Can. J. Math., Vol. XXXIX, No. 2, 1987, pp. 365-427

A CANCELLATION THEOREM FOR MODULES OVER THE GROUP C^*-ALGEBRAS OF CERTAIN NILPOTENT LIE GROUPS

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Introduction. In recent years, there has been a rapid growth of the K-theory of C^*-algebras. From a certain point of view, C^*-algebras can be treated as "non-commutative topological spaces", while finitely generated projective modules over them can be thought of as "non-commutative vector bundles". The K-theory of C^*-algebras [**30**] then generalizes the classical K-theory of topological spaces [**1**]. In particular, the K_0-group of a unital C^*-algebra A is the group "generated" by (or more precisely, the Grothendieck group of) the commutative semigroup of stable isomorphism classes of finitely generated projective modules over A with direct summation as the binary operation. The semigroup gives an order structure on $K_0(A)$ and is usually called the positive cone in $K_0(A)$.

Around 1980, the work of Pimsner and Voiculescu [**18**] and of A. Connes [**4**] provided effective ways to compute the K-groups of C^*-algebras. Then the classification of finitely generated projective modules over certain unital C^*-algebras up to stable isomorphism could be done by computing their K_0-groups as ordered groups. Later on, inspired by A. Connes's development of non-commutative differential geometry on finitely generated projective modules [**2**], the deeper question of classifying such modules up to isomorphism and hence the so-called cancellation question were raised (cf. [**21**]).

Although the cancellation question was studied and solved to some extent for commutative C^*-algebras by topologists in terms of vector bundles a long time ago [**11**], not much has been done for non-commutative C^*-algebras, except for Marc A. Rieffel's work [**21**], in which he completely solved the cancellation question for irrational rotation C^*-algebras (the "non-commutative tori") and classified all finitely generated projective modules over such algebras up to isomorphism.

In this paper, we are going to study the same question for the group C^*-algebras of nilpotent Lie groups. In the main result, a classical cancellation theorem for vector bundles over spheres is generalized to the case of finitely generated projective modules over certain "non-commutative spheres", namely, the unitized group C^*-algebras $C^*(G)^+$ of G in the

Received July 4, 1985 and in revised form February 19, 1986.

class Γ of nilpotent Lie groups of the form $\mathbf{R}^d \rtimes_\alpha \mathbf{R}$, in terms of the dimension of the fixed-point subspace of the coadjoint space $L(G)^*$. In particular, it follows that there are groups G of arbitrarily high dimension such that the cancellation law holds for projections over $C^*(G)^+$. This is in contrast to the case of abelian G: the cancellation law is known to fail for vector bundles over spheres of high dimensions. In addition, the finitely generated projective modules over $C^*(G)^+$ with G a simply connected nilpotent Lie group of dimension no greater than four are classified up to isomorphism.

We shall express $C^*(G)$ for G in Γ as an extension of $C^*(F)$ by $(C_0(\mathbf{R}^{d-1}) \oplus (C_0(\mathbf{R}^{d-1}))) \oplus \mathbf{K}$ for some F in Γ of lower dimension, where $d + 1$ is the dimension of G, and we shall compute the element corresponding to this extension in the relevant KK^1-group. This extends results of Fell, Kasparov, and Voiculescu for the Heisenberg Lie group.

Some new properties of connected stable rank and topological stable rank of C^*-algebras are derived, and the topological stable rank of $C^*(G)$ for G in Γ is computed; this generalizes a result of Rieffel for the Heisenberg Lie group. The connected stable rank of $C^*(G)$ for G in Γ is estimated.

Finally, two new stable ranks related to the cancellation problem are introduced, and for $C^*(G)$ with G in Γ, one of these is computed, and the other estimated.

Acknowledgement. I am grateful to my thesis advisor, Professor Marc A. Rieffel, for his patience, constant support and warm encouragement. I am also indebted to him for the suggestion of this topic, and for many helpful comments. I would also like to thank Professor Claude Schochet, Professor George A. Elliott, and the referee for reading the manuscript and making helpful comments.

1. Cancellation theorems for vector bundles over spheres. In this section, we shall give the topological background needed in this work, and an easy account of the cancellation property for vector bundles [**9**, **12**, **28**].

Before we go into the main topic of this section, we shall introduce some notations and clarify the meaning of "cancellation."

Let A be a C^*-algebra. We shall denote the $n \times n$ matrix algebra over A by $M_n(A)$; this is also a C^*-algebra, and in particular a normed space. For x in $M_n(A)$ and y in $M_m(A)$, we shall denote the element $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ of $M_{n+m}(A)$ by $x \oplus y$.

We shall denote the topological subspace of all self-adjoint idempotents in $M_n(A)$ by $\mathrm{Proj}_n(A)$, and regard $\mathrm{Proj}_n(A)$ as a subspace of $\mathrm{Proj}_{n+1}(A)$, by the obvious embedding that sends p in $\mathrm{Proj}_n(A)$ to $p \oplus 0$ in $\mathrm{Proj}_{n+1}(A)$. Let $\mathrm{Proj}_\infty(A)$ be the union of all $\mathrm{Proj}_n(A)$'s, or more precisely, let $\mathrm{Proj}_\infty(A)$ be the direct limit of the increasing sequence of topological space $\mathrm{Proj}_n(A)$. We shall call any element of $\mathrm{Proj}_\infty(A)$ a projection over A. From now on, it is understood that $\mathrm{Proj}_\infty(A)$ is filtered by $\mathrm{Proj}_n(A)$'s, and p in $\mathrm{Proj}_n(A)$ will always be identified with $p \oplus 0$ in $\mathrm{Proj}_{n+m}(A)$ for any n and m in $\mathbf{N}$.

Similarly, we shall denote the open subset of all invertible (or unitary) elements in $M_n(A)$ for any unital C^*-algebra A by $GL_n(A)$ (or $U_n(A)$), and view $GL_n(A)$ (or $U_n(A)$) as a subspace of $GL_{n+1}(A)$ (or $U_{n+1}(A)$) by the embedding that sends u in $GL_n(A)$ (or $U_n(A)$) to $u \oplus 1$ in $GL_{n+1}(A)$ (or $U_{n+1}(A)$). Let $GL_\infty(A)$ (or $U_\infty(A)$) be the direct limit of the increasing sequence of topological spaces $GL_n(A)$ (or $U_n(A)$). We shall call any element of $GL_\infty(A)$ (or $U_\infty(A)$) an invertible (or unitary) over A. From now on, it is understood that $GL_\infty(A)$ (or $U_\infty(A)$) is filtered by $GL_n(A)$'s (or $U_n(A)$'s) and u in $GL_n(A)$ (or $U_n(A)$) will always be identified with $u \oplus I_m$ in $GL_{n+m}(A)$ (or $U_{n+m}(A)$) for any n and m, where I_m is the identity matrix in $M_m(A)$.

When A is unital, we shall call two projections p and q in $\mathrm{Proj}_\infty(A)$ unitarily equivalent (or, simply, equivalent) over A, and write $p \simeq q$, if there exists u in $U_\infty(A)$ such that $upu^* = q$.

By definition, a finitely generated projective module over a unital C^*-algebra A is a direct summand of the free A-module A^n for some n. Let p be an element of $\mathrm{Proj}_n(A)$. Then we get

$$A^n = p(A^n) \oplus (1 - p)(A^n)$$

and $p(A^n)$ is a finitely generated projective module over A. It is well known that the map sending p in $\mathrm{Proj}_n(A)$ to the finitely generated projective module $p(A^n)$ over A induces a one-to-one correspondence between the unitary equivalence classes of projections over A and the isomorphism classes of finitely generated projective modules over A. Thus, in this work, we shall freely identify finitely generated projective modules over A with projections over A.

In doing K-theory for locally compact spaces, we shall usually consider the category of "pointed" topological spaces. Similarly, in defining the K-groups of a (not necessarily unital) C^*-algebra, we shall consider A^+, the algebra A with an identity adjoined. In other words, it is convenient to consider the category of "pointed" (or augmented) C^*-algebras, that is, pairs (A, Q) of a unital algebra A and an algebra homomorphism Q from A onto $\mathbf{C}$. (Note that $A = (\ker(Q))^+$ in this case.)

Let (A, Q) be a "pointed" C^*-algebra. Then for any projection p over A, say in $\mathrm{Proj}_n(A)$, we may define the dimension $\dim(p)$ of p with respect to Q to be the rank of the projection $Q(p)$ in $\mathrm{Proj}_n(\mathbf{C})$.

By Swan's theorem **[29]**, there is a one-to-one correspondence between the isomorphism classes of vector bundles over a compact space X and the isomorphism classes of finitely generated projective modules over $C(X)$. So we may identify a projection over $C(X)$ with a vector bundle over X.

Now a "pointed" commutative C^*-algebra $(C(X), Q)$ corresponds to a "pointed" space (X, x_0) in the sense that $Q(f) = f(x_0)$ for all f in $C(X)$. Under the above identifications, the dimension of a projection over $C(X)$ with respect to Q is equal to the dimension of the fibre of the corresponding vector bundle over X at x_0. When X is not connected, this example shows that the dimension of a projection over $C(X)$ depends upon the choice of Q (or, equivalently, of x_0).

In this work, we shall always view A^+ as the "pointed" C^*-algebra (A^+, Q_A) where Q_A is the canonical quotient map from A^+ to $\mathbf{C}$, sending the adjoined identity in A^+ to 1 in $\mathbf{C}$. The dimension of a projection over A^+ is understood to be with respect to Q_A.

Two projections p and q over a unital A are called stably equivalent over A, if the corresponding finitely generated projective modules are stably isomorphic, or, equivalently, if $p \oplus I_k$ and $q \oplus I_k$ are equivalent over A for some k, where I_k is the identity of $M_k(A)$.

1.1. *Definition.* Let A be any C^*-algebra. We shall say that *the cancellation law holds for projections of dimension* $\geqq n$ *over* A^+, if any two stably equivalent (over A^+) projections p and q of dimension $\geqq n$ over A^+ are equivalent (over A^+), that is, if $p \oplus I_k \simeq q \oplus I_k$ implies $p \simeq q$ for all projections p and q of dimension $\geqq n$ over A^+.

Now we recall a classical theorem (cf. Theorem 1.5 of chapter 8 of [**11**]) about the cancellation law for vector bundles, hence for finitely generated projective modules over commutative C^*-algebras by Swan's theorem [**29**]. Since the main result of this paper is based on a generalization of this theorem in the case of the d-sphere S^d, we shall present a proof (different from the one in [**11**]), both for the sake of completeness and for illustration of the idea used later in Sections 4, 5 and 6. By abuse of language, we shall denote the trivial n-dimensional complex vector bundle over a space simply by $\mathbf{C}^n$

1.2. THEOREM [**11**]. *If the compact space X is a CW complex of dimension d, and if E and F are complex vector bundles over X of (constant) dimension $\geqq d/2$ such that $E \oplus \mathbf{C}^n \simeq F \oplus \mathbf{C}^n$ for some n, then $E \simeq F$. That is, the cancellation law holds for projections of dimension $\geqq d/2$ over $C(X)$.*

Proof. We shall prove this by induction. Clearly the theorem holds for any 0-dimensional compact space X. Let us assume that it holds for X a CW complex of dimension $\leqq d - 1$.

For simplicity, we assume that X is gotten from its $(d - 1)$-skeleton X_{d-1} by attaching only one d-cell

$$D = \{x \text{ in } \mathbf{R}^d: |x| \leqq 1\},$$

say $X = X_{d-1} \cup_f D$. (The general case follows from the same argument.)

Let

$$D_+ = \{x \text{ in } \mathbf{R}^d\colon |x| \leqq 1/2\}, \quad D_- = \overline{D - D_+} \quad \text{and}$$

$$S = \partial D_+ = \partial D_- = S^{d-1}.$$

By the induction hypothesis,

$$E|_{X_{d-1}} \simeq F|_{X_{d-1}}.$$

Since X_{d-1} is a deformation retract of $Y = X_{d-1} \cup_f D_-$, we have $E|_Y \simeq F|_Y$ [**1**]. Clearly $E|_{D_+}$ and $F|_{D_+}$ are the trivial vector bundle $\mathbf{C}^k$ over D_+ since D_+ is contractible, where k is the dimension of E of F. So E and F are gotten by gluing $D_+ \times \mathbf{C}^k$ to $M = E|_Y = F|_Y$ along S through some clutching maps g and h from $S \times \mathbf{C}^k$ to $M|_S$. Since X is compact, we may endow M with a metric and assume that g and h are isometric. Then $g^{-1}h$ determines an element $[g^{-1}h]$ in $\pi_{d-1}(U_k(\mathbf{C}))$. Similarly, $E \oplus \mathbf{C}^n$ and $F \oplus \mathbf{C}^n$ are gotten from clutching maps $g \oplus i$ and $h \oplus i$ where i is the identity map on the trivial bundle $\mathbf{C}^n$ over S. It is routine to check that $E \oplus \mathbf{C}^n \simeq F \oplus \mathbf{C}^n$ implies that

$$[g^{-1}h] \oplus I_n = [(g \oplus i)^{-1}(h \oplus i)] = 0 \text{ in } \pi_{d-1}(U_{k+n}(\mathbf{C})).$$

It is well known that the map

$$\pi_p(U_q(\mathbf{C})) \to \pi_p(U_{q+1}(\mathbf{C}))$$

induced by inclusion of $U_q(\mathbf{C})$ into $U_{q+1}(\mathbf{C})$ is an isomorphism if $q \geqq (p + 1)/2$. Thus $[g^{-1}h] = 0$ in $\pi_{d-1}(U_k(\mathbf{C}))$ since $k \geqq d/2$. Again, it is routine to check that $[g^{-1}h] = 0$ implies that $E \simeq F$.

The above proof applied to the case $X = S^{d+1}$ shows that the cancellation law holds for all complex vector bundles over S^{d+1} if, and only if, the map

$$\pi_d(U_n(\mathbf{C})) \to \pi_d(U_{n+1}(\mathbf{C}))$$

is injective for all n in $\mathbf{N}$. So by the known data about $\pi_d(U_n(\mathbf{C}))$ for $d \leqq 3$ [**28**] and the facts that

$$\pi_{2m}(U_m(\mathbf{C})) = \mathbf{Z}/m!$$

(due to R. Bott),

$$\pi_{4m+1}(U_{2m}(\mathbf{C})) = \mathbf{Z}/2$$

[**32**], and

$$\lim_{\to n} \pi_d(U_n(\mathbf{C})) = \widetilde{K}^0(S^{d+1})$$

is always torsion free, we get the following corollary. Let $[t]$ denote the integral part of t in $\mathbf{R}$.

1.3 COROLLARY. (1) *The cancellation law holds for complex vector bundles of dimension* $\geqq [(d-1)/2]+1$ (*i.e., of dimension* $\geqq d/2$) *over* S^d.

(2) *The cancellation law holds for all complex vector bundles over* S^d *if* $d \leqq 4$, *but fails if* $d = 2m+3$ *or* $4m+2$ *for some* m *in* $\mathbf{N}$.

Remark. It is believed that the cancellation law always fails for vector bundles over S^d with $d \geqq 5$.

2. Group C^*-algebras of certain nilpotent Lie groups. In this work, we are mainly concerned with the nilpotent Lie groups in the class Γ, which is defined to be the collection of all nilpotent Lie groups of the form $\mathbf{R}^d \rtimes_\alpha \mathbf{R}$, that is, the semi-direct product of the abelian group $\mathbf{R}^d$ with $\mathbf{R}$ through an action α of $\mathbf{R}$ on $\mathbf{R}^d$. Since every simply connected nilpotent (or even solvable) Lie group can be gotten by taking a finite number of semi-direct products with $\mathbf{R}$, that is, can be written as

$$(\ldots((\mathbf{R}^d \rtimes_{\alpha_1} \mathbf{R}) \rtimes_{\alpha_2} \mathbf{R}) \rtimes_{\alpha_3} \ldots) \rtimes_{\alpha_n} \mathbf{R}$$

for some actions $\alpha_1, \alpha_2, \ldots, \alpha_n$, this work can be considered as a first attempt to study the cancellation property for more general nilpotent Lie groups.

It is well known that the group C^*-algebra [**17**] of a semi-direct product $N \rtimes_\alpha H$ is equal to the crossed product $C^*(N) \times_\alpha H$ where the H-action α on $C^*(N)$ is induced by the H-action α on N. Indeed, the group C^*-algebra $C^*(\mathbf{R}^d \rtimes_\alpha \mathbf{R})$ of the nilpotent Lie group $\mathbf{R}^d \rtimes_\alpha \mathbf{R}$ is equal to

$$C^*(\mathbf{R}^d) \times_\alpha \mathbf{R} = C_0(\mathbf{R}^d) \times_{\hat{\alpha}} \mathbf{R}$$

where $\hat{\alpha}$ is the induced action of $\mathbf{R}$ on $(\mathbf{R}^d)^* \simeq \mathbf{R}^d$, that is,

$$\langle \hat{\alpha}(t)(y_1, \ldots, y_d), (x_1, \ldots, x_d) \rangle$$
$$= \langle (y_1, \ldots, y_d), \alpha(t)(x_1, \ldots, x_d) \rangle$$

for all $(y_1, \ldots, y_d)$ in $(\mathbf{R}^d)^* \simeq \mathbf{R}^d$ and $(x_1, \ldots, x_d)$ in $\mathbf{R}^d$.

