda(X, Y)$;

https://doi.org/10.1017/S0017089516000604 Published online by Cambridge University Press

(ii) $\|T\|_{N_\lambda} = \inf \|\{f_n\}\|_\lambda^s \|\{y_n\}\|_{\lambda^\times}^w$, where $\{f_n\} \in \lambda^s(X^*)$ and $\{y_n\} \in (\lambda^\times)^w(Y)$ appears in the definition of $N_\lambda(X, Y)$;

(iii) $\|T\|_{QN_\lambda} = \inf \|\{f_n\}\|_\lambda^s$, where $\{f_n\} \in \lambda^s(X^*)$ appears in the definition of $QN_\lambda(X, Y)$;

(iv) $\|T\|_{k_\lambda} = \inf \|\{y_n\}\|_\lambda^s$, where $\{y_n\} \in \lambda^s(Y)$ appears in the definition of λ-compact operators.

It is known that $(\Pi_\lambda, \|.\|_{\Pi_\lambda})$, $(N_\lambda, \|.\|_{N_\lambda})$, $(QN_\lambda, \|.\|_{QN_\lambda})$ are quasi-Banach operator ideals and $(K_\lambda, \|.\|_{k_\lambda})$ is a quasi-normed operator ideal for suitable sequence spaces λ [**6,14,15**].

Let $\mathcal{A}$ be an operator ideal. For Banach spaces X and Y, recall the following classes of operators:

(i) $\mathcal{A}^d(X, Y) = \{T \in \mathcal{L}(X, Y) : T^* \in \mathcal{A}(Y^*, X^*)\}$,

(ii) $\mathcal{A}^{\text{sur}}(X, Y) = \{T \in \mathcal{L}(X, Y) : TQ_X \in \mathcal{A}(l_1(B_X), Y)\}$; where the canonical operator $Q_X : l_1(B_X) \to X$ is defined as $Q_X(\{\xi_x\}_{x \in B_X}) = \sum_{x \in B_X} \xi_x x$;

(iii) $\mathcal{A}^{\text{inj}}(X, Y) = \{T \in \mathcal{L}(X, Y) : J_Y T \in \mathcal{A}(X, l_\infty(B_{Y^*}))\}$, where the canonical operator $J_Y : Y \to l_\infty(B_{Y^*})$ is defined as $J_Y(y) = \{g(y)\}_{g \in B_{Y^*}}$;

(iv) $\mathcal{A}^{\min}(X, Y) = \{T \in \mathcal{L}(X, Y) : T = RT_0S$, for some $S \in \overline{\mathfrak{F}}(X, X_0)$, $T_0 \in \mathcal{A}(X_0, Y_0)$, $R \in \overline{\mathfrak{F}}(Y_0, Y)\}$, where $\overline{\mathfrak{F}}$ is the operator ideal of approximable operators;

(v) $\mathcal{A}^{\max}(X, Y) = \{T \in \mathcal{L}(X, Y) : RTS \in \mathcal{A}(X_0, Y_0)$, for any $S \in \overline{\mathfrak{F}}(X_0, X)$, $R \in \overline{\mathfrak{F}}(Y, Y_0)\}$.

Then $\mathcal{A}^d$, $\mathcal{A}^{\text{sur}}$, $\mathcal{A}^{\text{inj}}$, $\mathcal{A}^{\min}$ and $\mathcal{A}^{\max}$ equipped with corresponding norms are quasi-Banach operator ideals if $\mathcal{A}$ is so, cf.[**12**, Chapter 8].

An operator ideal $\mathcal{A}$ is said to be **surjective** if $\mathcal{A} = \mathcal{A}^{\text{sur}}$, **injective** if $\mathcal{A} = \mathcal{A}^{\text{inj}}$, **maximal** if $\mathcal{A} = \mathcal{A}^{\max}$ and **minimal** if $\mathcal{A} = \mathcal{A}^{\min}$.

An operator ideal $(\mathcal{A}, \|.\|_{\mathcal{A}})$ is said to have the l_∞-extension property if for a Banach space X, a set Γ and an operator $T \in \mathcal{A}(X, l_\infty(\Gamma))$, there is an operator $\tilde{T} \in \mathcal{A}(l_\infty(B_{X^*}), l_\infty(\Gamma))$ such that $T = \tilde{T} \circ i_X$, with $\|\tilde{T}\|_{\mathcal{A}} = \|T\|_{\mathcal{A}}$, where $i_X : X \hookrightarrow l_\infty(B_{X^*})$ is the Alaoglu embedding.

Regarding the injective hull of composition of two quasi-normed operator ideals, Karn and Sinha [**8**] proved.

LEMMA 2.1. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two operator ideals. If $\mathcal{A}_1$ has the l_∞-extension property, or if $\mathcal{A}_2$ is injective, then*

$$(\mathcal{A}_1 \circ \mathcal{A}_2)^{\text{inj}} = {\mathcal{A}_1}^{\text{inj}} \circ {\mathcal{A}_2}^{\text{inj}}.$$

3. λ-compact operators. Throughout this section, we consider that λ is a symmetric, normal sequence space equipped with a k-symmetric, monotone norm $\|.\|_\lambda$ such that $(\lambda, \|.\|_\lambda)$ is an AK-BK space. As the norm $\|.\|_\lambda$ is k-symmetric, we have $\|e^n\|_\lambda = \|e^n_\sigma\|_\lambda$, $\forall\, \sigma \in \Pi$ and so $0 < \sup_n \|e^n\|_\lambda < \infty$.