In order to understand the structure of $C^*(\mathbf{R}^d \rtimes_\alpha \mathbf{R})$ better, it is helpful to know certain general facts about the representation theory of nilpotent Lie groups. The main reference for this is [**14**]. (See also [**19**].) Kirillov classified the irreducible representations of any nilpotent Lie group G. Let us recall some facts from Kirillov's theory.

Any Lie group G has an adjoint action on its Lie algebra $L(G)$, denoted by

$$\mathrm{Ad}: G \to \mathrm{Aut}(L(G))$$

[**10**], which in turn induces a coadjoint action of G on the dual of $L(G)$, denoted by

$$\mathrm{coAd}: G \to \mathrm{Iso}(L(G)^*).$$

The orbits of the coadjoint action of G in $L(G)^*$ are called coadjoint orbits.

2.1. THEOREM [**14**]. *The equivalence classes of irreducible unitary representations of a given connected simply connected nilpotent Lie group G are in one-to-one correspondence with the coadjoint orbits of G.*

2.2. THEOREM [**14**]. *Given a connected simply connected nilpotent Lie group G, there are polynomial functions $p_0, p_1, \ldots, p_k$ on $L(G)^*$, invariant under the coadjoint action of G, such that the dense open subset*

$$S = \{x \text{ in } L(G)^*: p_0(x) \neq 0\}$$

of $L(G)^*$ is foliated by the coadjoint orbits of the form

$$\{x \text{ in } L(G)^*: p_i(x) = c_i \text{ for } i = l, \ldots, k\}$$

with c_i in $\mathbf{R}$.

2.3. *Definition* [**14**]. The coadjoint orbits contained in the set S defined in Theorem 2.2 are called *coadjoint orbits in general position.*

2.4. THEOREM [**14**]. *Every coadjoint orbit is even dimensional.*

2.5. THEOREM [**14**, **19**]. *Let G be a connected simply connected nilpotent Lie group. If S' is a coadjoint invariant subset of S in $L(G)^*$ with Euclidean measure zero, then the Plancherel measure on $\hat{G}$ is supported on* $(S - S')/\mathrm{coAd}$ *and hence the direct integral of the irreducible representations corresponding to the orbits in $S - S'$ is a faithful representation of $C^*(G)$.*

For nilpotent Lie groups G of the form $\mathbf{R}^d \rtimes_\alpha \mathbf{R}$, the orbit method of Kirillov can be simplified. Instead of working with the full coadjoint action of G on $L(G)^*$, we can work with just the action of the second factor $\mathbf{R}$ on the dual of the first factor $(\mathbf{R}^d)^* \simeq \mathbf{R}^d$, by Mackey's theory [**16**].

Before going further, we need to analyze the structure of the Lie algebra of $G = \mathbf{R}^d \rtimes_\alpha \mathbf{R}$. Recall that α is a group homomorphism from $\mathbf{R}$ to the group $GL_d(\mathbf{R})$ of linear isomorphisms on $\mathbf{R}^d$. Let

$$T = \frac{d}{dt}\alpha\bigg|_{t=0} \quad \text{in } M_d(\mathbf{R}).$$

Then

$$\alpha(t) = \operatorname{Exp}(tT) = \sum_{n=0}^{\infty} (t^n T^n)/(n!).$$

Since G is nilpotent, under a suitable change of basis for $\mathbf{R}^d$, we get T strictly upper triangular and $\alpha(t)$ upper triangular with entries in the diagonal equal to 1. More precisely, since $T^N = 0$ for some N, the Jordan canonical form of T is strictly upper triangular.

Now we assume that $G = \mathbf{R}^d \rtimes_\alpha \mathbf{R}$ is non-abelian, or equivalently α is non-trivial, that is, α is not the constant map sending each t to the identity element of $GL_d(\mathbf{R})$. In this case, the matrix T cannot be zero and hence has a non-zero entry above the main diagonal. So by a change of basis, we may assume $T_{ij} = 0$ for $i \geqq j$ and $T_{12} = 1$ with respect to the basis $\{e_1, \ldots, e_d\}$ of $\mathbf{R}^d$. Then since

$$\alpha(t) = \mathrm{Exp}(tT) = 1 + tT + (1/2!)t^2T^2 + \ldots,$$

we have $\alpha(t)_{ij}$ equal to 0 if $i > j$, to 1 if $i = j$, and to a polynomial $P_{ij}(t)$ if $i < j$; in particular we have $\alpha(t)_{12} = t$. In other words, we have $\alpha(t)e_1 = e_1$ and $\alpha(t)e_2 = te_1 + e_2$, while for $3 \leqq i \leqq d$, we have

$$\alpha(t)e_i = e_i + \sum_{j=1}^{i-1} p_{ji}(t)e_j.$$

Let $f_1, \ldots, f_d$ be the dual basis of $e_1, \ldots, e_d$ in $(\mathbf{R}^d)^* \simeq \mathbf{R}^d$. Then by the definition of the action $\hat{\alpha}$, $\hat{\alpha}(t)$ is the transpose $\alpha(t)^*$ of $\alpha(t)$, and so

$$\begin{aligned}&\hat{\alpha}(t)(y_1f_1 + \ldots + y_df_d)\\ &= y_1f_1 + (ty_1 + y_2)f_2 + \sum_{i=3}^{d} q_i(t, y_1, \ldots, y_d)f_i,\end{aligned}$$

where

$$q_i(t, y_1, \ldots, y_d) = y_i + \sum_{j=1}^{i-1} y_jp_{ji}(t) \quad \text{for } i = 3, 4, \ldots, d.$$

Let us state and prove a fact about crossed products of C^*-algebras with groups. Recall that a group homomorphism α from a locally compact group G to the automorphism group $\mathrm{Aut}(A)$ of a C^*-algebra A is called a strongly continuous G-action on A if for all x in A the map sending g in G to $(\alpha(g))(x)$ in A is continuous (in norm). For any given strongly continuous G-action α on A, we may construct the crossed product $A \times_\alpha G$.

2.6. THEOREM. *Let G be a locally compact group and α be a strongly continuous G-action on a C^*-algebra A. If J is an α-invariant closed ideal of A, then the following sequence is exact*

$$0 \to J \times_\alpha G \overset{\iota_*}{\to} A \times_\alpha G \overset{\pi_*}{\to} (A/J) \times_\alpha G \to 0,$$

where ι: $J \to A$ is the inclusion map and π: $A \to A/J$ is the quotient map.

Proof. (1) In order to prove exactness at $(A/J) \times_\alpha G$, we need only show that $\pi_*(A \times_\alpha G)$ is dense in $(A/J) \times_\alpha G$, since the image of a homomorphism of C^*-algebras is always closed [**17**]. Hence we only need to show that every element in $C_c(G, A/J)$, the (A/J)-valued compactly supported continuous functions on G, can be approximated by $\pi_*(C_c(G, A))$. Using a partition of unity, we can easily prove this fact.

(2) In order to show exactness at $J \times_\alpha G$, we need to prove that ι_* is injective. Recall that for each covariant representation (ϕ, ρ) of the dynamical system (J, G, α), we have a representation $\phi \times_\alpha \rho$ of $J \times_\alpha G$ such that

$$((\phi \times_\alpha \rho)(f))(\xi) = \int_G \phi(f(g))(\rho(g)(\xi))dg$$

for all f in $C_c(G, J)$ and ξ in the Hilbert space of the representation ϕ and ρ. And there is a covariant representation (ϕ', ρ') of the dynamical system (J, G, α) with ϕ' non-degenerate such that $\phi' \times_\alpha \rho'$ is a faithful representation of $J \times_\alpha G$ (cf. 7.6.4 of **[17]**).

Since ϕ' is non-degenerate, we may extend ϕ' to a representation ϕ'' of A by setting

$$\phi''(a) = \text{strong-lim}_\lambda\, \phi'(au_\lambda)$$

for a in A and $\{u_\lambda\}_\lambda$ an approximate identity for J (cf. 2.10.4 of **[6]**). We claim that (ϕ'', ρ') is a covariant representation of (A, G, α). Indeed

$$\rho(g)\phi''(a)\rho(g)^{-1} = \phi''(\alpha(g)(a)),$$

since

$$\text{strong-lim}_\lambda\, \phi'(\alpha(g)(u_\lambda)) = \text{id}$$

$(\{\alpha(g)(u_\lambda)\}_\lambda$ is also an approximate identity for J, and ϕ' is non-degenerate).

Clearly

$$(\phi'' \times_\alpha \rho) \circ \iota_* = \phi' \times_\alpha \rho.$$

Hence ι_* must be injective because $\phi' \times_\alpha \rho$ is injective.

(3) In order to prove exactness at $A \times_\alpha G$, we need only show that $\mathrm{Ker}(\pi_*)$ is contained in $\mathrm{Im}(\iota_*)$, since

$$(\pi_* \circ \iota_*)(C_c(G, J)) = 0$$

and so $\pi_* \circ \iota_* = 0$. By (2) above, we may regard $J \times_\alpha G$ as an ideal of $A \times_\alpha G$ since $C_c(G, J)$ is clearly an ideal of $C_c(G, A)$. Let δ be a faithful representation of $A \times_\alpha G/J \times_\alpha G$, lifting to a representation δ' of $A \times_\alpha G$. Since every representation of $A \times_\alpha G$ comes from a covariant representation of (A, G, α), we may assume that $\delta' = \phi \times_\alpha \rho$ for some covariant representation (ϕ, ρ) of (A, G, α).

Since

$$0 = \delta'(J \times_\alpha G) = (\phi \times_\alpha \rho)(J \times_\alpha G),$$

we get $\phi(J) = 0$, and hence ϕ gives rise to a representation ψ of A/J such that $\psi \circ \pi = \phi$. It is easy to see that (ψ, ρ) is a covariant representation of $(A/J, G, \alpha)$, and that

$$(\psi \times_\alpha \rho) \circ \pi_* = \phi \times_\alpha \rho = \delta'$$

is a representation of $A \times_\alpha G$ with kernel equal to $J \times_\alpha G$. So $\mathrm{Ker}(\pi_*)$ is contained in $J \times_\alpha G = \mathrm{Im}(\iota_*)$.

Remark. (1) The above theorem was proved for the case $G = \mathbf{R}$ in [**4**]. (2) The idea of the above proof is buried somewhere in [**8**].

Now let us return to the nilpotent Lie group $G = \mathbf{R}^d \rtimes_\alpha \mathbf{R}$, with α non-trivial. We shall denote the linear span of $f_2, \dots, f_d$ in $(\mathbf{R}^d)^* \simeq \mathbf{R}^d$ by W and the complement $(\mathbf{R}^d)^* - W$ by V. Then both W and V are invariant under the action $\hat{\alpha}$. Therefore by Theorem 2.6, we get an exact sequence of C^*-algebras

$$\begin{aligned} 0 \to C_0(V) \times_{\hat{\alpha}} \mathbf{R} &\to C_0(\mathbf{R}^d) \times_{\hat{\alpha}} \mathbf{R} \\ &= C^*(G) \to C_0(W) \times_{\hat{\alpha}} \mathbf{R} \to 0. \end{aligned}$$

Since e_1 is in the center of G, the quotient group $G/\mathbf{R}e_1$ is isomorphic to $\mathbf{R}^{d-1} \rtimes_\beta \mathbf{R}$ where the $\mathbf{R}$-action β on

$$\mathbf{R}^{d-1} \simeq \mathbf{R}^d/\mathbf{R}e_1 \simeq \mathbf{R}e_2 + \mathbf{R}e_3 + \dots + \mathbf{R}e_d$$

is gotten from α in a natural way. In fact, the infinitesimal generator of the action β, namely

$$S = \frac{d}{dt}\beta\Big|_{t=0},$$

is gotten from T by deleting the first column and the first row, and similarly $\beta(t)$ is gotten from $\alpha(t)$ in the same way for all t in $\mathbf{R}$. Clearly we may identify the dual of

$$\mathbf{R}^d/\mathbf{R}e_1 \simeq \mathbf{R}e_2 + \mathbf{R}e_3 + \dots + \mathbf{R}e_d$$

with

$$W = \mathbf{R}f_2 + \mathbf{R}f_3 + \dots + \mathbf{R}f_d$$

in an obvious way by regarding $f_2, f_3, \dots, f_d$ as the dual basis of $e_2, e_3, \dots, e_d$. Since $\hat{\beta}(t)$ is the transpose of $\beta(t)$ and $\hat{\alpha}(t)$ is the transpose of $\alpha(t)$, we get $\hat{\beta}(t)$ from $\hat{\alpha}(t)$ by deleting the first column and row, hence $\hat{\beta}(t) = \hat{\alpha}(t)|_W$ under the above identification of the dual of $\mathbf{R}^d/\mathbf{R}e_1$ with W.

Let $F = \mathbf{R}^{d-1} \times_\beta \mathbf{R}$. (Note that F is a quotient of G, not a subgroup of G.) Then

$$C^*(F) = C_0(\mathbf{R}^{d-1}) \times_{\hat{\beta}} \mathbf{R} \simeq C_0(W) \times_{\hat{\alpha}} \mathbf{R}.$$

We remark that β may be trivial, in which case F is abelian and hence isomorphic to $\mathbf{R}^d$, so $C^*(F) \simeq C_0(\mathbf{R}^d)$. But in any case, F is in the class Γ.

Now we analyze the crossed product $C_0(V) \times_{\hat{\alpha}} \mathbf{R}$, where

$$V = \{ (y_1, \dots, y_d) \text{ in } \mathbf{R}^d | y_1 \neq 0\}$$

is the complement $\mathbf{R}^d - W$ of W in $\mathbf{R}^d \simeq (\mathbf{R}^d)^*$. Since

$$\hat{\alpha}(t)(y_1, \dots, y_d) = (y_1, ty_1 + y_2, q_3(t, y_1, \dots, y_d), \dots, q_d(t, y_1, \dots, y_d))$$

and, for $y_1 \neq 0$, we have

$$|ty_1 + y_2| \rightarrow \infty \text{ as } |t| \rightarrow \infty,$$

each $\hat{\alpha}$-orbit in V is an algebraic curve expanding to infinity and the isotropy subgroup of each $(y_1, \dots, y_d)$ in V is the trivial subgroup $\{0\}$.

We claim that the union X of

$$X_+ = \{ (y_1, y_2, \dots, y_d) \text{ in } \mathbf{R}^d | y_1 > 0 \text{ and } y_2 = 0\}$$

and

$$X_- = \{ (y_1, y_2, \dots, y_d) \text{ in } \mathbf{R}^d | y_1 < 0 \text{ and } y_2 = 0\}$$

is a transversal to the orbits in V. In fact, we have a map ϕ from V to X, sending $(y_1, \dots, y_d)$ to

$$(y_1, 0, q_3(-y_2/y_1, y_1, \dots, y_d), \dots, q_d(-y_2/y_1, y_1, \dots, y_d)),$$

which is a submersion of V onto X with the inverse image of each point in X being its $\hat{\alpha}$-orbit. Indeed, we have $\phi \circ \iota = \mathrm{id}_X$, where ι is the inclusion map from X into V, by a straight computation. It is also easy to check that y and $\phi(y)$ are in the same $\hat{\alpha}$-orbit

$$(\phi(y) = \hat{\alpha}(-y_2/y_1) \cdot y)$$

for all y in V, and that if $\phi(y) \neq \phi(y')$ for y and y' in V, then y and y' are not in the same $\hat{\alpha}$-orbit.