Regarding λ-compact operators, we prove.

PROPOSITION 3.1. *Let $(\lambda, \|.\|_\lambda)$ be a symmetric, normal, AK-BK space with a k-symmetric, monotone norm. Then, every λ-compact operator is compact.*

Proof. Let $T \in K_\lambda(X, Y)$. We can find $\bar{y} = \{y_n\} \in \lambda^s(Y)$ such that $T(B_X) \subseteq \{\sum_{n \geq 1} \alpha_n y_n : \bar{\alpha} \in B_{\lambda^\times}\}$.

https://doi.org/10.1017/S0017089516000604 Published online by Cambridge University Press

As $(\lambda, \|.\|_\lambda)$ is c_0-invariant [**19**, Theorem 1], there exist $\overline{z} = \{z_n\} \in c_0(Y)$ and $\overline{\beta} = \{\beta_n\} \in \lambda$ such that $y_n = \beta_n z_n$. This would imply

$$T(B_X) \subseteq \{\sum_{n\geq 1} \gamma_n(\|\overline{\beta}\|_\lambda z_n) : \overline{\gamma} \in B_{l_1}\} = \overline{co}\{\|\beta\|_\lambda z_n\}.$$

Thus, by Grothendieck's characterization of compactness, we have $T \in K(X, Y)$. □

Now, we prove that the quasi-norm $\|\cdot\|_{k_\lambda}$ is in fact a factorization norm which follows from the following natural factorizations of λ-compact operators. Though the result is not needed in the sequel, it is important on its own.

LEMMA 3.2. *Let $(\lambda, \|.\|_\lambda)$ be as in Proposition 3.1 and reflexive. Let $T \in \mathcal{L}(X, Y)$. Then, the following statements are equivalent:*
(*i*) *T is λ-compact operator.*
(*ii*) *There are $\overline{y} = \{y_n\} \in \lambda^s(Y)$ and $S_{\overline{y}} \in \mathcal{L}(R(\overline{y}), X^*)$ such that $T^* = S_{\overline{y}} \circ E^*_{\overline{y}}$.*
(*iii*) *There are $\overline{y} \in B_{l_\infty(Y)}$, $\overline{\gamma} = \{\gamma_n\} \in \lambda$ and $S_{\overline{y}} \in \mathcal{L}(R(\overline{y}), X^*)$ such that $T^* = S_{\overline{y}} \circ M^{\overline{y}}_{\overline{\gamma}} \circ E^*_{\overline{y}}$.*
(*iv*) *There are $\overline{y} \in B_{c_0^s(Y)}$, $\overline{\beta} = \{\beta_n\} \in \lambda$ and $S_{\overline{y}} \in \mathcal{L}(R(\overline{y}), X^*)$ such that $T^* = S_{\overline{y}} \circ M^{\overline{y}}_{\overline{\beta}} \circ E^*_{\overline{y}}$.*

Here, $R(\overline{y}) = \{\{f(y_n)\} : f \in Y^*\}$, $E_{\overline{y}} : \lambda^\times \to Y$ given by $E_{\overline{y}}(\overline{\alpha}) = \sum_{n\geq 1} \alpha_n y_n$, $M_{\overline{\gamma}} : l_\infty \to \lambda$ defined as $M_{\overline{\gamma}}(\overline{\alpha}) = \{\gamma_n \alpha_n\}$, and $M^{\overline{y}}_{\overline{\gamma}} = M_{\overline{\gamma}}/R(\overline{y})$.

The proof is omitted since this is on the same line as given in **[16, pp. 20–21]**.

PROPOSITION 3.3. *Let $(\lambda, \|.\|_\lambda)$ be as in Lemma 3.2 and $T \in \mathcal{L}(X, Y)$. Define $k_\lambda(.)$ on $K_\lambda(X, Y)$ as*

$$k_\lambda(T) = \inf\{\|S_{\overline{y}}\| \|\overline{y}\|^s_\lambda : \ T^* = S_{\overline{y}} \circ E^*_{\overline{y}} \ \text{as in Lemma } 3.2(ii)\}.$$

Then, for $T \in K_\lambda(X, Y)$, we have

$$\|T\|_{k_\lambda} = k_\lambda(T).$$

Proof. Though the argument is same as given in [**3**, Proposition 3.15], we sketch a proof of a one-sided ineqality for the sake of completeness. Let $T \in K_\lambda(X, Y)$. Given $\varepsilon > 0$, we can find $\overline{y} = \{y_n\} \in \lambda^s(Y)$ such that $T(B_X) \subseteq \{\sum_{n\geq 1} \alpha_n y_n : \overline{\alpha} \in B_{\lambda^\times}\}$ and $\|\overline{y}\|^s_\lambda < \|T\|_{k_\lambda} + \varepsilon$. As $(\lambda, \|.\|_\lambda)$ is reflexive, $\{\sum_{n\geq 1} \alpha_n y_n : \overline{\alpha} \in B_{\lambda^\times}\}$ is norm-closed. Thus, $\|T^* f\| \leq \|\{f(y_n)\}\|_\lambda$. Hence,

$$\|S_{\overline{y}}\| = \sup_{\|\{f(y_n)\}\|_\lambda \leq 1} \|T^* f\| \leq 1.$$

So we have $k_\lambda(T) \leq \|\overline{y}\|^s_\lambda < \|T\|_{k_\lambda} + \varepsilon$. □

As a consequence of [**18**, page 13, Condition(S)] and the definition of λ-compact operators, we have the following proposition.

PROPOSITION 3.4. *Let $(\lambda, \|.\|_\lambda)$ be as in Proposition 3.1. Then, $K_\lambda = K^{\text{sur}}_\lambda$ as quasi-normed operator ideals.*

Applying the above result, we prove.

PROPOSITION 3.5. *Let* $(\lambda, \|.\|_\lambda)$ *be as in Proposition 3.1. Then,* K_λ^d *is injective operator ideal and* $\|T\|_{k_\lambda^d} = \|T\|_{(k_\lambda^d)^{\text{inj}}}$, *for* $T \in K_\lambda^d(X, Y)$.

Proof. As K_λ is a surjective operator ideal, K_λ^d becomes an injective operator ideal, cf.[**12**, 4.7.18, Proposition 2], that is, $K_\lambda^d(X, Y) = (K_\lambda^d)^{\text{inj}}(X, Y)$.

Let $T \in K_\lambda^d(X, Y)$. Since $K_\lambda^d(X, Y) \subseteq (K_\lambda^d)^{\text{inj}}(X, Y)$, we have $T \in (K_\lambda^d)^{\text{inj}}(X, Y)$. Then,

$$\|T\|_{(k_\lambda^d)^{\text{inj}}} = \|T^*(J_Y)^*\|_{k_\lambda} \leq \|(J_Y)^*\| \|T^*\|_{k_\lambda} \leq \|T^*\|_{k_\lambda} = \|T\|_{k_\lambda^d}.$$

Now we want to show $\|T\|_{k_\lambda^d} \leq \|T\|_{(k_\lambda^d)^{\text{inj}}}$, for $T \in K_\lambda^d(X, Y) = (K_\lambda^d)^{\text{inj}}(X, Y)$. For $\varepsilon > 0$, we can find $\overline{f} = \{f_n\} \in \lambda^s(X^*)$ such that

$$T^*(J_Y)^*(B_{(Y^{\text{inj}})^*}) \subseteq \{\textstyle\sum_{n\geq 1} \alpha_n f_n : \overline{\alpha} \in B_{\lambda^\times}\} \text{ with } \|\overline{f}\|_\lambda^s < \|T^*(J_Y)^*\|_{k_\lambda} + \varepsilon.$$

As $J_Y^* : (Y^{\text{inj}})^* \to Y^*$ is a metric surjection, J_Y^* transforms the open unit ball of $(Y^{\text{inj}})^*$ onto the open unit ball of Y^*. So we have

$$T^*(B_{Y^*}) \subseteq \{\sum_{n\geq 1} \alpha_n f_n : \overline{\alpha} \in B_{\lambda^\times}\}.$$

This implies $\|T^*\|_{k_\lambda} \leq \|\overline{f}\|_\lambda^s < \|T^*(J_Y)^*\|_{k_\lambda} + \varepsilon$.

As $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0$, we have $\|T\|_{k_\lambda^d} \leq \|T\|_{(k_\lambda^d)^{\text{inj}}}$. This completes the proof. □

The main result of this section is the following theorem.

THEOREM 3.6. *Let* $(\lambda, \|.\|_\lambda)$ *be as in Proposition 3.1. Then,* $K_\lambda^d = N_\lambda^{\text{inj}}$ *as quasi-Banach operator ideals.*

To prove Theorem 3.6, we need the following lemma, which is essentially [**14**, Theorem 7]. We provide a proof for the sake of completeness.

LEMMA 3.7. *Let* $(\lambda, \|.\|_\lambda)$ *be as in Proposition 3.1. Then,* $QN_\lambda = N_\lambda^{\text{inj}}$ *as quasi-Banach operator ideals.*

Proof. It is well known that for any Banach space Z, the map $J_Z : Z \to Z^{\text{inj}}$ is an isometry. Hence, $QN_\lambda = QN_\lambda^{\text{inj}}$. Also, Z^{inj} has l_∞-extension property. Thus, as a consequence of [**14**, Theorem 9] and injectivity of the operator ideal QN_λ, we have $QN_\lambda(X, Y) = N_\lambda^{\text{inj}}(X, Y)$ for any pair of Banach spaces X and Y.