Thus X is the orbit space $V/\hat{\alpha}(\mathbf{R})$ and V is a foliated manifold with the $\hat{\alpha}$-orbits as leaves; moreover the foliation on V comes from submersion of V onto X. So either by Corollary 15 of **[8]** or by Connes's theory of C^*-algebras of foliations **[3]**, we get

$$C_0(V) \times_{\hat{\alpha}} \mathbf{R} \simeq C_0(X) \otimes \mathbf{K} \simeq C_0(X_+) \otimes \mathbf{K} \oplus C_0(X_-) \otimes \mathbf{K}$$

where $\mathbf{K}$ is the albegra of compact operators on a separable Hilbert space. Since X_- and X_- are homeomorphic to $\mathbf{R}^{d-1}$, we have

$$C_0(X_+) \otimes \mathbf{K} \simeq C_0(X_-) \otimes \mathbf{K} \simeq C_0(\mathbf{R}^{d-1}) \otimes \mathbf{K}.$$

Summarizing the above discussion, we get

2.7. THEOREM. *For any non-abelian* $G = \mathbf{R}^d \rtimes_\alpha \mathbf{R}$ *in* Γ (*that is,* α *non-trivial*), *there are a* (*possibly abelian*) *group* F *in* Γ, *which is the quotient group of* G *by a* 1-*dimensional central subgroup, and an exact sequence of* C^*-*algebras*

$$0 \to D = D_+ \oplus D_- \xrightarrow{\iota} C^*(G) \xrightarrow{\pi} C^*(F) \to 0,$$

where

$$D_+ \simeq D_- \simeq C_0(\mathbf{R}^{d-1}) \otimes \mathbf{K}$$

and π *is induced by the quotient map from* G *to* F. *Thus* $C^*(G)$ *is an extension of* $C^*(F)$ *by*

$$(C_0(\mathbf{R}^{d-1}) \oplus C_0(\mathbf{R}^{d-1})) \otimes \mathbf{K}.$$

Remark. (1) This theorem generalizes the well-known description of the group C^*-algebra of the (3-dimensional) Heisenberg Lie group as an extension of $C_0(\mathbf{R}^2)$ by $C_0(\mathbf{R} - \{0\}) \otimes \mathbf{K}$ [7, **15**].

(2) For general nilpotent Lie groups, it is still possible to describe $C^*(G)$ as a sequence of extensions by algebras of the form $C_0(X) \otimes \mathbf{K}$. But we shall not pursue this matter here.

The $\hat{\alpha}$-orbits in $(\mathbf{R}^d)^* \simeq \mathbf{R}^d$ are exactly the intersections of the coadjoint orbits in $L(G)^* \simeq \mathbf{R}^{d+1}$ with $\mathbf{R}^d$, since $\mathrm{coAd}(\mathbf{R}^d)$ acts on $(\mathbf{R}^d)^* \simeq \mathbf{R}^d$ trivially ($\mathbf{R}^d$ is abelian). Either by invoking Kirillov's description of orbits of the first kind and the second kind with respect to a codimension one Lie subalgebra, or by making computations directly, we can conclude that the coadjoint orbits are either the products of 1-dimensional $\hat{\alpha}$-orbits in $\mathbf{R}^d$ with $\mathbf{R}$, or single points. In particular, the open subset $V \times \mathbf{R}$ in $\mathbf{R}^{d+1} \simeq L(G)^*$ is composed of 2-dimensional coadjoint orbits of the form (1-dimensional $\hat{\alpha}$-orbit in V) $\times$ $\mathbf{R}$. This dense open subset of orbits is co-null for Euclidean measure. Hence, by Theorem 2.5, the group C^*-algebra $C^*(G)$ is contained in $C_b(X) \otimes \mathbf{K}$; that is, the elements of $C^*(G)$ can be realized as bounded continuous $\mathbf{K}$-valued functions on X. However, the boundary behaviour of such functions around

$$\{(0, 0, y_3, \dots, y_d) \mid y_3, \dots, y_d \text{ are in } \mathbf{R}\}$$

is still unclear. (Cf. Theorem 5.3.)

Summarizing the above, we get

2.8. PROPOSITION. *Let* G *and* F *be as in* 2.7. *Then the short exact sequence*

$$0 \to C_0(X) \otimes \mathbf{K} \to C^*(G) \to C^*(F) \to 0$$

is an essential extension. In other words, $C^*(G)$ *may be considered as a subalgebra of* $C_b(X) \otimes \mathbf{K}$, *containing* $C_0(X) \otimes \mathbf{K}$ *as an ideal.*

Now we introduce an invariant of Lie groups, which, for groups in Γ, is closely related to the cancellation property for modules over the group C^*-algebra and to various notions of stable rank of this C^*-algebra, as explained later in Section 3.

2.9. *Definition.* For any Lie group G, we define $r(G)$ to be $\dim((L(G)^*)^G)$, that is, the dimension of the fixed point subspace of $L(G)^*$ under the coadjoint action of G.

2.10. PROPOSITION. *Let G and F be as in* 2.7. *Then*

$$\dim(F) = \dim(G) - 1 \quad \textit{and} \quad r(F) = r(G).$$

Proof. Since F is the quotient of G by a 1-dimensional central subgroup, we have $\dim(F) = \dim(G) - 1$.

Because V contains only 1-dimensional $\hat{\alpha}$-orbits, we know that all the zero-dimensional $\hat{\alpha}$-orbits in $\mathbf{R}^d$ are contained in W and, from the proof of Theorem 2.7, they are exactly the zero-dimensional $\hat{\beta}$-orbits for $F = \mathbf{R}^{d-1} \times_\beta \mathbf{R}$. Moreover, from the proof of Proposition 2.8, we know that the set of zero-dimensional coadjoint orbits of G is the product of the set of zero-dimensional $\hat{\alpha}$-orbits with $\mathbf{R}$. So

$$\begin{aligned} r(G) &= \dim(\{y \text{ in } (\mathbf{R}^d)^* \simeq \mathbf{R}^d | \hat{\alpha}(t)(y) = y \text{ for all } t \text{ in } \mathbf{R}\} \times \mathbf{R}) \\ &= \dim(\{y \text{ in } W | \hat{\beta}(t)(y) = y \text{ for all } t \text{ in } \mathbf{R}\} \times \mathbf{R}) = r(F). \end{aligned}$$

Remark. (1) In a sense, Theorem 2.7 combined with Proposition 2.10 allows us to use induction on $\dim(G)$ for G in Γ, in order to prove various properties of $C^*(G)$.

(2) The group F in 2.7 could be abelian, and in that case,

$$r(G) = r(F) = \dim(F).$$

(3) The only G in Γ with $r(G) = 1$ is $\mathbf{R}$ [**26**].

2.11. PROPOSITION. *Given $n \geqq m \geqq 2$, we can always find G in Γ with $\dim(G) = n$ and $r(G) = m$.*

Proof. Taking any strictly upper triangular T in $M_{n-1}(\mathbf{C})$, we may define

$$\alpha(t) = \operatorname{Exp}(tT) \quad \text{for } t \text{ in } \mathbf{R}$$

and set

$$G = \mathbf{R}^{n-1} \rtimes_\alpha \mathbf{R},$$

which is in Γ. Then $\hat{\alpha}(t) = \operatorname{Exp}(tT^*)$ and

$$\begin{aligned} &\{y \text{ in } \mathbf{R}^{n-1} | \hat{\alpha}(t)y = y \text{ for all } t \text{ in } \mathbf{R}\} \\ &\qquad = \{y \text{ in } \mathbf{R}^{n-1} | T^*(y) = 0\} = \operatorname{Ker}(T^*). \end{aligned}$$

So

$$r(G) = \dim(\mathrm{Ker}(T^*) \times \mathbf{R}) = \dim(\mathrm{Ker}(T^*)) + 1.$$

Pick T in $M_{n-1}(\mathbf{C})$ such that

$$T(y_1, \ldots, y_{n-1}) = (y_m, y_{m+1}, \ldots, y_{n-1}, 0, \ldots, 0).$$

Then

$$\dim(\mathrm{Ker}(T^*)) = m - 1.$$

And the G gotten from this T has

$$r(G) = (m - 1) + 1 = m \quad \text{and} \quad \dim(G) = n.$$

Remark. Note that for each m greater than 1, we have infinitely many G in Γ with $r(G) = m$, and the dimension of G can be arbitrarily high. However, for each m in $\mathbf{N}$, there are only finitely many G in Γ with $\dim(G) \leqq m$, since each G in Γ, say

$$G = \mathbf{R}^d \rtimes_\alpha \mathbf{R},$$

is determined by the Jordan canonical form of $\left.\frac{d}{dt}\alpha\right|_{t=0}$ (up to a permutation of the blocks in the main diagonal) and the entries of Jordan canonical forms are either 0 or 1.

Before we close this section, we shall discuss a special kind of group automorphism on non-abelian G's in Γ, which will be needed later in Section 5 to prove the main theorem (Theorem 5.4) of this work and to compute the element of

$$KK^1(C^*(F), C_0(\mathbf{R}^{d-1}) \oplus C_0(\mathbf{R}^{d-1}))$$

corresponding to the extension gotten in Theorem 2.7. (For definitions and results of KK-groups and extensions, see [**13**, **23**].)

Let $G = \mathbf{R}^d \rtimes_\alpha \mathbf{R}$ be non-abelian in Γ. Since

$$\alpha(t)(-x) = -\alpha(t)(x) \quad \text{for all } x \text{ in } \mathbf{R}^d \text{ and } t \text{ in } \mathbf{R},$$

we get a well-defined group automorphism h_G of period two on G by sending, (x, t) in $\mathbf{R}^d \rtimes_\alpha \mathbf{R}$ to $(-x, t)$. Then h_G induces an automorphism $C^*(h_G)$ of $C^*(G)$, which sends f in $C_C(G)$ to $f \circ h_G$ in $C_c(G)$. Since

$$(f \circ h_G)(x, t) = f(-x, t) \quad \text{for all } (x, t) \text{ in } G,$$

we see that, under the isomorphism

$$C^*(G) \simeq C^*(\mathbf{R}^d) \times_\alpha \mathbf{R} \simeq C_0((\mathbf{R}^d)^*) \times_{\hat{\alpha}} \mathbf{R},$$

$C^*(h_G)$ can be identified with the automorphism on $C_0((\mathbf{R}^d)^*) \times_{\hat{\alpha}} \mathbf{R}$ gotten by sending y in $(\mathbf{R}^d)^* \simeq \mathbf{R}^d$ to $-y$ and sending t in $\mathbf{R}$ to t.

Since V and W in $(\mathbf{R}^d)^* \simeq \mathbf{R}^d$ are invariant under the map sending y to $-y$, the map $C^*(h_G)$ restricted to $D \simeq C_0(X) \otimes \mathbf{K}$ gives an automorphism of D. Moreover, since $(y, 0, y_3, \ldots, y_d)$ is sent to $(-y, 0, -y_3, \ldots, -y_d)$ under h_G, the orbit spaces X_+ and X_- are interchanged under h_G. So $C^*(h_G)$ sends D_+ to D_- and vice versa. From the fact that $h_G^2 = \mathrm{id}_G$, we even get

$$C^*(h_G)|_{D_-} = (C^*(h_G)|_{D_+})^{-1}.$$

So we may identify D_+ with D_- under $C^*(h_G)$ and then identify them with $C_0(\mathbf{R}^{d-1}) \otimes \mathbf{K}$. In this way, we may identify $C^*(h_G)|_D$ with the automorphism of

$$(C_0(\mathbf{R}^{d-1}) \otimes \mathbf{K}) \oplus (C_0(\mathbf{R}^{d-1}) \otimes \mathbf{K})$$

sending (f_+, f_-) to (f_-, f_+).

Summarizing, we get

2.12. PROPOSITION. *For non-abelian G in Γ, there is an automorphism h of $C^*(G)$ such that under suitable identification of D_+ and D_- with* $C_0(\mathbf{R}^{d-1}) \otimes \mathbf{K}$, *we have the following commutative diagram*:

$$\begin{tikzcd}
(C_0(\mathbf{R}^{d-1}) \otimes \mathbf{K}) \oplus (C_0(\mathbf{R}^{d-1}) \otimes \mathbf{K}) \simeq D \arrow[r] \arrow[d, "\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}"] & C^*(G) \arrow[d, "h"] \\
(C_0(\mathbf{R}^{d-1}) \otimes \mathbf{K}) \oplus (C_0(\mathbf{R}^{d-1}) \otimes \mathbf{K}) \simeq D \arrow[r] & C^*(G)
\end{tikzcd}$$

3. Topological and connected stable ranks of $C^*(G)$ for G in Γ. In this section, we shall recall some facts about topological stable rank and connected stable rank, invented by Marc A. Rieffel [**20**], we shall derive a few useful facts about stable ranks, and we shall compute the stable ranks of $C^*(G)$ for G in Γ. We shall also introduce and study two new stable ranks, which will be used in Section 5 to describe the cancellation property.

3.1. *Definition* [**20**]. Let A be a unital C^*-algebra. The set $Lg_n(A)$ of *left unimodular n-rows* is defined to be $\{ (a_1, \ldots, a_n)$ in $A^n | a_1, \ldots, a_n$ generate A as a left ideal in $A \}$. Similarly $Rg_n(A)$ is defined to be $\{ (a_1, \ldots, a_n)$ in $A^n | a_1, \ldots, a_n$ generate A as a right ideal in $A \}$.

So $(a_1, \ldots, a_n)$ is in $Lg_n(A)$ (or $Rg_n(A)$) if and only if there exist $b_1, \ldots, b_n$ in A such that $b_1a_1 + \ldots + b_na_n = 1$ (or $a_1b_1 + \ldots + a_nb_n = 1$).

3.2. *Definition* [**20**]. The *topological stable rank* $\mathrm{tsr}(A)$ of a unital C^*-algebra is the least integer n in $\mathbf{N}$ such that $Lg_n(A)$ in dense in A^n.

For unital A, we have $\mathrm{tsr}(A) = \mathrm{tsr}(A^+)$. Thus for a general C^*-algebra A, we may define the topological stable rank $\mathrm{tsr}(A)$ of A to be $\mathrm{tsr}(A^+)$.

3.3. *Definition* [**20**]. The *connected stable rank* $\mathrm{csr}(A)$ of a unital C^*-algebra A is the least integer n in $\mathbf{N}$ such that for all $k \geqq n$, we have $GL_k(A)^\circ$, the connected component of the identity in $GL_k(A)$, acting transitively on $Lg_k(A)$.

For unital A, we have $\mathrm{csr}(A) = \mathrm{csr}(A^+)$. Thus for a general C^*-algebra A, we may define the connected stable rank $\mathrm{csr}(A)$ of A to be $\mathrm{csr}(A^+)$.

3.4. THEOREM [**20**]. (1) *For a compact space X, we have*

$$\mathrm{tsr}(C(X)) = [\dim(X)/2] + 1.$$

(2) *For any C^*-algebra A, we have* $\mathrm{tsr}(A \otimes \mathbf{K})$ *equal to* 1 *if* $\mathrm{tsr}(A) = 1$ *and equal to* 2 *if* $\mathrm{tsr}(A) \geqq 2$.

3.5. THEOREM [**20**]. *For any C^*-algebra A, we have*

$$\mathrm{csr}(A) \leqq \mathrm{tsr}(A) + 1.$$

3.6. PROPOSITION [**20**]. *If A is a unital C^*-algebra, then* $\mathrm{csr}(A)$ *is equal to the least integer n in $\mathbf{N}$ such that $Lg_k(A)$ is connected for all $k \geqq n$.*

3.7. THEOREM [**20**]. *If J is a closed ideal of a C^*-algebra A, then*

$$\max\{\mathrm{tsr}(J), \mathrm{tsr}(A/J)\} \leqq \mathrm{tsr}(A)$$
$$\leqq \max\{\mathrm{tsr}(J), \mathrm{tsr}(A/J), \mathrm{csr}(A/J)\}.$$

3.8. THEOREM [**20**]. *If A is unital and* $\mathrm{csr}(A) \leqq n$, *then the canonical homomorphism from $GL_{n-1}(A)$ to $GL_n(A)/GL_n(A)^\circ$, sending x in $GL_{n-1}(A)$ to the coset $(x \oplus 1)GL_n(A)^\circ$, is surjective. In particular, the canonical homomorphism from $GL_{\mathrm{csr}(A)-1}(A)$ to $K_1(A)$ is surjective.*

As an example, we shall compute the stable ranks of $C_0(\mathbf{R}^d)$. Before doing that, we shall give a topological interpretation of $\mathrm{csr}(C(X))$ for a compact space X.

Note that

$$Lg_n(C(X)) = (\mathbf{C}^n - \{0\})^X$$
$$= \{f \mid f \text{ is a continuous map from } X \text{ to } \mathbf{C}^n - \{0\}\}.$$

Since S^{2n-1} is a deformation retract of $\mathbf{C}^n - \{0\}$, we get

$$[X, \mathbf{C}^n - \{0\}] = [X, S^{2n-1}],$$

the $(2n - 1)$-th cohomotopy group of X. By the definition of $\mathrm{csr}(C(X))$, we have $\mathrm{csr}(C(X)) = n$ if, and only if, for all $k \geqq n$ the topological space $Lg_k(C(X))$ is connected. But

$$\pi_0(Lg_k(C(X))) = \pi_0((\mathbf{C}^k - \{0\})^X) = [X, \mathbf{C}^k - \{0\}]$$
$$= [X, S^{2k-1}].$$

So $\operatorname{csr}(C(X)) = n$ if, and only if, $[X, S^{2k-1}]$ is trivial for all $k \geqq n$.