For proving $\|T\|_{QN_\lambda} = \|T\|_{N_\lambda^{\text{inj}}}$, $T \in QN_\lambda(X, Y)$, we consider $S \in N_\lambda(X, Y)$. For given $\varepsilon > 0$, there exists $\{f_n\} \in \lambda^s(X^*)$ and $\{y_n\} \in (\lambda^\times)^w(Y)$ such that

$$Sx = \sum_{n\geq 1} f_n(x) y_n, \text{ for } x \in X \text{ with } \|\{f_n\}\|_\lambda^s \|\{y_n\}\|_{\lambda^\times}^w < \|S\|_{N_\lambda} + \varepsilon.$$

https://doi.org/10.1017/S0017089516000604 Published online by Cambridge University Press

For $g \in B_{Y^*}$, we have

$$\begin{aligned}|g(Sx)| &\leq \sum_{n\geq 1} |f_n(x)||g(y_n)| \\ &= \|\{f_n(x)\}\|_\lambda \sum_{n\geq 1} \frac{|f_n(x)|}{\|\{f_n(x)\}\|_\lambda} |g(y_n)| \\ &\leq \|\{f_n(x)\}\|_\lambda \|\{y_n\}\|^w_{\lambda^\times} \\ &\leq \|\{f_n\}\|^s_\lambda \|\{y_n\}\|^w_{\lambda^\times} < \|S\|_{N_\lambda} + \varepsilon.\end{aligned}$$

This would imply

$$\|Sx\| = \sup_{g\in B_{Y^*}} |g(Sx)| \leq \|\{f_n(x)\}\|_\lambda \|\{y_n\}\|^w_{\lambda^\times}.$$

Thus, $\|S\|_{QN_\lambda} \leq \|\{f_n\}\|^s_\lambda \|\{y_n\}\|^w_{\lambda^\times} < \|S\|_{N_\lambda} + \varepsilon$. As $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0$, we have $\|S\|_{QN_\lambda} \leq \|S\|_{N_\lambda}$.

As $N_\lambda(X, Y) \subseteq QN_\lambda(X, Y)$, we have $N^{\mathrm{inj}}_\lambda(X, Y) \subseteq QN^{\mathrm{inj}}_\lambda(X, Y) = QN_\lambda(X, Y)$ and so

$$\|T\|_{QN^{\mathrm{inj}}_\lambda} = \|J_Y T\|_{QN_\lambda} \leq \|J_Y T\|_{N_\lambda} = \|T\|_{N^{\mathrm{inj}}_\lambda}.$$

For the other inequality, that is, $\|T\|_{N^{\mathrm{inj}}_\lambda} \leq \|T\|_{QN_\lambda}$, let $T \in QN_\lambda(X, Y)$. For given $\varepsilon > 0$, we can find $\bar{f} = \{f_n\} \in \lambda^s(X^*)$ such that $\|Tx\| \leq \|\{f_n(x)\}\|_\lambda$ with $\|\{f_n\}\|^s_\lambda < \|T\|_{QN_\lambda} + \varepsilon$, for every $x \in X$. We define a map $T_0 : X \to \lambda$ as

$$T_0(x) = \{f_n(x)\} = \sum_{n\geq 1} f_n(x)e^n$$

for $x \in X$. As $\{f_n\} \in \lambda^s(X^*)$ and $\{e^n\} \in (\lambda^\times)^w(\lambda)$, $T_0 \in N_\lambda(X, \lambda)$. Also, $\|Tx\| \leq \|T_0x\|$. Thus, $T \in N^{\mathrm{inj}}_\lambda(X, Y)$ and $\|T\|_{N^{\mathrm{inj}}_\lambda} \leq \|T_0\|_{N_\lambda}$ by [**12**, Proposition 8.4.4]. As

$$\|T_0\|_{N_\lambda} \leq \|\{f_n\}\|^s_\lambda \|\{e^n\}\|^w_{\lambda^\times} \leq \|\{f_n\}\|^s_\lambda < \|T\|_{QN_\lambda} + \varepsilon,$$

we have $\|T\|_{N^{\mathrm{inj}}_\lambda} < \|T\|_{QN_\lambda} + \varepsilon$. As $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0$, we have $\|T\|_{N^{\mathrm{inj}}_\lambda} \leq \|T\|_{QN_\lambda}$. This completes the proof. □

Proof of Theorem 3.6 Since $N_\lambda(X, Y) \subseteq K^d_\lambda(X, Y)$, cf.[**6**, Theorem 4.1], we have $N^{\mathrm{inj}}_\lambda(X, Y) \subseteq (K^d_\lambda)^{\mathrm{inj}}(X, Y) = K^d_\lambda(X, Y)$.

For showing $K^d_\lambda(X, Y) \subseteq N^{\mathrm{inj}}_\lambda(X, Y)$, we consider $T \in K^d_\lambda(X, Y)$. Then, there exists $\{f_n\} \in \lambda^s(X^*)$ such that $T^*(B_{Y^*}) \subseteq \{\sum_{n\geq 1} \alpha_n f_n : \bar{\alpha} \in B_{\lambda^\times}\}$.

If $Tx = 0$ for $x \in X$, we have $0 = \|Tx\| \leq \|\{f_n(x)\}\|_\lambda$. If $Tx \neq 0$, we have $0 \neq \|Tx\| = \sup_{g\in B_{Y^*}} |g(Tx)|$. For $g \in B_{Y^*}$, we have

$$|g(Tx)| = |T^*g(x)| = |\sum_{n\geq 1} \alpha_n f_n(x)|, \text{ for some } \bar{\alpha} \in B_{\lambda^\times}.$$

https://doi.org/10.1017/S0017089516000604 Published online by Cambridge University Press

This would imply

$$\begin{aligned}|g(Tx)| &= |\sum_{n\geq 1}\alpha_n f_n(x)| \\ &\leq \|\{f_n(x)\}\|_\lambda \sum_{n\geq 1}|\alpha_n|\frac{|f_n(x)|}{\|\{f_n(x)\}\|_\lambda} \\ &\leq \|\{f_n(x)\}\|_\lambda \|\overline{\alpha}\|_{\lambda^\times} \\ &\leq \|\{f_n(x)\}\|_\lambda.\end{aligned}$$

Hence, $\|Tx\| \leq \|\{f_n(x)\}\|_\lambda$. So $T \in QN_\lambda(X, Y) = N_\lambda^{\text{inj}}(X, Y)$ and so $K_\lambda^d(X, Y) \subseteq N_\lambda^{\text{inj}}(X, Y)$.

For proving the equality of the norms $\|.\|_{k_\lambda^d}$ and $\|.\|_{N_\lambda^{\text{inj}}}$, we consider $T \in K_\lambda^d(X, Y)$. For $\varepsilon > 0$, there exists $\overline{f} = \{f_n\} \in \lambda^s(X^*)$ such that

$$T^*(B_{Y^*}) \subseteq \{\sum_{n\geq 1}\alpha_n f_n : \overline{\alpha} \in B_{\lambda^\times}\} \text{ and } \|\overline{f}\|_\lambda^s < \|T^*\|_{k_\lambda} + \varepsilon.$$

For $g \in B_{Y^*}$, we have

$$|g(Tx)| \leq \sum_{n\geq 1}|\alpha_n||f_n(x)|, \text{ for some } \overline{\alpha} \in B_{\lambda^\times}.$$

This implies

$$|g(Tx)| \leq \|\{f_n(x)\}\|_\lambda.$$

Thus, $\|Tx\| \leq \|\{f_n(x)\}\|_\lambda$, for every $x \in X$. Hence, $\|T\|_{QN_\lambda} = \|T\|_{N_\lambda^{\text{inj}}} \leq \|\overline{f}\|_\lambda^s < \|T^*\|_{k_\lambda} + \varepsilon$. As $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0$, we have $\|T\|_{N_\lambda^{\text{inj}}} \leq \|T^*\|_{k_\lambda} = \|T\|_{k_\lambda^d}$.

For the other inequality, let $T \in N_\lambda^{\text{inj}}(X, Y)$. This would imply $J_Y T \in N_\lambda(X, Y^{\text{inj}})$. Thus, $J_Y T \in K_\lambda^d(X, Y^{\text{inj}})$. Now we want to prove $\|J_Y T\|_{k_\lambda^d} \leq \|J_Y T\|_{N_\lambda}$, that is, $\|T\|_{(k_\lambda^d)^{\text{inj}}} \leq \|T\|_{N_\lambda^{\text{inj}}}$. For proving this ineqality, let $S \in N_\lambda(X, Y)$. Given $\varepsilon > 0$, there exists $\{f_n\} \in \lambda^s(X^*)$ and $\{y_n\} \in (\lambda^\times)^w(Y)$ such that

$$Sx = \sum_{n\geq 1} f_n(x)y_n \text{ and } \|\{f_n\}\|_\lambda^s \|\{y_n\}\|_{\lambda^\times}^w < \|S\|_{N_\lambda} + \varepsilon.$$

For $g \in B_{Y^*}$ and $x \in X$, we have

$$(S^*g)x = g(Sx) = \sum_{n\geq 1} f_n(x)g(y_n) = (\sum_{n\geq 1} g(y_n)f_n)x.$$

Since $\{\frac{g(y_n)}{\|\{y_n\}\|_{\lambda^\times}^w}\} \in B_{\lambda^\times}$, we have $S^*(B_{Y^*}) \subseteq \lambda - co\{\|\{y_n\}\|_{\lambda^\times}^w f_n\}$. So

$$\|S\|_{k_\lambda^d} = \|S^*\|_{k_\lambda} \leq \|\{f_n\}\|_\lambda^s \|\{y_n\}\|_{\lambda^\times}^w < \|S\|_{N_\lambda} + \varepsilon.$$

Thus, $\|S\|_{k_\lambda^d} \leq \|S\|_{N_\lambda}$ as $\varepsilon \to 0$.

Hence, for $T \in N_\lambda^{\text{inj}}(X, Y)$ we have $T \in (K_\lambda^d)^{\text{inj}}(X, Y)$ and so

$$\|J_Y T\|_{k_\lambda^d} \leq \|J_Y T\|_{N_\lambda} = \|T\|_{N_\lambda^{\text{inj}}}.$$

Applying Proposition 3.5, we get $\|T\|_{k_\lambda^d} = \|T\|_{N_\lambda^{\text{inj}}}$.

Since $(N_\lambda, \|.\|_{N_\lambda})$ is a quasi-Banach operator ideal, its injective hull $(K^d_\lambda, \|.\|_{k^d_\lambda})$ would be a quasi-Banach operator ideal, cf.[**12**, Theorem 8.4.2].