Example. (1) By Theorem 3.4, we have

$$\operatorname{tsr}(C(S^d)) = [d/2] + 1.$$

From the classical results of homotopy theory and the above interpretation of $\operatorname{csr}(C(X))$, we get $\operatorname{csr}(C(S^d))$ equal to 2 if $d = 1$, equal to 1 if $d = 2$, and equal to $[(d + 1)/2] + 1$ if $d \geqq 3$. In fact, we have

$$[S^1, S^{2k-1}] = \pi_1(S^{2k-1})$$

equal to $\mathbf{Z}$ if $k = 1$ and equal to 0 if $k \geqq 2$, while

$$[S^2, S^{2k-1}] = \pi_2(S^{2k-1}) = 0 \quad \text{for all } k \text{ in } \mathbf{N}.$$

So we get

$$\operatorname{csr}(C(S^1)) = 2 \quad \text{and} \quad \operatorname{csr}(C(S^2)) = 1.$$

It is well known that $\pi_n(S^{n-1})$ equals $\mathbf{Z}_2$ if $n \geqq 4$ and equals $\mathbf{Z}$ if $n = 3$, while $\pi_n(S^n) = \mathbf{Z}$ for all n in $\mathbf{N}$ (cf. Section 21 of [**28**]). So for $d \geqq 3$, we get

$$[S^d, S^{2k-1}] = \pi_d(S^{2k-1})$$

equal to 0 if $2k - 1 > d$ and not equal to 0 if $2k - 1$ is d or $d - 1$. Hence

$$\operatorname{csr}(C(S^d)) = [(d + 1)/2] + 1 \quad \text{for } d \geqq 3.$$

(2) From 3.4, we get $\operatorname{tsr}(C(\mathbf{T}^2)) = 2$ where $\mathbf{T}^2$ is the 2-torus. By Theorem 3.5, we have

$$\operatorname{csr}(C(\mathbf{T}^2)) \leqq 3.$$

But it can be shown that

$$[\mathbf{T}^2, S^{2k-1}] = 0 \quad \text{if } k \geqq 2 \text{ and } [\mathbf{T}^2, S^1] \neq 0,$$

by elementary arguments from homotopy theory. So

$$\operatorname{csr}(C(\mathbf{T}^2)) = 2.$$

(More generally, we have

$$\operatorname{csr}(C(\mathbf{T}^d)) = [(d + 1)/2] + 1$$

for any d-torus with d in $\mathbf{N}$.)

Remark. (1) Since $C_0(\mathbf{R})$ is a quotient of $C_0(\mathbf{R}^2)$ and

$$\operatorname{csr}(C_0(\mathbf{R})) = \operatorname{csr}(C(S^1)) = 2 > 1 = \operatorname{csr}(C(S^2)) = \operatorname{csr}(C_0(\mathbf{R}^2)),$$

https://doi.org/10.4153/CJM-1987-018-7 Published online by Cambridge University Press

we find that $\max\{\mathrm{csr}(J), \mathrm{csr}(A/J)\} \leqq \mathrm{csr}(A)$ does not hold in general for closed ideals J of A. However, we do have the following analogue, for connected stable rank, of half of Theorem 3.7.

(2) If X is a contractible compact space, then

$$\mathrm{csr}(C(X)) = 1,$$

while

$$\mathrm{tsr}(C(X)) = [\dim(X)/2] + 1.$$

This simple observation tells us that $\mathrm{csr}(A)$ can be much smaller than $\mathrm{tsr}(A)$ in general. For example, we have

$$\mathrm{tsr}(C(B^d)) = [d/2] + 1$$

while $\mathrm{csr}(C(B^d)) = 1$ for the closed unit ball B^d in $\mathbf{R}^d$.

3.9. THEOREM. *If J is a closed ideal of a C^*-algebra A, then*

$$\mathrm{csr}(A) \leqq \max\{\mathrm{csr}(J), \mathrm{csr}(A/J)\}.$$

Proof. Since $\mathrm{csr}(A) = \mathrm{csr}(A^+)$, we may assume that A is unital.

Let n be the maximum of $\mathrm{csr}(J)$ and $\mathrm{csr}(A/J)$. With elements of $Lg_k(A)$ considered as column vectors and elements of $GL_k(A)$ considered as k by k matrices over A, we have $GL_k(A)$ acting on $Lg_k(A)$ by left multiplication.

What we want to show is that for all $k \geqq n$ and any $(a_1, \dots, a_k)$ in $Lg_k(A)$, there is some W in $GL_k(A)^\circ$ such that

$$W \cdot (a_1, \dots, a_k) = (1, 0, \dots, 0).$$

Let π be the quotient map from A to A/J. Given $k \geqq n$ and $(a_1, \dots, a_k)$ in $Lg_k(A)$, since $n \geqq \mathrm{csr}(A/J)$, we can find T in $GL_k(A/J)^\circ$ such that

$$T \cdot (\pi(a_1), \dots, \pi(a_k)) = (1, 0, \dots, 0) \quad \text{in } (A/J)^k.$$

($(\pi(a_1), \dots, \pi(a_k))$ is in $Lg_k(A/J)$ since

$$\pi(b_1)\pi(a_1) + \dots + \pi(b_k)\pi(a_k) = 1$$

if $b_1a_1 + \dots + b_ka_k = 1$ for $b_1, \dots, b_k$ in A.)

Since T is in $GL_k(A/J)^\circ$, we can find S in $GL_k(A)^\circ$ such that $\pi(S) = T$ (cf. [**30**]). Thus

$$\pi(S \cdot (a_1, \dots, a_k) - (1, 0, \dots, 0)) = (0, \dots, 0);$$

that is,

$$S \cdot (a_1, \dots, a_k) = (1 + x_1, x_2, \dots, x_k)$$

for some x_i in J.

We claim that $(1 + x_1, x_2, \dots, x_k)$ is in $Lg_k(J^+)$. In fact, since

$$(1 + x_1, \dots, x_k) = S \cdot (a_1, \dots, a_k)$$

is in $Lg_k(A)$, there are $b_1, \dots, b_k$ in A such that

$$b_1(1 + x_1) + b_2x_2 + \dots + b_kx_k = 1.$$

So $b_1 = 1 - b_1x_1 - b_2x_2 - \dots - b_kx_k$ is in J^+. Let $\{u_\lambda\}$ be an approximate identity for J. Then $b_1(1 + x_1) + b_2(u_\lambda x_2) + \dots + b_k(u_\lambda x_k)$ approaches

$$b_1(1 + x_1) + b_2x_2 + \dots + b_kx_k = 1$$

as λ goes to infinity. Hence for λ large enough, we get $b_1(1 + x_1) + (b_2u_\lambda)x_2 + \dots + (b_ku_\lambda)x_k$ close to 1, hence invertible in J^+. Let b in J^+ be the inverse of $b_1(1 + x_1) + (b_2u_\lambda)x_2 + \dots + (b_ku_\lambda)x_k$. Then we get

$$(bb_1)(1 + x_1) + (bb_2u_\lambda)x_2 + \dots + (bb_ku_\lambda)x_k = 1.$$

Since $bb_1, bb_2u_\lambda, \dots, bb_ku_\lambda$ are all in J^+, we find that $(1 + x_1, x_2, \dots, x_k)$ is indeed in $Lg_k(J^+)$.

Now since $k \geqq n \geqq \operatorname{csr}(J^+)$, there is some S' in $GL_k(J^+)^\circ$ such that

$$S' \cdot (1 + x_1, x_2, \dots, x_k) = (1, 0, \dots, 0).$$

Let $W = S' \cdot S$, which is in $GL_k(A)^\circ$. Then

$$\begin{aligned} W \cdot (a_1, \dots, a_k) &= S' \cdot (S \cdot (a_1, \dots, a_k)) \\ &= S' \cdot (1 + x_1, x_2, \dots, x_k) = (1, 0, \dots, 0). \end{aligned}$$

Remark. The above proof works if A is a Banach algebra and J is a closed ideal of A with an approximate identity.

The following theorem is an analogue, for connected stable rank, of 3.4(2). Before we state and prove it, we need to recall some properties of $GL_k(A)^\circ$ and $Lg_k(A)$. For unital A, the subset $Lg_k(A)$ of A^k is open, and two elements of $Lg_k(A)$ are in the same connected component (that is, they are connected by a path in $Lg_k(A)$) if, and only if, there is some T in $GL_k(A)^\circ$ sending one to the other. The map sending $(a_1, \dots, a_k)$ to $(a_1, \dots, a_{j-1}, ba_i + a_j, a_{j+1}, \dots, a_k)$ with $i \neq j$ and b in A is an operator in $GL_k(A)^\circ$ (it is an elementary row operation). Also the map sending $(a_1, \dots, a_k)$ to $(a_1, \dots, a_{i-1}, ba_i, a_{i+1}, \dots, a_k)$, with b in $GL_1(A)^\circ$, is in $GL_k(A)^\circ$.

3.10. THEOREM. *For any C^*-algebra A, we have* $\operatorname{csr}(A \otimes \mathbf{K}) \leqq 2$.

Proof. By Theorems 3.4 and 3.5, we have

$$\operatorname{csr}(A \otimes \mathbf{K}) \leqq \operatorname{tsr}(A \otimes \mathbf{K}) + 1 \leqq 3.$$

So we only need to show that $GL_2((A \otimes \mathbf{K})^+)^\circ$ acts transitively on $Lg_2((A \otimes \mathbf{K})^+)$.

When A is unital, the proof is easier, but we shall just consider the general case.

Let $(\lambda + x, \mu + y)$ be in $Lg_2((A \otimes \mathbf{K})^+)$ with λ, μ in $\mathbf{C}$ and x, y in $A \otimes \mathbf{K}$. We need to show that $(\lambda + x, \mu + y)$ is connected to $(1, 0)$ in $Lg_2((A \otimes \mathbf{K})^+)$. It is clear that either $\lambda \neq 0$ or $\mu \neq 0$, since $A \otimes \mathbf{K}$ is a proper ideal of $(A \otimes \mathbf{K})^+$. Clearly we can find T in $GL_2(\mathbf{C})^\circ = GL_2(\mathbf{C})$, which is regarded as a subgroup of $GL_2((A \otimes \mathbf{K})^+)^\circ$, such that

$$T \cdot (\lambda, \mu) = (1, 0),$$

and hence such that

$$T \cdot (\lambda + x, \mu + y) = (1 + x', y')$$

for some x' and y' in $A \otimes \mathbf{K}$.

Thus we only need to show that every $(1 + x, y)$ in $Lg_2((A \otimes \mathbf{K})^+)$ is connected to $(1, 0)$. Consider $A \otimes \mathbf{K}$ as filtrated by $M_n(A)$'s in the canonical way, that is, identifying w in $M_n(A)$ with $w \oplus 0$ in $A \otimes \mathbf{K}$ where 0 is the zero matrix of infinite size. Since $Lg_2((A \otimes \mathbf{K})^+)$ is open in $((A \otimes \mathbf{K})^+)^2$, we can approximate x and y by a and b in $M_n(A)$ respectively, for some large n, so closely that $(1 + a, b)$ is connected to $(1 + x, y)$ in $Lg_2((A \otimes \mathbf{K})^+)$.

Let c, d be in $M_n(A)$ and s, t be in $\mathbf{C}$ such that

$$(s + c)(1 + a) + (t + d)b = 1.$$

Pick any v in $M_n(A)$. By an elementary row operation, it is easy to see that $(1 + a, b)$ is connected in $Lg_2((A \otimes \mathbf{K})^+)$ to

$$\left(1 + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ v(s + c) & 0 \end{pmatrix}\left(1 + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) + \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\right)$$

$$= \left(1 + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & 0 \\ v(s + c)(1 + a) & 0 \end{pmatrix}\right),$$

where 0 is the zero matrix in $M_n(A)$. Since

$$1 + \begin{pmatrix} 0 & 0 \\ v(t + d) & 0 \end{pmatrix}$$

is in $GL_1((A \otimes \mathbf{K})^+)^\circ$, we get, by the remark before this theorem, that

$$\left(1 + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \left(1 + \begin{pmatrix} 0 & 0 \\ v(t + d) & 0 \end{pmatrix}\right)\begin{pmatrix} b & 0 \\ v(s + c)(1 + a) & 0 \end{pmatrix}\right)$$

is connected to

$$\left(1 + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & 0 \\ v(s + c)(1 + a) & 0 \end{pmatrix}\right),$$

where the former is equal to

$$\left(1 + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & 0 \\ v(s + c)(1 + a) + v(t + d)b & 0 \end{pmatrix}\right)$$
$$= \left(1 + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & 0 \\ v & 0 \end{pmatrix}\right).$$

Now by another elementary row operation, we find that

$$\left(1 + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & 0 \\ v & 0 \end{pmatrix}\right)$$

is connected to

$$\left(\left(1 + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) + \begin{pmatrix} 0 & -a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ v & 0 \end{pmatrix}, \begin{pmatrix} b & 0 \\ v & 0 \end{pmatrix}\right)$$
$$= \left(1 + \begin{pmatrix} a - av & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & 0 \\ v & 0 \end{pmatrix}\right)$$

in $Lg_2((A \otimes \mathbf{K})^+)$.

Since $M_n(A)$ is a C^*-algebra, it has an approximate identity. So we may choose v in $M_n(A)$ such that $a - av$ is close to 0 and

$$1 + \begin{pmatrix} a - av & 0 \\ 0 & 0 \end{pmatrix}$$

sits in $GL_1((A \otimes \mathbf{K})^+)^\circ$.

Let w_t be a path in $GL_1((A \otimes \mathbf{K})^+)$ such that

$$w_0 = 1 + \begin{pmatrix} a - av & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad w_1 = 1.$$

Then clearly

$$\left(w_t, \begin{pmatrix} b & 0 \\ v & 0 \end{pmatrix}\right)$$

is in $Lg_2((A \otimes \mathbf{K})^+)$ for all t, and

$$\left(1 + \begin{pmatrix} a - av & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & 0 \\ v & 0 \end{pmatrix}\right)$$

is connected to

$$\left(1, \begin{pmatrix} b & 0 \\ v & 0 \end{pmatrix}\right)$$

in $Lg_2((A \otimes \mathbf{K})^+)$.

By an elementary row operation, it is clear that

$$\left(1, \begin{pmatrix} b & 0 \\ v & 0 \end{pmatrix}\right)$$

is connected to $(1, 0)$ in $Lg_2((A \otimes \mathbf{K})^+)$.

Summarizing the above discussion, we get that every $(1 + x, y)$ in $Lg_2((A \otimes \mathbf{K})^+)$ is connected to $(1, 0)$ in $Lg_2((A \otimes \mathbf{K})^+)$. So $Lg_2((A \otimes \mathbf{K})^+)$ is connected and hence

$$\mathrm{csr}((A \otimes \mathbf{K})^+) \leqq 2.$$

Remark. (1) Combining Theorems 3.8 and 3.10, we get a well-known fact, namely that every element of $K_1((A \otimes \mathbf{K})^+)$ $(= K_1(A))$ is represented by a unitary element in $(A \otimes \mathbf{K})^+$, since the canonical map from $U_1((A \otimes \mathbf{K})^+)$ to $K_1((A \otimes \mathbf{K})^+)$ is surjective (cf. **[5]**).

(2) Let n be a non-negative integer. Then $\mathrm{csr}(C_0(\mathbf{R}^n) \otimes \mathbf{K})$ equals 1 if n is even and equals 2 if n is odd. In particular, we have

$$\mathrm{csr}(\mathbf{K}) = 1 \quad \text{and} \quad \mathrm{csr}(C_0(\mathbf{R}) \otimes \mathbf{K}) = 2$$

although

$$\mathrm{tsr}(C_0(\mathbf{R})) = 1 = \mathrm{tsr}(C_0(\mathbf{R}) \otimes \mathbf{K})$$

by 3.4 (cf. **[26]**).

(3) If X is a contractible space (or a disjoint union of finitely many contractible compact spaces), then

$$\mathrm{csr}(C(X) \otimes \mathbf{K}) = 1 \quad \textbf{[26]}.$$

We are going to prove another theorem about connected stable rank, namely Theorem 3.17, but we need a few lemmas before we can prove it.

In what follows, we shall denote an element $(a_1, \ldots, a_n)$ of A^n by $\mathbf{a}$ and $b_1a_1 + \ldots + b_na_n$ by $(\mathbf{b}|\mathbf{a})$ for elements $\mathbf{b}$ and $\mathbf{a}$ of A^n. We shall also denote $|a_1| + |a_2| + \ldots + |a_n|$ by $|\mathbf{a}|$.

3.11. LEMMA. *Two elements* $\mathbf{a}$ *and* $\mathbf{a}'$ *are connected by a path* $\mathbf{a}(t)$ *in* $Lg_n(A)$ (*or* $Rg_n(A)$) *with* $\mathbf{a} = \mathbf{a}(0)$ *and* $\mathbf{a}' = a(1)$ *if, and only if, there is a path* T_t *in* $GL_n(A)^\circ$ *such that* $T_0 = I_n$ *and* $T_t \cdot \mathbf{a} = \mathbf{a}(t)$ (*or* $\mathbf{a} \cdot T_t = \mathbf{a}(t)$) *for all* t *in* $[0, 1]$.

Proof. Since $[0, 1]$ is compact, the lemma follows easily from the following assertion:

For all $\mathbf{a}$ in $Lg_n(A)$, there is a small neighborhood U of $\mathbf{a}$ in A^n and a continuous map T from U to $GL_n(A)^\circ$ such that

$$T(\mathbf{x}) \cdot \mathbf{a} = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } U \text{ and } T(\mathbf{a}) = I_n.$$

In fact, we may fix an element $\mathbf{b}$ in A^n such that $(\mathbf{b}|\mathbf{a}) = 1$ and define $S(\mathbf{x})$ to be the operator in $M_n(A)$ sending $\mathbf{c}$ in A^n to

$$(\mathbf{x} - \mathbf{a}) \cdot (\mathbf{b}|\mathbf{c}) = ((x_1 - a_1) \cdot (\mathbf{b}|\mathbf{c}), \ldots, (x_n - a_n) \cdot (\mathbf{b}|\mathbf{c})).$$

Then since

$$|(\mathbf{x} - \mathbf{a}) \cdot (\mathbf{b}|\mathbf{c})| \leqq |\mathbf{x} - \mathbf{a}| \cdot |(\mathbf{b}|\mathbf{c})| \leqq |\mathbf{x} - \mathbf{a}| \cdot |\mathbf{c}| \cdot M$$

where

$$M = \sup\{ |b_i|: i = 1, 2, \dots, n\},$$

we get

$$|S(\mathbf{x})| \leqq M \cdot |\mathbf{x} - \mathbf{a}|.$$

So for $\mathbf{x}$ in a suitably small neighborhood U of $\mathbf{a}$, the norm of $S(\mathbf{x})$ is small and hence the operator $T(\mathbf{x}) = S(\mathbf{x}) + I_n$ is in $GL_n(A)^\circ$. Notice that

$$T(\mathbf{x}) \cdot \mathbf{a} = S(\mathbf{x}) \cdot \mathbf{a} + \mathbf{a} = (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{b}|\mathbf{a}) + \mathbf{a}$$
$$= (\mathbf{x} - \mathbf{a}) + \mathbf{a} = \mathbf{x}.$$

This proves the assertion, and the lemma follows.

3.12. LEMMA. *Let J be a closed ideal of a unital A and let π be the quotient map from A to A/J. Assume that* $\mathrm{csr}(A) < \infty$ *and that* $\mathbf{a}(t)$ *is a path in* $Lg_n(A)$ *for some n. If*

$$(\mathbf{b}|\mathbf{a}(0)) = 1 \quad \textit{and} \quad (\mathbf{d}|\pi(\mathbf{a}(1))) = 1$$

for some $\mathbf{b}$ *in* A^n *and* $\mathbf{d}$ *in* $(A/J)^n$, *then there is a path* $\mathbf{b}(t)$ *in* A^n *such that*

$$(\mathbf{b}(t)|\mathbf{a}(t)) = 1$$

for all t in $[0, 1]$ *with* $\mathbf{b}(0) = \mathbf{b}$ *and* $\pi(\mathbf{b}(1)) = \mathbf{d}$.

Proof. Case (1): $n \geqq \mathrm{csr}(A)$. By Lemma 3.11, we have paths T_t and T_t' in $GL_n(A)^\circ$ such that

$$T_t \cdot \mathbf{a}(0) = \mathbf{a}(t) \quad \text{for all } t \text{ in } [0, 1],$$

with

$$T_0 = I_n = T_0' \quad \text{and} \quad T_1' \cdot \mathbf{a}(1) = (1, 0, \dots, 0).$$

So using T_t and T_t', we may construct a homotopy F from $[0, 1]^2$ into $GL_n(A)^\circ$ such that

$$F(0, t) = I_n \quad \text{and} \quad F(1, t) \cdot \mathbf{a}(t) = (1, 0, \dots, 0)$$

for all t in $[0, 1]$; that is, the map sending (s, t) in $[0, 1]^2$ to $F(s, t) \cdot \mathbf{a}(t)$ gives a deformation of the path $\mathbf{a}(t)$ to the point $(1, 0, \dots, 0)$ in $Lg_n(A)$.

Let

$$\mathbf{c} = \mathbf{b} \cdot F(1, 0)^{-1} \quad \text{and} \quad \mathbf{d}' = \mathbf{d} \cdot \pi(F(1, 1)^{-1}).$$

Then

$$c_1 = (\mathbf{c}|(1, 0, \dots, 0)) = (\mathbf{b}|\mathbf{a}(0)) = 1$$

and similarly $d_1' = 1$. By considering elementary matrices, we can easily see that, for any $(1, x_2, \ldots, x_n)$ in $Lg_n(A)$, there is a path W_t in $GL_n(A)^\circ$ such that $(1, x_2, \ldots, x_n) \cdot W_t$ is of the form $(1, *, *, \ldots, *)$ for all t in $[0, 1]$, with

$$(1, x_2, \ldots, x_n) \cdot W_1 = (1, 0, \ldots, 0) \quad \text{and} \quad W_0 = I_n.$$

Now pick $\mathbf{c}'$ in A^n such that

$$c'_1 = 1 \quad \text{and} \quad \pi(\mathbf{c}') = \mathbf{d}'.$$

Then there is a path S_t in $GL_n(A)^\circ$ such that

$$(\mathbf{c} \cdot S_t | (1, 0, \ldots, 0)) = 1$$

for all t in $[0, 1]$, with

$$S_0 = \text{id} \quad \text{and} \quad \mathbf{c} \cdot S_1 = \mathbf{c}'.$$

Now we set

$$V_t = F(1, 0)^{-1} \cdot S_t \cdot F(1, t).$$

Then

$$\pi(\mathbf{b} \cdot V_1) = \pi(\mathbf{c} \cdot S_1 \cdot F(1, 1)) = \pi(\mathbf{c}' \cdot F(1, 1))$$
$$= d' \cdot \pi(F(1, 1)) = \mathbf{d}.$$

Moreover,

$$(\mathbf{b} \cdot V_t | \mathbf{a}(t)) = (\mathbf{c} \cdot S_t \cdot F(1, t) | \mathbf{a}(t))$$
$$= (\mathbf{c} \cdot S_t | F(1, t) \cdot \mathbf{a}(t)) = (\mathbf{c} \cdot S_t | (1, 0, \ldots, 0)) = 1.$$

So the path $\mathbf{b}(t) = \mathbf{b} \cdot V_t$ is what we are looking for.

Case (2): $n < \text{csr}(A)$. Let $\text{csr}(A) = N$. Setting $a'(t)_i$ to be $a(t)_i$ if $1 \leqq i \leqq n$ and to be 0 if $n < i \leqq N$, we get $\mathbf{a}'(t)$ in $Lg_N(A)$ for all t in $[0, 1]$. Let b'_i be b_i if $1 \leqq i \leqq n$ and be 0 if $n < i \leqq N$. Then

$$(\mathbf{b}' | \mathbf{a}'(0)) = (\mathbf{b} | \mathbf{a}(0)) = 1.$$

Similarly let d'_i be d_i if $1 \leqq i \leqq n$ and be 0 if $n < i \leqq N$. Then

$$(\mathbf{d}' | \pi(\mathbf{a}'(1))) = (\mathbf{d} | \pi(\mathbf{a}(1))) = 1.$$

So we may apply case (1) to get a path $\mathbf{b}'(t)$ in $Lg_N(A)$ such that $\mathbf{b}'(0) = \mathbf{b}'$ while

$$\pi(\mathbf{b}'(1)) = \mathbf{d}' \quad \text{and} \quad (\mathbf{b}'(t) | \mathbf{a}'(t)) = 1$$

for all t in $[0, 1]$. Since $a'(t)_i = 0$ for $i > n$, we get

$$(\mathbf{b}(t) | \mathbf{a}(t)) = 1 \quad \text{if } b(t)_i = b'(t)_i \text{ for } 1 \leqq i \leqq n.$$

Clearly $\mathbf{b}(t)$ is the path we are looking for.

3.13. COROLLARY. (1) *Let A be a unital C^*-algebra with* $\mathrm{csr}(A) < \infty$. *If* $\mathbf{a}(t)$ *is a path in* $Lg_n(A)$ *for some n with*

$$(\mathbf{b}|\mathbf{a}(0)) = 1 = (\mathbf{c}|\mathbf{a}(1))$$

for some $\mathbf{b}$ *and* $\mathbf{c}$ *in* A^n, *then there is a path* $\mathbf{b}(t)$ *in* A^n *connecting* $\mathbf{b}$ *and* $\mathbf{c}$, *and*

$$(\mathbf{b}(t)\,|\mathbf{a}(t)) = 1 \text{ for all } t \text{ in } [0, 1].$$

(2) *Let J be a closed ideal of a unital C^*-algebra A with* $\mathrm{csr}(A) < \infty$ *and denote by π the quotient map from A to A/J. If*

$$(\mathbf{d}|\pi(\mathbf{a})) = 1$$

for some $\mathbf{d}$ *in* $(A/J)^n$ *and some* $\mathbf{a}$ *in* $Lg_n(A)$, *then there is* $\mathbf{c}$ *in* A^n *such that*

$$\pi(\mathbf{c}) = \mathbf{d} \quad \text{and} \quad (\mathbf{c}|\mathbf{a}) = 1.$$

Given two surjective unital homomorphisms $\phi{:}A \to C$ and $\psi{:}B \to C$ between unital C^*-algebras, we shall call the subalgebra

$$D = \{ (a, b) \text{ in } A \oplus B | \phi(a) = \psi(b) \}$$

of $A \oplus B$ the pull-back of (A, ϕ, B, ψ, C) and denote it by $A \oplus_C B$.

3.14. LEMMA. *Let D be the pull-back of (A, ϕ, B, ψ, C), and let $((a_1, b_1), \ldots, (a_n, b_n))$ be in D^n for some n in* $\mathbf{N}$. *If* $\mathrm{csr}(A) < \infty$, *then $((a_1, b_1), \ldots, (a_n, b_n))$ is in $Lg_n(D)$ if, and only if, $(a_1, \ldots, a_n)$ is in $Lg_n(A)$ and $(b_1, \ldots, b_n)$ is in $Lg_n(B)$.*

Proof. We shall denote $((a_1, b_1), \ldots, (a_n, b_n))$ by $(\mathbf{a}, \mathbf{b})$.

(1) If $(\mathbf{a}, \mathbf{b})$ is in $Lg_n(D)$, then for some $(\mathbf{a}', \mathbf{b}')$ in D^n, we have

$$(1, 1) = ((\mathbf{a}', \mathbf{b}')\,|\,(\mathbf{a}, \mathbf{b})) = ((\mathbf{a}'|\mathbf{a}), (\mathbf{b}'|\mathbf{b})),$$

that is, $(\mathbf{a}'|\mathbf{a}) = 1$ and $(\mathbf{b}'|\mathbf{b}) = 1$. Hence $\mathbf{a}$ is in $Lg_n(A)$ and $\mathbf{b}$ is in $Lg_n(B)$.

(2) If $\mathbf{a}$ is in $Lg_n(A)$ and $\mathbf{b}$ is in $Lg_n(B)$, then we have

$$(\mathbf{b}'|\mathbf{b}) = 1 \quad \text{for some } \mathbf{b}' \text{ in } B^n.$$

So by 3.13 and the fact that

$$\begin{aligned} &\psi(b_1') \cdot \phi(a_1) + \ldots + \psi(b_n') \cdot \phi(a_n) \\ &= \psi(b_1') \cdot \psi(b_1) + \ldots + \psi(b_n') \cdot \psi(b_n) \\ &= \psi((\mathbf{b}'|\mathbf{b})) = \psi(1) = 1, \end{aligned}$$

we have $(\mathbf{a}'|\mathbf{a}) = 1$ for some $\mathbf{a}'$ in A^n such that $\phi(\mathbf{a}') = \psi(\mathbf{b}')$. Thus $(\mathbf{a}', \mathbf{b}')$ is in D^n and

$$((\mathbf{a}', \mathbf{b}')\,|\,(\mathbf{a}, \mathbf{b})) = ((\mathbf{a}'|\mathbf{a}), (\mathbf{b}'|\mathbf{b})) = (1, 1).$$

Hence $(\mathbf{a}, \mathbf{b})$ is indeed in $Lg_n(D)$.

There is a more intrinsic description of pull-backs. Let p_1 and p_2 be the projections from D to A and B respectively, that is,

$$p_1((a, b)) = a \quad \text{and} \quad p_2((a, b)) = b \quad \text{for } (a, b) \text{ in } D.$$

Then $A \simeq D/J$ and $B \simeq D/K$ where $J = \mathrm{Ker}(p_1)$ and $K = \mathrm{Ker}(p_2)$. Moreover, $J \cdot K = 0$, because if (a, b) is in $J \cdot K$ then (a, b) is both in J and in K; hence

$$a = p_1((a, b)) = 0 \quad \text{and} \quad b = p_2((a, b)) = 0.$$

And it is easy to see that

$$C \simeq D/(J + K),$$

since

$$\mathrm{Ker}(\phi \circ p_1) = J + K$$

(if (a, b) is in $\mathrm{Ker}(\phi \circ p_1)$, then

$$\psi(b) = \phi(a) = \phi(p_1((a, b))) = 0,$$

whence $(a, 0)$ and $(0, b)$ are in D, and $(a, b) = (a, 0) + (0, b)$ is in $J + K$). Conversely, if we have two closed ideals J and K in a C^*-algebra D such that $J \cdot K = 0$, then D is the pull-back of

$$(D/J, p_1, D/K, p_2, D/(J + K)),$$

where p_1 and p_2 are the quotient maps from D/J and D/K to $D/(J + K)$ respectively. We would like to thank Marc A. Rieffel for pointing out this fact, which simplifies our original statement of the following proposition.

Analyzing the properties needed to prove the next theorem (namely, Theorem 3.17), we found the following technical proposition.

3.15. Proposition. *Let $\{J_\lambda\}_{\lambda \in \Lambda}$ be a net of closed ideals (ordered by inclusion) of a unital C^*-algebra A and J be the closure of the union of the J_λ's. If K_λ, λ in Λ, are closed ideals of A such that $J_\lambda \cdot K_\lambda = 0$ for all λ in Λ, then*

$$\mathrm{trs}(A) = \max\{\mathrm{tsr}(A/J), \mathrm{tsr}(A/K_\lambda) \,|\lambda \textit{ in } \Lambda\}.$$

Proof. By Theorem 3.7, we clearly have $\mathrm{tsr}(A) \geqq n$ where n is

$$\max\{\mathrm{tsr}(A/J), \mathrm{tsr}(A/K_\lambda) \,|\lambda \text{ in } \Lambda\}.$$

Clearly we may assume $n < \infty$. So by Theorem 3.5, we have

$$\mathrm{csr}(A/K_\lambda) \leqq \mathrm{tsr}(A/K_\lambda) + 1 < \infty \quad \text{for all } \lambda.$$

Let $\mathbf{a}$ be any element of A^n and let ϵ be any positive real number.

Since $\operatorname{tsr}(A/J) \leqq n$, we can find $\mathbf{a}'$ in A^n such that $\pi(\mathbf{a}')$ is in $Lg_n(A/J)$ and

$$|\pi(\mathbf{a}' - \mathbf{a})| < \epsilon/2,$$

where π is the quotient map from A to A/J. So there is some λ_o in Λ such that $\pi_{\lambda_o}(\mathbf{a}')$ is in $Lg_n(A/J_{\lambda_o})$ and

$$|\pi_{\lambda_o}(\mathbf{a}' - \mathbf{a})| < \epsilon/2,$$

since

$$\lim_\lambda |\pi_\lambda(x)| = |\pi(x)| \quad \text{for all } x \text{ in } A.$$

(Here π_λ is the quotient map from A to A/J_λ).