This completes the proof. □

REMARK: As $(K^d_\lambda, \|.\|_{k^d_\lambda})$ is an injective quasi-Banach operator ideal, we see that $(K^{dd}_\lambda, \|.\|_{k^{dd}_\lambda})$ is a surjective quasi-Banach operator ideal. Recall that $(K_\lambda, \|.\|_\lambda)$ is a surjective quasi-normed operator ideal. Though, we have not be able to show its completeness, we believe it to be true. We may observe that $K_\lambda \subseteq K^{dd}_\lambda$, for $(\lambda, \|.\|_\lambda)$ as in Proposition 3.1. In fact, $K_\lambda \subseteq QN^d_\lambda = K^{dd}_\lambda$. We expect that $K_\lambda = K^{dd}_\lambda$ as quasi-Banach operator ideals.

Regarding the factorization of λ-compact operators, we prove the following.

PROPOSITION 3.8. *Let* $(\lambda, \|.\|_\lambda)$ *be as in Proposition 3.1. Then,*

$$K \circ (K^d_\lambda)^{\max} \circ \overline{\mathfrak{F}} \subseteq K^d_\lambda \subseteq K \circ (K^d_\lambda)^{\max} \circ K.$$

For proving the proposition we need the following

LEMMA 3.9. *Let* $(\lambda, \|.\|_\lambda)$ *be as in Proposition 3.1. Then,* N_λ *is a minimal operator ideal.*

Proof. Let $T \in N_\lambda(X, Y)$. Then, there exists $\{f_n\} \in \lambda^s(X^*)$ and $\{y_n\} \in (\lambda^\times)^w(Y)$ such that $Tx = \sum_{n\geq 1} f_n(x)y_n$, for $x \in X$. As λ is c_0-invariant, we can find $\{\alpha_n\} \in c_0$ and $\{\beta_n\} \in \lambda$ such that $\|f_n\| = \alpha_n\beta_n$. We define $\{\gamma_n\} \in c_0$, where $\gamma_n = (\alpha_n)^{1/2}$. The operator $T \in N_\lambda(X, Y)$ can be factorized as $T = Q_1Q_2DP_2P_1$, where $P_1 \in \mathcal{L}(X, l_\infty)$, $P_2 \in \mathcal{L}(l_\infty, l_\infty)$, $D \in \mathcal{L}(l_\infty, \lambda)$, $Q_2 \in \mathcal{L}(\lambda, \lambda)$, $Q_1(\lambda, Y)$ are defined as $P_1(x) = \{(f_n/\|f_n\|)x\}$, $P_2(\{\eta_n\}) = \{\gamma_n\eta_n\}$, $D(\{\xi_n\}) = \{\gamma_n^{-2}\|f_n\|\xi_n\}$, $Q_2(\{\delta_n\}) = \{\gamma_n\delta_n\}$, $Q_1(\{\mu_n\}) = \sum_{n\geq 1} \mu_n y_n$, respectively. Since l_∞ and λ have approximation property, we know that the operators $P_2P_1 \in \overline{\mathfrak{F}}(X, l_\infty)$ and $Q_1Q_2 \in \overline{\mathfrak{F}}(\lambda, Y)$. As the operator $D \in (l_\infty, \lambda)$ is a diagonal operator, we have $D \in N_\lambda(l_\infty, \lambda)$. Therefore, $T \in \overline{\mathfrak{F}} \circ N_\lambda \circ \overline{\mathfrak{F}}(X, Y) = N^{\min}_\lambda(X, Y)$. Thus, $N_\lambda(X, Y) = N^{\min}_\lambda(X, Y)$. □

Proof of Proposition 3.8. As N_λ is a minimal operator ideal, applying [**12**, Proposition 8.7.15] we have

$$(N^{\max}_\lambda)^{\min} = \overline{\mathfrak{F}} \circ N^{\max}_\lambda \circ \overline{\mathfrak{F}} = N^{\min}_\lambda = N_\lambda.$$

Since K is injective as well as enjoys l_∞-extension property [**9**] (see also [**20**, Theorem 4.2]), using Lemma 2.1, we get

$$\begin{aligned} K^d_\lambda = N^{\text{inj}}_\lambda &\subseteq (\overline{\mathfrak{F}} \circ N^{\max}_\lambda \circ \overline{\mathfrak{F}})^{\text{inj}} \\ &\subseteq (K \circ N^{\max}_\lambda \circ K)^{\text{inj}} \\ &= K \circ (N^{\max}_\lambda)^{\text{inj}} \circ K \\ &= K \circ (N^{\text{inj}}_\lambda)^{\max} \circ K \\ &= K \circ (K^d_\lambda)^{\max} \circ K. \end{aligned}$$

For the reverse inclusion, we have

$$K^d_\lambda = N^{\text{inj}}_\lambda = (\overline{\mathfrak{F}} \circ N^{\max}_\lambda \circ \overline{\mathfrak{F}})^{\text{inj}} = K \circ (N^{\max}_\lambda \circ \overline{\mathfrak{F}})^{\text{inj}}.$$

Since $(N_\lambda^{\max})^{\text{inj}} \circ \overline{\mathfrak{F}} \subseteq (N_\lambda^{\max} \circ \overline{\mathfrak{F}})^{\text{inj}}$, it follows that

$$K \circ (K_\lambda^d)^{\max} \circ \overline{\mathfrak{F}} \subseteq K \circ (N_\lambda^{\max} \circ \overline{\mathfrak{F}})^{\text{inj}} = N_\lambda^{\text{inj}} = K_\lambda^d.$$

This completes the proof. □

As a consequence of this result, we prove the following factorization of K_λ^d.