Now let B be the quotient A/K_{λ_o}, D be the quotient A/J_{λ_o}, and C be the quotient $A/(K_{\lambda_o} + J_{\lambda_o})$. Then by the remark before 3.15, we have A isomorphic to the pull-back of (B, ϕ, D, ψ, C), where ϕ and ψ are the quotient maps from B and D to C respectively, by identifying a in A with $(\beta(a), \delta(a))$ in the pull-back, where β and δ are the quotient maps from A to B and D respectively. Note also that

$$|a| = \max\{ |\beta(a)|, |\delta(a)| \} \quad \text{for all } a \text{ in } A.$$

Since

$$\psi(\delta(\mathbf{a})) = \phi(\beta(\mathbf{a})) \text{ and } |\psi(\delta(\mathbf{a}' - \mathbf{a}))| < \epsilon/2,$$

we can find $\mathbf{b}'$ in B^n such that

$$\phi(\mathbf{b}') = \psi(\delta(\mathbf{a}')) \quad \text{and} \quad |\mathbf{b}' - \beta(\mathbf{a})| < \epsilon/2.$$

Since

$$\operatorname{tsr}(B) = \operatorname{tsr}(A/K_{\lambda_o}) \leqq n$$

by assumption, we can find $\mathbf{b}$ in $Lg_n(B)$ such that

$$|\mathbf{b} - \mathbf{b}'| < \min\left\{\frac{\epsilon}{2}, \eta\right\},$$

where η is a positive real number such that if

$$|\mathbf{d} - \delta(\mathbf{a}')| < \eta \quad \text{with } \mathbf{d} \text{ in } D^n,$$

then $\mathbf{d}$ is in $Lg_n(D)$. (Such η exists because $Lg_n(D)$ is open in D^n and $\delta(\mathbf{a}')$ is in $Lg_n(D)$.) Then

$$|\phi(\mathbf{b}) - \psi(\delta(\mathbf{a}'))| = |\phi(\mathbf{b} - \mathbf{b}')| < \min\left\{\frac{\epsilon}{2}, \eta\right\}.$$

So we can find $\mathbf{d}$ in D^n such that

$$\psi(\mathbf{d}) = \phi(\mathbf{b}) \quad \text{and} \quad |\mathbf{d} - \delta(\mathbf{a}')| < \min\left\{\frac{\epsilon}{2}, \eta\right\} \leqq \eta,$$

whence $\mathbf{d}$ is in $Lg_n(D)$.

Thus (b_i, d_i) is in the pull-back A of (B, Φ, D, Ψ, C) for $i = 1, 2, \ldots, n$. By 3.14 and the assumption that

$$\mathrm{csr}(B) = \mathrm{csr}(A/K_{\lambda_\circ}) < \infty,$$

we find that $(\mathbf{b}, \mathbf{d})$ is in $Lg_n(A)$ since $\mathbf{b}$ and $\mathbf{d}$ are in $Lg_n(B)$ and $Lg_n(D)$, respectively. Moreoever, since

$$|\mathbf{b} - \beta(\mathbf{a})| \leqq |\mathbf{b} - \mathbf{b}'| + |\mathbf{b}' - \beta(\mathbf{a})| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \text{ and}$$

$$|\mathbf{d} - \delta(\mathbf{a})| \leqq |\mathbf{d} - \delta(\mathbf{a}')| + |\delta(\mathbf{a}' - \mathbf{a})| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

we get

$$|(\mathbf{b}, \mathbf{d}) - \mathbf{a}| < \epsilon \quad \text{with } (\mathbf{b}, \mathbf{d}) \text{ in } Lg_n(A).$$

So we get $Lg_n(A)$ dense in A^n. Hence $\mathrm{tsr}(A) \leqq n$. Thus $\mathrm{tsr}(A) = n$.

3.16. COROLLARY. *If A is the pull-back of (B, ϕ, D, ψ, C) for some B, ϕ, D, ψ and C, then we have*

$$\mathrm{tsr}(A) = \max\{\mathrm{tsr}(B), \mathrm{tsr}(D)\}.$$

In the following, we denote the space of bounded continuous functions on a topological space X by $C_b(X)$.

3.17. THEOREM. *Let X be a locally compact topological space. If A is a C^*-subalgebra of $C_b(X) \otimes \mathbf{K}$ containing $J = C_0(X) \otimes \mathbf{K}$, then*

$$\mathrm{tsr}(A) = \max\{\mathrm{tsr}(A/J), \mathrm{tsr}(J)\}.$$

(*Note that* $\mathrm{tsr}(J)$ *equals* 1 *if* $\dim(X^+) \leqq 1$ *and equals* 2 *if* $\dim(X^+) \geqq 2$, *by Theorem* 3.4.)

Proof. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an increasing net of relatively compact, open subsets (ordered by inclusion) of X such that X is the union of U_λ's. So

$$J = \varinjlim_\lambda (C_0(U_\lambda) \otimes \mathbf{K}).$$

Let $\bar{U}_\lambda$ be the closure of U_λ in X, and J_λ be $C_0(U_\lambda) \otimes \mathbf{K}$. Setting

$$K_\lambda = \{f \text{ in } A | f(x) = 0 \text{ for all } x \text{ in } \bar{U}_\lambda\},$$

we get $J_\lambda \cdot K_\lambda = 0$ and

$$A/K_\lambda \simeq C(\bar{U}_\lambda) \otimes \mathbf{K} \simeq J/C_0(X - \bar{U}_\lambda) \otimes \mathbf{K}.$$

Thus, applying Proposition 3.15, we get

$$\begin{aligned}\mathrm{tsr}(A) &= \max\{\mathrm{tsr}(A/J), \mathrm{tsr}(A/K_\lambda) | \lambda \text{ in } \Lambda\} \\ &\leqq \max\{\mathrm{tsr}(A/J), \mathrm{tsr}(J)\}\end{aligned}$$

since A/K_λ is a quotient of J. But by Theorem 3.7, we have

$$\text{tsr}(A) \geqq \max\{\text{tsr}(A/J), \text{tsr}(J)\},$$

so we get the conclusion

$$\text{tsr}(A) = \max\{\text{tsr}(A/J), \text{tsr}(J)\}.$$

Remark. This theorem gives a partial answer to a question raised by Marc. A. Rieffel (Question 4.15 in [**20**]), concerning when

$$\text{tsr}(A) = \max\{\text{tsr}(A/J), \text{tsr}(J)\},$$

where J is an ideal in A.

Now we are ready to compute $\text{tsr}(C^*(G))$ and estimate $\text{csr}(C^*(G))$ for G in Γ.

3.18 THEOREM. *If G is an element in Γ, then we have*

$$\text{csr}(C^*(G)) \leqq \max\{\text{csr}(C_0(\mathbf{R}^{r(G)})), 2\} \leqq [(r(G) + 1)/2] + 1,$$

and

$$\text{tsr}(C^*(G)) = \text{tsr}(C_0(\mathbf{R}^{r(G)})) = [r(G)/2] + 1.$$

Proof. By the remark (3) following Proposition 2.10, the only G in Γ with $r(G) = 1$ is $\mathbf{R}$. We may therefore assume that $r(G) \geqq 2$. Then

$$\text{tsr}(C_0(\mathbf{R}^{r(G)})) = [r(G)/2] + 1 \geqq 2.$$

Case 1. If G is abelian, say $G = \mathbf{R}^d$ ($d \geqq 2$), then

$$r(G) = d \quad \text{and} \quad C^*(G) = C_0(\mathbf{R}^d).$$

So

$$\text{csr}(C^*(G)) = \text{csr}(C_0(\mathbf{R}^d)) \leqq \max\{\text{csr}(C_0(\mathbf{R}^{r(G)})), 2\} \quad \text{and}$$

$$\text{tsr}(C^*(G)) = \text{tsr}(C_0(\mathbf{R}^d)) = \text{tsr}(C_0(\mathbf{R}^{r(G)})).$$

Case 2. If G is not abelian, say $G = \mathbf{R}^d \times_\alpha \mathbf{R}$ with α non-trivial, then by 2.7 and 2.10, we can find F (which may be abelian) in G with

$$r(F) = r(G) \geqq 2 \quad \text{and} \quad \dim(F) = \dim(G) - 1$$

such that there is an exact sequence of C^*-algebras

$$0 \to (C_0(\mathbf{R}^{d-1}) \oplus C_0(\mathbf{R}^{d-1})) \otimes \mathbf{K} \to C^*(G) \to C^*(F) \to 0.$$

So we get

$$\begin{aligned} \text{csr}(C^*(G)) &\leqq \max\{\text{csr}((C_0(\mathbf{R}^{d-1}) \oplus C_0(\mathbf{R}^{d-1})) \otimes \mathbf{K}), \\ &\qquad\qquad \text{csr}(C^*(F))\} \\ &\leqq \max\{2, \max\{2, \text{csr}(C_0(\mathbf{R}^{r(F)}))\}\} \\ &= \max\{2, \text{csr}(C_0(\mathbf{R}^{r(G)}))\}, \end{aligned}$$

by Theorem 3.9 and the induction hypothesis for F. Similarly,

$$\begin{aligned}\operatorname{tsr}(C^*(G)) &= \max\{\operatorname{tsr}(C^*(F)),\\ &\qquad \operatorname{tsr}((C_0(\mathbf{R}^{d-1}) \oplus C_0(\mathbf{R}^{d-1})) \otimes \mathbf{K})\}\\ &= \max\{\operatorname{tsr}(C_0(\mathbf{R}^{r(F)})),\\ &\qquad \operatorname{tsr}((C_0(\mathbf{R}^{d-1}) \oplus C_0(\mathbf{R}^{d-1})) \otimes \mathbf{K})\}\\ &= \max\{\operatorname{tsr}(C_0(\mathbf{R}^{r(G)})), 2\}\\ &= \operatorname{tsr}(C_0(\mathbf{R}^{r(G)})),\end{aligned}$$

by Theorem 3.17, the induction hypothesis for F, and the fact that

$$\operatorname{tsr}(C_0(\mathbf{R}^{r(G)})) \geqq 2$$

together with Proposition 2.8.

Now we introduce two new stable ranks.

3.19 *Definition*. Let A be a C^*-algebra (maybe unital). The *cancellation stable rank*, $\operatorname{cansr}(A^+)$, of A^+ is the least non-negative integer n such that the cancellation law holds for projections of dimension $\geqq n$ over A^+.

3.20. *Definition*. For a unital C^*-algebra A, the *surjective K_1 stable rank* $\mathrm{sK_1sr}(A)$ is the least non-negative integer n such that the canonical homomorphism from $U_n(A)$ to $K_1(A)$ is surjective. (By definition, $U_0(A) = \{1\}$.)

3.21. PROPOSITION. (1) $\operatorname{cansr}(C(S^d))$ *is the least non-negative integer n such that, for all $k \geqq n$, the canonical homomorphism from $\pi_{d-1}(U_k(\mathbf{C}))$ to $\pi_{d-1}(U_{k+1}(\mathbf{C}))$ is injective.*

(2) $\operatorname{cansr}(C(S^d)) \leqq \operatorname{tsr}(C(S^{d-1}))$.

(3) $\mathrm{sK_1sr}(C(S^d))$ *is the smallest non-negative integer n such that, for all $[d/2] + 1 > k \geqq n$, the canonical homomorphism from $\pi_d(U_k(\mathbf{C}))$ to $\pi_d(U_{[d/2]+1}(\mathbf{C}))$ is surjective.*

(4) $\mathrm{sK_1sr}(A^+) \leqq \operatorname{csr}(A^+) - 1$, *for any C^*-algebra A.*

Proof. (1) Since isomorphism classes of vector bundles of dimension k over S^d correspond to elements of $\pi_{d-1}(U_k(\mathbf{C}))$, and the stabilization "$\oplus I_m$" corresponds to the canonical homomorphism from $\pi_{d-1}(U(k))$ to $\pi_{d-1}(U(k + m))$, the property that stable isomorphism implies isomorphism for all vector bundles of dimension $\geqq n$ over S_d is equivalent to the property that the canonical homomorphism from $\pi_{d-1}(U(k))$ to $\pi_{d-1}(U(k + 1))$ is injective for all $k \geqq n$. So by Swan's theorem [**29**], we get the conclusion.

(2) Apply Proposition 1.2 and the fact that

$$\operatorname{tsr}(C(S^{d-1})) = [(d - 1)/2] + 1.$$

(3) Since $\pi_d(U_m(\mathbf{C}))$ stabilizes at $m = [d/2] + 1$ as we have seen in Section 1, and

$$\pi_d(U_k(\mathbf{C})) = U_k(C(S^d)),$$

we see that $\mathrm{sK_1sr}(C(S^d)) = n$ is equivalent to the surjectivity of the canonical map from $\pi_d(U_k(\mathbf{C}))$ to

$$\pi_d(U_{[d/2]+1}(\mathbf{C})) \simeq K_1(C(S^d))$$

for all $[d/2] + 1 \geqq k \geqq n$ by definition.

(4) Apply Theorem 3.8.

Remark. Since $U_0(A)$ is defined to be $\{1\}$, it is clear that the following three conditions are equivalent: (a) $\mathrm{sK_1sr}(A) = 0$, (b) $K_1(A) = 0$, (c) $U_\infty(A)$ is connected.

Using known data from homotopy theory, we can compute $\mathrm{sK_1sr}(C(S^d))$ and $\mathrm{cansr}(C(S^d))$ for d small.

3.22. *Example.* Using data in Chapter 11 of [**9**] (or Sections 23-25 of [**28**]), we get the following sequences:

$$\pi_1(U_1(\mathbf{C})) = \mathbf{Z} \xrightarrow{\simeq} \pi_1(U_2(\mathbf{C})) = \mathbf{Z} \xrightarrow{\simeq} \pi_1(U_3(\mathbf{C})) = \mathbf{Z} \xrightarrow{\simeq} \dots$$

$$\pi_2(U_1(\mathbf{C})) = 0 \xrightarrow{\simeq} \pi_2(U_2(\mathbf{C})) = 0 \xrightarrow{\simeq} \pi_2(U_3(\mathbf{C})) = 0 \xrightarrow{\simeq} \dots$$

$$\pi_3(U_1(\mathbf{C})) = 0 \rightarrow \pi_3(U_2(\mathbf{C})) = \mathbf{Z} \xrightarrow{\simeq} \pi_3(U_3(\mathbf{C})) = \mathbf{Z} \xrightarrow{\simeq} \dots$$

$$\pi_4(U_1(\mathbf{C})) = 0 \rightarrow \pi_4(U_2(\mathbf{C})) = \mathbf{Z}_2 \xrightarrow{0} \pi_4(U_3(\mathbf{C})) = 0 \xrightarrow{\simeq} \dots$$

$$\pi_5(U_1(\mathbf{C})) = 0 \rightarrow \pi_5(U_2(\mathbf{C})) = \mathbf{Z}_2 \xrightarrow{0} \pi_5(U_3(\mathbf{C})) = \mathbf{Z} \xrightarrow{\simeq} \dots$$

From the above sequences, we get, by definition,

$$\mathrm{cansr}(C(S^1)) = 0, \ \mathrm{sK_1sr}(C(S^1)) = 1.$$

$$\mathrm{cansr}(C(S^2)) = 0, \ \mathrm{sK_1sr}(C(S^2)) = 0.$$

$$\mathrm{cansr}(C(S^3)) = 0, \ \mathrm{sK_1sr}(C(S^3)) = 2.$$

$$\mathrm{cansr}(C(S^4)) = 0, \ \mathrm{sK_1sr}(C(S^4)) = 0.$$

$$\mathrm{cansr}(C(S^5)) = 3, \ \mathrm{sK_1sr}(C(S^5)) = 3.$$

Remark. Note that

$$\mathrm{sK_1sr}(C(S^{2d})) = 0 \quad \text{for all } d,$$

since $K_1(C(S^{2d})) = 0$.

3.23. LEMMA. *Let J be a closed ideal of A and denote by π the quotient map from A to A/J. If $K_1(\pi)$ from $K_1(A^+)$ to $K_1(A^+/J)$ is the zero map then*

$$\mathrm{sK_1sr}(A^+) \leqq \mathrm{sK_1sr}(J^+).$$

Proof. Let u be an element of $U_\infty(A^+)$. Then since $[\pi(u)] = 0$ in $K_1(A^+/J)$, we get that $\pi(u)$ is in the connected component $U_m(A^+/J)^\circ$ for some m, hence can be lifted to some element of $U_m(A^+)^\circ$, say u'. Thus u is connected to $u'^{-1}u$ in $U_\infty(A^+)$. Let $v = u'^{-1}u$. Then

$$\pi(v) = \pi(u')^{-1}\pi(u) = \pi(u)^{-1}\pi(u) = I_m,$$

so v is in $U_m(J^+)$.

Let $n = \mathrm{sK_1sr}(J^+)$. Then since the canonical map from $U_n(J^+)$ to $K_1(J^+)$ is surjective, we get v connected to some w in $U_n(J^+)$ by a path in $U_\infty(J^+)$. So u is connected to w by a path in $U_\infty(A^+)$.

Thus we have proved that for any u in $U_\infty(A^+)$, there is some w in $U_n(J^+)$ (hence in $U_n(A^+)$) such that u and w are connected by a path in $U_\infty(A^+)$. In other words, the canonical map from $U_n(A^+)$ to $K_1(A^+)$ is surjective. So $\mathrm{sK_1sr}(A^+) \leqq n = \mathrm{sK_1sr}(J^+)$.

3.24. THEOREM. *For any non-abelian G in Γ, we have*

$$\mathrm{sK_1sr}(C^*(G)^+) = \begin{cases} 0 \text{ if } \dim(G) \text{ is even,} \\ 1 \text{ if } \dim(G) \text{ is odd.} \end{cases}$$

Proof. Applying 2.7, we get an exact sequence

$$0 \to D \xrightarrow{\iota} C^*(G) \xrightarrow{\pi} C^*(F) \to 0$$

for some F in Γ.

By Connes's Thom isomorphism theorem, either

$$K_1(C^*(G)) = 0 \text{ or } K_1(C^*(F)) = 0.$$

Thus the map $K_1(\pi)$ from

$$K_1(C^*(G)^+) = K_1(C^*(G))$$

to

$$K_1(C^*(F)^+) = K_1(C^*(F))$$

is always zero. So we may apply Lemma 3.23 to get

$$\mathrm{sK_1sr}(C^*(G)^+) \leqq \mathrm{sK_1sr}(D^+).$$

By 3.21(4) and 3.10, we get

$$\mathrm{sK_1sr}(D^+) \leqq \mathrm{csr}(D^+) - 1 \leqq 2 - 1 = 1.$$

Hence we get

$$\mathrm{sK_1sr}(C^*(G)^+)) \leqq 1.$$

Clearly the canonical map from

$$U_0(C^*(G)^+) = \{1\}$$

to

$$K_1(C^*(G)^+) = \pi_0(U_\infty(C^*(G)^+))$$

is surjective if and only if $U_\infty(C^*(G)^+)$ is connected, that is,

$$K_1(C^*(G)^+) = 0.$$

But by Connes's Thom isomorphism theorem, we know that $K_1(C^*(G)^+)$ equals 0 if $\dim(G)$ is even, and equals $\mathbf{Z}$ if $\dim(G)$ is odd. The conclusion follows.

3.25. THEOREM. *Let X be a locally compact space with the property that, for any given compact subset K of X, there is an open and relatively compact subset Y of X such that K is contained in Y and $X - Y$ is a retract of X. If A is a C^*-subalgebra of $C_b(X) \otimes \mathbf{K}$, containing $C_0(X) \otimes \mathbf{K}$, and $K^1(X^+) = 0$ (where X^+ is the one-point compactification of X), then*

$$\mathrm{sK_1sr}(A^+) \leqq \mathrm{sK_1sr}(A^+/C_0(X) \otimes \mathbf{K}).$$

Proof. Let $J = C_0(X) \otimes \mathbf{K}$. Then

$$\begin{aligned} K_1(J^+) &= K_1(J) = K_1(C_0(X)) \\ &= K_1(C(X^+)) = K^1(X^+) = 0. \end{aligned}$$

If $\mathrm{sK_1sr}(A^+/J) = 0$ then $K_1(A^+/J) = 0$, and then by the exact sequence

$$0 = K_1(J) \to K_1(A^+) \to K_1(A^+/J) = 0,$$

we get

$$K_1(A) = K_1(A^+) = 0.$$

Hence $\mathrm{sK_1sr}(A^+) = 0$.

Thus we may assume

$$\mathrm{sK_1sr}(A^+/J) \geqq 1;$$

in particular, X is not compact. (If X is compact, then

$$J = C_0(X) \otimes \mathbf{K} = A = C_b(X) \otimes \mathbf{K}$$

and hence

$$A^+/J = J^+/J = \mathbf{C}.$$

But $K_1(\mathbf{C}) = 0$, so

$$\mathrm{sK}_1\mathrm{sr}(A^+/J) = \mathrm{sK}_1\mathrm{sr}(\mathbf{C}) = 0.)$$

Let $n = \mathrm{sK}_1\mathrm{sr}(A^+/J) \geqq 1$, and denote by π the quotient map from A to A/J. Given any u in $U_\infty(A^+)$, we have a v in $U_n(A^+/J)$ such that $\pi(u)$ is connected to v by a path in $U_\infty(A^+/J)$, since

$$\mathrm{sK}_1\mathrm{sr}(A^+/J) = n.$$

Since $v\pi(u)^{-1}$ in $U_\infty(A^+/J)^\circ$ may be lifted to some element u' in $U_\infty(A^+)^\circ$, we have u connected to $u'u$ by a path in $U_\infty(A^+)$, and

$$\pi(u'u) = v\pi(u)^{-1}\pi(u) = v \quad \text{in } U_\infty(A^+/J).$$

With m such that u and u' are in $U_m(A^+)$ and $m > n$,

$$\pi(u'u) = v \oplus I_{m-n} \quad \text{in } U_m(A^+/J).$$

Let w be an element in $M_n(A^+)$ such that $\pi(w) = v$. Then

$$\pi(u'u - (w \oplus I_{m-n})) = 0 \text{ in } M_m(A^+/J),$$

and so $u'u - (w \oplus I_{m-n})$ is in $M_m(J)$.

Let $h = u'u - (w \oplus I_{m-n})$. Then

$$u'u = (w \oplus I_{m-n}) + h$$

with h in $M_m(J)$. Since $h(x)$ goes to 0 as x goes to infinity, we may find an open and relatively compact subset Y of X and a retraction r from X to $X - Y$ such that

$$|((u'u)|_{X-Y} - (w \oplus I_{m-n})|_{X-Y})| < 1/2,$$

and hence $w|_{X-Y}$ is in $GL_n(A^+/C_0(Y) \otimes \mathbf{K})$ since $(u'u)|_{X-Y}$ is a unitary over $A^+/C_0(Y) \otimes \mathbf{K}$.

Note that $Q_{\mathbf{K}} \circ w$ is a constant $M_n(\mathbf{C})$-valued function on X, so that

$$Q_{\mathbf{K}} \circ (w \circ r) = Q_{\mathbf{K}} \circ w$$

and $w \circ r - w$ is in $M_n(C_b(X) \otimes \mathbf{K})$. Moreover,

$$(w \circ r - w)(x) = 0 \quad \text{for all } x \text{ in } X - Y,$$

so $w \circ r - w$ is in $M_n(J)$, and hence $w \circ r$ is in $M_n(A^+)$. Similarly, $(w|_{X-Y})^{-1} \circ r$ is in $M_n(A^+)$ and

$$((w|_{X-Y})^{-1} \circ r)(w \circ r) = 1 = (w \circ r)((w|_{X-Y})^{-1} \circ r).$$

So $w \circ r$ is in $GL_n(A^+)$.

Let

$$w' = (w \circ r)((w \circ r)^*(w \circ r))^{-1/2}$$

be the unitarization of $w \circ r$. Then

$$\pi(w') = \pi(w \circ r)(\pi(w \circ r)^*\pi(w \circ r))^{-1/2}$$

$$= \pi(w)(\pi(w)^*\pi(w))^{-1/2} = v(v^*v)^{-1/2} = v,$$

since $w \circ r - w$ is in $M_n(J)$.

Thus

$$\pi((w' \oplus I_{m-n})^{-1}(u'u)) = (v \oplus I_{m-n})^{-1}(v \oplus I_{m-n}) = I_m,$$

so $(w' \oplus I_{m-n})^{-1}(u'u)$ is in $U_m(J^+)$. Since

$$\pi_0(U_\infty(J^+)) = K_1(J^+) = 0,$$

we get $(w' \oplus I_{m-n})^{-1}(u'u)$ connected to I_m by a path in $U_\infty(J^+)$; hence $u' \cdot u$ is connected to w' by a path in $U_\infty(A^+)$.

So u is connected to w' by a path in $U_\infty(A^+)$, and w' is in $U_n(A^+)$. Hence $[w'] = [u]$ in $K_1(A^+)$. Thus the canonical map from $U_n(A^+)$ to $K_1(A^+)$ is surjective, and hence

$$\mathrm{sK_1sr}(A^+) \leqq n = \mathrm{sK_1sr}(A^+/J).$$

4. The cancellation property for projections over $(C_0(\mathbf{R}^d) \otimes \mathbf{K})^+$. In this section, we shall prove that the cancellation law holds for all finitely generated projective modules over $(C_0(X) \otimes \mathbf{K})^+$ when X is a disjoint union of finitely many Euclidean spaces.

It is helpful at this point to recall basic definitions and facts about K-theory of topological spaces or C^*-algebras. The main references are [**1**, **12**, **30**].

Using the notation and terminology introduced in Section 1, we have the well-known fact that two projections over a unital C^*-algebra A are unitarily equivalent over A if and only if they are in the same path component of $\mathrm{Proj}_\infty(A)$ [**28**]. The K_0-group $K_0(A)$ of A is defined to be the Grothendieck group of the semi-group of (unitary) equivalence classes of projections over A, or equivalently, the Grothendieck group of the semi-group $\pi_0(\mathrm{Proj}_\infty(A))$ of connected components of $\mathrm{Proj}_\infty(A)$, where the binary operation is the direct sum "$\oplus$" of matrices over A as defined in Section 1.

Similarly the K_1-group $K_1(A)$ of A is defined to be the group $\pi_0(U_\infty(A))$ of connected components of $U_\infty(A)$, where the binary operation is again the direct sum of matrices over A (or, equivalently, the matrix product, since the path

$$\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix},$$

with t in $[0, \pi/2]$, connects $uv \oplus 1$ with $u \oplus v$).

For a general C^*-algebra A, the K_0-group $K_0(A)$ of A is defined to be the kernel of the homomorphism from $K_0(A^+)$ to $K_0(\mathbf{C}) \simeq \mathbf{Z}$ induced by the quotient map Q_A from A^+ to $\mathbf{C}$, while the K_1-group $K_1(A)$ of A is simply set to be $K_1(A^+)$.

If $A = C(X)$ for some compact space X, then by regarding elements of $\mathrm{Proj}_\infty(C(X))$ as $\mathrm{Proj}_\infty(\mathbf{C})$-valued continuous functions on X, we get $K_0(C(X))$ equal to the Grothendieck group of

$$[X, \mathrm{Proj}_\infty(\mathbf{C})] = \lim_{\to n} [X, \mathrm{Proj}_n(\mathbf{C})],$$

where $[Y, Z]$ means the set of homotopy classes of maps from Y to Z. By identifying a $\mathrm{Proj}_n(\mathbf{C})$-valued continuous function on X with a vector sub-bundle of the n-dimensional trivial bundle $X \times \mathbf{C}^n$ over X, we get

$$K_0(C(X)) = K^0(X)$$

where $K^0(X)$ is the classical K^0-group of compact spaces as introduced in [**1**]. Similarly, we have

$$K^1(X) = K_1(C(X)).$$

The following lemma is well known and a simple proof can be found in [**13**].

4.1. Lemma. *Given any unital C^*-algebra A, there is a continuous map τ_A from the set $\{(p, q) \mid p$ and q are in $\mathrm{Proj}_1(A)$ and $|p - q| < 1\}$ to $U_1(A)$ such that*

$$\tau_A(p, q)p(\tau_A(p, q))^{-1} = q$$

for all (p, q) in the domain of τ_A and $\tau_A(p, p) = 1$ for all p.

Remark. τ_A is natural with respect to A, that is, if ϕ is a unital homomorphism from A to B, then

$$\phi(\tau_A(p, q)) = \tau_B(\phi(p), \phi(q))$$

for (p, q) in the domain of τ_A.

4.2. Lemma. *Given a continuous map p from $\mathbf{R}^d$ to $\mathrm{Proj}_n(A^+)$ for some C^*-algebra A and a point x_0 in $\mathbf{R}^d$, we can find a continuous map u from $\mathbf{R}^d$ to $U_n(A^+)$ such that*

$$u(x)p(x)u(x)^{-1} = p(x_0)$$

for all x in $\mathbf{R}^d$ and $u(x_0) = I_n$. Moreover, if $Q_A \circ p$ is a constant function, then u can be chosen so that

$$(Q_A \circ u)(x) = I_n \quad \text{for all } x \text{ in } \mathbf{R}^d.$$

Proof. Without loss of generality, we may assume $x_0 = 0$. Let r denote a non-negative integer, and B_r the open ball of radius r in $\mathbf{R}^d$ centered at the origin x_0.

We shall show that any continuous $U_n(A^+)$-valued function on the closure $\bar{B}_r$ of B_r, satisfying the conditions

$$(***)\quad \begin{cases} u(0) = I_n \text{ and } u(x)p(x)u(x)^{-1} = p(0), \\ (Q_A \circ u)(x) = I_n \text{ if } Q_A \circ p \text{ is a constant function,} \end{cases}$$

for all x in $\bar{B}_r$, can be extended to a continuous $U_n(A^+)$-valued function on $\bar{B}_{r+1}$ satisfying $(***)$ for all x in $\bar{B}_{r+1}$. Then by induction on r, we get u defined on $\mathbf{R}^d$, satisfying $(***)$ for all x in $\mathbf{R}^d$.

Let u be a $U_n(A^+)$-valued continuous function on $\bar{B}_r$, satisfying $(***)$.

Since p is uniformly continuous on the compact ball $\bar{B}_{r+1}$, we can find m in $\mathbf{N}$ such that if x and y are in $\bar{B}_{r+1}$ with $|x - y| < 1/m$, then

$$|p(x) - p(y)| < 1.$$

We shall extend u step by step to $\bar{B}_{r+(i/m)}$ for all $i = 1, 2, \ldots, m$.

So let us assume that u has been extended to $\bar{B}_{r+((i-1)/m)}$ and satisfies $(***)$ for all x in $\bar{B}_{r+((i-1)/m)}$. For x in $\mathbf{R}^d$ with

$$r + (i - 1)/m \leqq |x| \leqq r + (i/m),$$

we define

$$u(x) = u((r + (i - 1)/m)x/|x|)\tau_{M_n(A^+)}(p(x), p((r + (i - 1)/m)x/|x|));$$

this is meaningful, since

$$|x - (r + (i - 1)/m)x/|x|| \leqq 1/m,$$

and hence

$$|p(x) - p((r + (i - 1)/m)x/|x|)| < 1,$$

so Lemma 4.1 can be applied. Note that u is well defined, since

$$\tau_{M_n(A^+)}(p(x), p(x)) = I_n$$

for x in $\mathbf{R}^d$ with $|x| = r + (i - 1)/m$.

It is routine to check that the extended u satisfies $(***)$ for x in $\bar{B}_{r+(i/m)}$ by the remark after Lemma 4.1.

In dealing with unitaries over A^+, it is sometimes convenient to consider only "normalized" unitaries, that is, the elements of

$$V_n(A^+) = \{u \text{ in } U_n(A^+) \,|\, Q_A(u) = I_n \text{ in } U_n(\mathbf{C})\}.$$

Clearly every unitary u in $U_n(A^+)$ can be "normalized", that is, multiplied by $Q_A(u)^{-1}$, since

$$Q_A(Q_A(u)^{-1}u) = I_n.$$

We shall denote $\lim_{\to n} V_n(A^+)$ by $V_\infty(A^+)$.

Since $U_n(\mathbf{C})$ is connected and the map sending u in $U_n(A^+)$ to $Q_A(u^{-1})u$ is a retraction, the imbedding of $V_n(A^+)$ into $U_n(A^+)$ induces an isomorphism from $\pi_0(V_n(A^+))$ to $\pi_0(U_n(A^+))$. Hence

$$\pi_0(V_\infty(A^+)) = \pi_0(U_\infty(A^+)).$$

Similarly, it is convenient to work with "standard" projections, that is, the elements of

$$S\mathrm{Proj}_n(A^+) = \{p \text{ in } \mathrm{Proj}_n(A^+) : Q_A(p) = I_{\dim(p)} \oplus O_{n-\dim(p)}\}.$$

(Here, we denote the $n \times n$ zero matrix by O_n.) We shall denote $\lim\limits_{\to n}$ $S\mathrm{Proj}_n(A^+)$ by $S\mathrm{Proj}_\infty(A^+)$.

4.3. PROPOSITION. *The imbedding of* $S\mathrm{Proj}_n(A^+)$ *into* $\mathrm{Proj}_n(A^+)$ *induces an isomorphism between* $\pi_0(S\mathrm{Proj}_n(A^+))$ *and* $\pi_0(\mathrm{Proj}_n(A^+))$. *Hence*

$$\pi_0(S\mathrm{Proj}_\infty(A^+)) = \pi_0(\mathrm{Proj}_\infty(A^+)).$$

Proof. (1) For any p in $\mathrm{Proj}_n(A^+)$, there is a u in $U_n(A^+)^\circ$ such that

$$Q_A(upu^{-1}) = I_k \oplus O_{n-k}$$

where $k = \dim(p)$.

In fact, since $Q_A(p)$ is a rank k projection in $\mathrm{Proj}_n(\mathbf{C})$, we can find u in $U_n(\mathbf{C}) = U_n(\mathbf{C})^\circ$ such that

$$uQ_A(p)u^{-1} = I_k \oplus O_{n-k}.$$

Regarding $U_n(\mathbf{C})^\circ$ as a subset of $U_n(A^+)^\circ$, we get u in $U_n(A^+)^\circ$ and

$$Q_A(upu^{-1}) = uQ_A(p)u^{-1} = I_k \oplus O_{n-k}$$

as required.