PROPOSITION 3.10. *Let $(\lambda, \|.\|_\lambda)$ be as in Proposition 3.1. If λ is also perfect and the norm $\|.\|_\lambda$ of λ has norm iteration property, the following holds*

$$K_\lambda^d \subseteq \Pi_\lambda \circ K.$$

To prove it, we need the following two lemmas.

LEMMA 3.11. *Let $(\lambda, \|.\|_\lambda)$ be as in Proposition 3.1. Then, $T \in \Pi_\lambda(X, Y)$ if and only if for any $x_1, x_2, x_3, \ldots, x_n \in X$ and $g_1, g_2, g_3, \ldots., g_n \in Y^*$ there exists a constant $C > 0$ such that*

$$\begin{aligned}&\|\{g_1(Tx_1), g_2(Tx_2), \ldots, g_n(Tx_n), 0, 0, \ldots\}\|_\lambda \\ &\quad \leq C\|\{x_1, x_2, \ldots., x_n, 0, 0, \ldots\}\|_\lambda^w \sup_{1\leq i\leq n} \|g_i\|.\end{aligned}$$

Proof. Let $T \in \mathcal{L}(X, Y)$ satisfy the given criterion and $x_1, x_2, \ldots, x_n \in X$. WLOG we consider $Tx_i \neq 0$ for $i = 1, 2, \ldots, n$. By the Hann-Banach theorem, for each $1 \leq i \leq n$, we can find $g_i \in Y^*$ with $\|g_i\| = 1$ such that $\|Tx_i\| = g_i(Tx_i)$. For $i > n$, we consider $g_i = 0$. Thus, by given hypothesis, we have

$$\begin{aligned}&\|\{\|Tx_1\|, \|Tx_2\|, \ldots, \|Tx_n\|, 0, 0, \ldots\}\|_\lambda \\ &\quad = \|\{g_1(Tx_1), g_2(Tx_2), \ldots, g_n(Tx_n), 0, 0, \ldots\}\|_\lambda \\ &\quad \leq C\|\{x_1, x_2, \ldots., x_n, 0, 0, \ldots\}\|_\lambda^w \sup_{1\leq i\leq n} \|g_i\| \\ &\quad = C\|\{x_1, x_2, \ldots., x_n, 0, 0, \ldots\}\|_\lambda^w.\end{aligned}$$

This would imply $T \in \Pi_\lambda(X, Y)$.

For the converse, let $T \in \Pi_\lambda(X, Y)$, $x_1, x_2, \ldots, x_n \in X$ and $g_1, g_2, \ldots, g_n \in Y^*$. Then, $(g_i/\|g_i\|)Tx_i \leq \|Tx_i\|$ for each $i = 1, 2, \ldots, n$. Since λ is normal and $\|.\|_\lambda$ is monotone, we have

$$\begin{aligned}&\|\{g_1(Tx_1), g_2(Tx_2), \ldots, g_n(Tx_n), 0, 0, \ldots\}\|_\lambda \\ &\quad \leq \|\{\|g_1\|\|Tx_1\|, \|g_2\|\|Tx_2\|, \ldots, \|g_n\|\|Tx_n\|, 0, 0, \ldots\}\|_\lambda \\ &\quad \leq \sup_{1\leq i\leq n} \|g_i\|\|\{\|Tx_1\|, \|Tx_2\|, \ldots, \|Tx_n\|, 0, 0, \ldots\}\|_\lambda.\end{aligned}$$

As $T \in \Pi_\lambda(X, Y)$, from the above inequality, we can prove that T satisfies the required criterion. □

LEMMA 3.12. *Let $(\lambda, \|.\|_\lambda)$ be as in Proposition 3.1. Then, Π_λ is a maximal operator ideal.*

Proof. Though the proof is same as [**12**, 17.1.3], we prove the maximality of Π_λ for the sake of completeness. From the definition of a maximal operator ideal, it is clear that

https://doi.org/10.1017/S0017089516000604 Published online by Cambridge University Press

$\Pi_\lambda \subseteq \Pi_\lambda^{\max}$. For the reverse inclusion, let $T \in \Pi_\lambda^{\max}(X, Y)$. Let $x_1, x_2, \ldots, x_n \in X$ and $g_1, g_2, \ldots, g_n \in Y^*$. We define $X_0 = \operatorname{span}\{x_1, x_2, \ldots, x_n\}$. The operator $J_{X_0}^X : X_0 \to X$ is defined as $J_{X_0}^X(x) = x$ for $x \in X_0$. Defining the subset $Y_0 = \cap_{i=1}^n \ker g_i$ of Y, we consider the quotient space Y/Y_0. We have $\dim(Y/Y_0) < \infty$. We consider the quotient operator $Q_{Y_0}^Y : Y \to Y/Y_0$, which is a metric surjection. Hence, for each $i = 1, 2, \ldots, n$, we can find $G_i \in (Y/Y_0)^*$ such that $(Q_{Y_0}^Y)^*(G_i) = g_i$ and $\|(Q_{Y_0}^Y)^*(G_i)\| = \|g_i\| = \|G_i\|$, cf.[**12**, Proposition B.3.3, Proposition B.3.10.2]. Since $T \in \Pi_\lambda^{\max}(X, Y)$, applying [**12**, Theorem 8.7.5] and Lemma 3.11, we can find a constant $C > 0$ such that

$$\begin{aligned}
&\|\{g_1(Tx_1), g_2(Tx_2), \ldots, g_n(Tx_n), 0, 0, \ldots\}\|_\lambda \\
&= \|\{G_1(Q_{Y_0}^Y T J_{X_0}^X x_1), G_2(Q_{Y_0}^Y T J_{X_0}^X x_2), \ldots, G_n(Q_{Y_0}^Y T J_{X_0}^X x_n), 0, 0, \ldots\}\|_\lambda \\
&\le C\|\{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}\|_\lambda^w \sup_{1\le i\le n} \|G_i\| \\
&= C\|\{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}\|_\lambda^w \sup_{1\le i\le n} \|g_i\|.
\end{aligned}$$

This would imply $T \in \Pi_\lambda(X, Y)$ by Lemma 3.11. So $\Pi_\lambda^{\max}(X, Y) \subseteq \Pi_\lambda(X, Y)$. This completes the result. □

Proof of Proposition 3.10. Since $K_\lambda^d \subseteq \Pi_\lambda$ whenever $\|.\|_\lambda$ has norm-iteration property, cf.[**6**, Proposition 4.3] and Π_λ is a maximal operator ideal by Lemma 3.12, we have

$$\begin{aligned}
K_\lambda^d &\subseteq K \circ (K_\lambda^d)^{\max} \circ K \\
&\subseteq (K_\lambda^d)^{\max} \circ K \\
&\subseteq (\Pi_\lambda)^{\max} \circ K \\
&= \Pi_\lambda \circ K.
\end{aligned}$$

Hence, the proof. □

REMARK: At present, we do not know a sequence space other than l_p, for which $\Pi_\lambda \circ K \subseteq K_\lambda^d$ holds.

ACKNOWLEDGEMENTS. The authors are thankful to the referee for her/his valuable suggestions.

REFERENCES

1. K. Ain, R. Lillemets and E. Oja, Compact operators which are defined by l_p-spaces, *Quaest. Math.* **35** (2012), 145–159.

2. A. Defant and K. Floret, *Tensor norms and operator ideals* (North Holland Mathematics Studies 176: North-Holland, Amsterdam, 1993).

3. J. M. Delgado, C. Piñeiro and E. Serrano, Operators whose adjoints are quasi-p-nuclear, *Studia Math.* **197**(3) (2010), 291–304.

4. J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators* (Cambridge Stud. Adv. Math., vol 43, Cambridge Univ. Press, Cambridge, 1995).

5. A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, *Memo. Amer. Math. Soc.* **16** (1955).

6. M. Gupta and A. Bhar, On λ-compact operators, *Indian J. Pure Appl. Math.* **44**(3) (2013), 355–374.

7. P. K. Kamthan and M. Gupta, *Sequence Spaces and Series*, Lecture Notes in Pure and Applied Mathematics, vol. 65 (Marcel Dekker, Inc., New York, 1981).

8. A. K. Karn and D. P. Sinha, *Compactness and an approximation property related to an operator ideal*, arXiv:1207.1947.

9. J. Lindenstrauss, Extension of compact operators, *Mem. Amer. Math. Soc.* **48** (1964).

10. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces - sequence spaces* (Springer-Verlag, Berlin, 1977).

11. F. Ortel, Composition of operator ideals and their regular hulls, *Acta. Univ. Carolin. Math. Phys.* **36** (1995), 69–72.

12. A. Pietsch, *Operator Ideals* (North Holland Mathematical Library 20, North Holland Publishing Company: North Holland, Amsterdam, 1980).

13. A. Pietsch, The ideal of p-compact operators and its maximal hull, *Proc. Amer. Math. Soc.* **142** (2014), 519–530.

14. M. S. Ramanujan, Generalized nuclear maps in normed linear spaces, *J. Reine Angew Math.* **244** (1970), 190–197.

15. M. S. Ramanujan, Absolutely λ-summing operators, λ a symmetric sequence space, *Math Z.* **144** (1970), 187–193.

16. D. P. Sinha and A. K. Karn, Compact operators whose adjoints factor through subspaces of l_p, *Studia Math.* **150** (2002), 17–33.

17. D. P. Sinha and A. K. Karn, Compact operators which factor through subspaces of l_p, *Math. Nachr.* **281** (2008), 412–423.

18. I. Stephani, Generating systems of sets and quotients of surjective operator ideals, *Math. Nachr.* **99** (1980), 13–27.

19. G. R. Walker, Compactness of λ-nuclear operators, *Michigan Math. J.* **23** (1976), 167–172.

20. M. Zippin, Extension of bounded linear operators, in *Handbook of geometry of Banach spaces,* vol. 2 (Johnson, W. B. and Lindenstrauss, J., Editors) (Elsevier, North Holland, Amsterdam, 2003), 1703–1741.