(2) We claim that if p_0 and p_1 are in $S\mathrm{Proj}_n(A^+)$ and are in the same connected component of $\mathrm{Proj}_n(A^+)$, then they are in the same connected component of $S\mathrm{Proj}_n(A^+)$.

Indeed, let p_0 and p_1 be connected by a path p_t in $\mathrm{Proj}_n(A^+)$. Then $Q_A(p_t)$ is a path in $\mathrm{Proj}_n(\mathbf{C})$, hence of constant rank, so $\dim(p_t) = \dim(p_0) = k$ for all t, for some k.

By Lemma 4.1, we can find a path u_t in $U_n(\mathbf{C})$ such that

$$u_tQ_A(p_t)u_t^{-1} = Q_A(p_0) = I_k \oplus O_{n-k}$$

for all t in $[0, 1]$. Thus

$$u_1(I_k \oplus O_{n-k})u_1^{-1} = u_1Q_A(p_1)u_1^{-1} = I_k \oplus O_{n-k},$$

so $u_1 = a \oplus b$ for some a in $U_k(\mathbf{C})$ and b in $U_{n-k}(\mathbf{C})$. Let a_t and b_t, with t in $[1, 2]$, be paths in $U_k(\mathbf{C})$ and $U_{n-k}(\mathbf{C})$ respectively, such that $a_1 = a$, $b_1 = b$, $a_2 = I_n$ and $b_2 = I_{n-k}$. *Then*

$$\begin{aligned}(a_t \oplus b_t)Q_A(p_1)(a_t \oplus b_t)^{-1} &= (a_t \oplus b_t)(I_k \oplus O_{n-k})(a_t^{-1} \oplus b_t^{-1})\\ &= I_k \oplus O_{n-k},\end{aligned}$$

and

https://doi.org/10.4153/CJM-1987-018-7 Published online by Cambridge University Press

$$(a_2 \oplus b_2)p_1(a_2 \oplus b_2)^{-1} = I_n p_1 I_n^{-1} = p_1.$$

Now set q_t to be $u_t p_t u_t^{-1}$ if t is in $[0, 1]$, and to be $(a_t \oplus b_t)p_1(a_t \oplus b_t)^{-1}$ if t is in $[1, 2]$. Note that

$$u_1 p_1 u_1^{-1} = (a \oplus b)p_1(a \oplus b)^{-1} = (a_1 \oplus b_1)p_1(a_1 \oplus b_1)^{-1}.$$

Then $p_0 = u_0 p_0 u_0^{-1}$ is connected to $p_1 = q_2$ by the path q_t in $S\mathrm{Proj}_n(A^+)$ with t in $[0, 2]$.

Thus, as long as we are only concerned with the connected components of $\mathrm{Proj}_\infty(A^+)$, i.e., with the equivalence classes of projections over A^+, we only need to investigate standard projections.

An important feature of the K-theory of C^*-algebras is stability, that is, $K_*(A) \simeq K_*(A \otimes \mathbf{K})$ for any C^*-algebra A and the isomorphism is induced by the canonical imbedding of A into $A \otimes \mathbf{K}$, sending x in A to $x \otimes e_{11}$ where $\{e_{ij}\}$ are matrix units of $\mathbf{K}$. Actually, for unital A, we have $K_0(A)$ isomorphic to the Grothendieck group of $\mathrm{Proj}_1(A \otimes \mathbf{K})$/unitary equivalence, and $K_1(A)$ isomorphic to $\pi_0(U_1(A \otimes \mathbf{K})^+)$. In [**5**], the K-groups are indeed defined in this way. Combining these ideas, we have the following Proposition 4.4.

Realizing elements of $A \otimes \mathbf{K}$ as matrices with infinitely many entries in A, we can embed $V_\infty((A \otimes \mathbf{K})^+)$ into $V_1((A \otimes \mathbf{K})^+)$. More precisely, let $\{e_{i,j}\}$ be a set of matrix units for $\mathbf{K}$, and m be a bijection from $\mathbf{N} \times \mathbf{N}$ to $\mathbf{N}$. Then for v in $V_n((A \otimes \mathbf{K})^+)$, say $v = I_n + w$ with w in $M_n(A \otimes \mathbf{K})$, we define

$$\eta(v) = I_1 + \sum (e_{h,h} w_{i,j} e_{k,k}) e_{m(i,h),m(j,k)}$$

in $V_1((A \otimes \mathbf{K})^+)$, where the summation is over all possible i, j, h, and k in $\mathbf{N}$ (with $1 \leqq i, j \leqq n$) and $w_{i,j}$ is the (i, j)-th entry of w as an $n \times n$ matrix over $A \otimes \mathbf{K}$. Then η is an embedding of $V_\infty((A \otimes \mathbf{K})^+)$ into $V_1((A \otimes \mathbf{K})^+)$.

4.4. PROPOSITION. *Let η_n be the embedding η restricted to $V_n((A \otimes \mathbf{K})^+)$. Then $\pi_0(\eta_n)$ is an isomorphism from $\pi_0(V_n((A \otimes \mathbf{K})^+))$ to $\pi_0(V_1((A \otimes \mathbf{K})^+))$. In particular, the inclusion of $V_n((A \otimes \mathbf{K})^+)$ into $V_\infty((A \otimes \mathbf{K})^+)$ induces an isomorphism from $\pi_0(V_n((A \otimes \mathbf{K})^+))$ to*

$$\pi_0(V_\infty((A \otimes \mathbf{K})^+)) = K_1((A \otimes \mathbf{K})^+),$$

and $\pi_0(\eta)$ is an isomorphism from

$$\pi_0(V_\infty((A \otimes \mathbf{K})^+)) \simeq K_1(A \otimes \mathbf{K})$$

to

$$\pi_0(V_1((A \otimes \mathbf{K})^+)) \simeq K_1(A).$$

A detailed proof can be found in [**26**].

https://doi.org/10.4153/CJM-1987-018-7 Published online by Cambridge University Press

Let A be a C^*-algebra. It is often useful to regard elements of $(C_0(X) \otimes A)^+$ as those A^+-valued continuous functions on X which are constant modulo A and converge to that constant at infinity. In this interpretation, we can describe the projections and unitaries over $(C_0(X) \otimes A)^+$ (or $(C_b(X) \otimes A)^+$) in the following way.

$\mathrm{Proj}_n((C_0(X) \otimes A)^+) = \{p|p$ is a $\mathrm{Proj}_n(A^+)$-valued continuous function on X, such that $Q_A \circ p$ is a constant $\mathrm{Proj}_n(\mathbf{C})$-valued function and $p - Q_A \circ p$ vanishes at infinity$\}$.

$U_n((C_0(X) \otimes A)^+) = \{u|u$ is a $U_n(A^+)$-valued continuous function on X, such that $Q_A \circ u$ is a constant $U_n(\mathbf{C})$-valued function and $u - Q_A \circ u$ vanishes at infinity$\}$.

In the rest of this section, we shall use X to denote a disjoint union of finitely many Euclidean spaces of positive dimensions, say $\mathbf{R}^{d(1)}, \ldots, \mathbf{R}^{d(m)}$ with $d(i)$ in $\mathbf{N}$. Then for any element x in X we shall denote the norm of x, inherited from the Euclidean space in which x sits, by $|x|$. So

$$S = \{x \text{ in } X:|x| = 1\}$$

is the disjoint union of the unit spheres in $\mathbf{R}^{d(1)}, \ldots, \mathbf{R}^{d(m)}$, and

$$B = \{x \text{ in } X:|x| < 1\}$$

is the disjoint union of the open unit balls in $\mathbf{R}^{d(1)}, \ldots, \mathbf{R}^{d(m)}$. Also the vector space structure is preserved on each Euclidean space and so we may do addition or scalar multiplication with elements of X. Finally we set

$$S' = S - \{(1, 0, \ldots, 0) \text{ in } \mathbf{R}^{d(i)}:i = 1, 2, \ldots, m\}$$

and for x in $\mathbf{R}^{d(i)}$ we define $e_1(x)$ to be the point $(1, 0, \ldots, 0)$ in $\mathbf{R}^{d(i)}$.

Now we proceed to study the cancellation property for $(C_0(X) \otimes \mathbf{K})^+$.

First, let us observe that 0 is the only projection of dimension zero over $(C_0(X) \otimes \mathbf{K})^+$. Indeed, let p be in $\mathrm{Proj}_n((C_0(X) \otimes A)^+)$ such that $\dim(p) = 0$. Then p is in $\mathrm{Proj}_n((C_0(X) \otimes A)$, that is, $p(x)$ is in $\mathrm{Proj}_n(A)$ for all x in X and $p(x)$ converges to 0 as x goes to infinity. But since every non-zero projection has norm one and each component of X is non-compact, we get $|p(x)| = 0$ for all x in X, that is, $p(x) = 0$ for all x in X. Thus $p = 0$ as desired. (Note that if X is replaced by a compact set Y, there may be a lot of non-zero projections of dimension zero over $(C(Y) \otimes A)^+$. For example, every projection in $\mathbf{K}$ is a projection of dimension zero over $\mathbf{K}^+$ and there are infinitely many non-zero projections in $\mathbf{K}$.)

In what follows, we shall only consider projections of dimension greater than zero over $(C_0(X) \otimes A)^+$.

For each p in $S\mathrm{Proj}_n((C_0(X) \otimes A)^+)$, we define q_p in

$$S\mathrm{Proj}_n((C_0(X) \otimes A)^+)$$

in the following way. Let $k = \dim(p)$. For all y in $\mathbf{R}^{d(i)}$ and for any $1 \leqq i \leqq m$, we set $q_p(y)$ equal to $p(x)$ if $y = x/(1 + |x|)$, and equal to $I_k \oplus O_{n-k}$ if $|y| \geqq 1$. Since $p(x)$ converges to $I_k \oplus O_{n-k}$ as x goes to the infinity, q_p is continuous. Since the map sending x to $1/(1 + |x|)$ is homotopic to the identity map of X, we see that p is connected to q_p by a path in

$$S\mathrm{Proj}_n((C_0(X) \otimes A)^+).$$

Moreover, if p_0 and p_1 are connected by a path p_t in

$$S\mathrm{Proj}_n((C_0(X) \otimes A)^+),$$

then q_{p_0} and q_{p_1} are connected by the path q_{p_t}. (Actually, all we need from q_p are the properties that q_p is connected to p in

$$S\mathrm{Proj}_n((C_0(X) \otimes A)^+)$$

and that $q_p - (I_k \oplus O_{n-k})$ vanishes outside a compact set in X.)

By Lemma 4.2, for each p of dimension k in

$$S\mathrm{Proj}_n((C_0(X) \otimes A)^+)$$

there is a u in $V_n((C_b(X) \otimes A)^+)$ such that

$$u(x)q_p(x)u(x)^{-1} = q_p(e_1(x)) = I_k \oplus O_{n-k} \quad \text{and}$$

$$u(e_1(x)) = I_n \quad \text{for all } x \text{ in } X.$$

Thus for x in X with $|x| \geqq 1$, we have

$$u(x)(I_k \oplus O_{n-k})u(x)^{-1} = I_k \oplus O_{n-k},$$

so that

$$u(x) = w(x) \oplus v(x) \text{ for some } w \text{ in } V_k((C_b(X - B) \otimes A)^+) \text{ and}$$

$$v \text{ in } V_{n-k}((C_b(X - B) \otimes A)^+).$$

Clearly

$$w(e_1(x)) = I_k \text{ for all } x \text{ in } X.$$

(Note that w and v are defined only outside the union of open unit balls in X.)

It can be shown that the class $[w|_{S'}]$ in

$$\pi_0(V_\infty((C_0(S') \otimes A)^+)) = K_1(C_0(S') \otimes A),$$

determined by the restriction of w to S', is independent of the choice of u, and that the map T from

$$\pi_0(\mathrm{Proj}_\infty((C_0(X) \otimes A)^+))$$

to $K_1(C_0(S') \otimes A)$, sending $[p]$ to $[w|_{S'}]$, is a homomorphism of semigroups, which sends I_r to 0 for all r in $\mathbf{N}$. Hence T induces a homomorphism from $K_0((C_0(X) \otimes A)^+)$ to $K_1(C_0(S') \otimes A)$. (See [**26**] for details.)

In the following theorem, we shall consider the special case that A is a stable C^*-algebra, that is, an algebra of the form $A \otimes \mathbf{K}$.

4.5. THEOREM. *Let X be a disjoint union of finitely many Euclidean spaces of positive dimensions. The cancellation law holds for all projections over $(C_0(X) \otimes (A \otimes \mathbf{K}))^+$.*

Proof. Let S' and B be as defined above, that is, the disjoint unions of all unit spheres with $(1, 0, \dots, 0)$ removed, and all open unit balls, respectively.

If p_0 and p_1 are in $S\mathrm{Proj}_\infty((C_0(X) \otimes A \otimes \mathbf{K})^+)$ and $I_r \oplus p_0$ is unitarily equivalent to $I_r \oplus p_1$ for some r in $\mathbf{N}$, or, equivalently,

$$[I_r \oplus p_0] = [I_r \oplus p_1] \quad \text{in } \pi_0(S\mathrm{Proj}_\infty((C_0(X) \otimes A \otimes \mathbf{K})^+)),$$

then, clearly,

$$T(p_0) = T(p_1) \quad \text{and} \quad \dim(p_0) = \dim(p_1).$$

By the fact that the only projection of dimension zero over $(C_0(X) \otimes A \otimes \mathbf{K})^+$ is 0 as we observed before, we may assume that

$$k = \dim(p_0) = \dim(p_1) > 0.$$

Let n be an integer in $\mathbf{N}$ such that p_0 and p_1 are contained in $S\mathrm{Proj}_n((C_0(X) \otimes A \otimes \mathbf{K})^+)$, and for $i = 0, 1$, let u_i be an element in $V_n((C_b(X) \otimes A \otimes \mathbf{K})^+)$ such that

$$u_i(x)q_{p_i}(x)u_i(x)^{-1} = I_k \oplus O_{n-k} \quad \text{and}$$

$$u_i(e_1(x)) = I_n \quad \text{for all } x \text{ in } X.$$

Then there are w_i in $V_k((C_b(X - B) \otimes A \otimes \mathbf{K})^+)$ and v_i in $V_{n-k}((C_b(X - B) \otimes A \otimes \mathbf{K})^+)$ such that

$$u_i(x) = w_i(x) \oplus v_i(x) \quad \text{for all } x \text{ in } X - B.$$

Clearly

$$[w_0|_{S'}] = T(p_0) = T(p_1) = [w_1|_{S'}]$$

in

$$\pi_0(V_\infty((C_0(S') \otimes A \otimes \mathbf{K})^+)) = K_1(C_0(S') \otimes A \otimes \mathbf{K}).$$

Now by Proposition 4.4, the inclusion of $V_k((C_0(S') \otimes A \otimes \mathbf{K})^+)$ into $V_\infty((C_0(S') \otimes A \otimes \mathbf{K})^+)$ induces an isomorphism from

$$\pi_0(V_k((C_0(S') \otimes A \otimes \mathbf{K})^+))$$

to

$$\pi_0(V_\infty((C_0(S') \otimes A \otimes \mathbf{K})^+)),$$

so we get

$$[w_0|_{S'}] = [w_1|_{S'}] \quad \text{in } \pi_0(V_k((C_0(S') \otimes A \otimes \mathbf{K})^+)).$$

So there is a path w_t in $V_k((C_0(S') \otimes A \otimes \mathbf{K})^+)$ connecting $w_0|_{S'}$ and $w_1|_{S'}$, and a path v_t in $V_{n-k}((C_0(S') \otimes A \otimes \mathbf{K})^+)$ connecting $v_0|_{S'}$ and $v_1|_{S'}$, where t varies from 0 to 1.

We know that

$$u_1^{-1}u_0q_{p_0}u_0^{-1}u_1 = u_1^{-1}(I_k \oplus O_{n-k})u_1 = q_{p_1}$$

in $M_n((C_b(X) \otimes A \otimes \mathbf{K})^+)$, but we do not know if $u_1^{-1}u_0$ is in $U_n((C_0(X) \otimes A \otimes \mathbf{K})^+)$ or not. So we modify it in the following way. Let

$$u(x) = \begin{cases} u_1(x)^{-1}u_0(x) & \text{if } x \text{ is in } B, \\ w_1(x/|x|)^{-1}w_{|x|-1}(x/|x|) \oplus v_1(x/|x|)^{-1}v_{|x|-1}(x/|x|) & \\ & \text{if } 1 \leqq |x| \leqq 2, \\ I_n & \text{if } |x| > 2. \end{cases}$$

Then u is in $V_n((C_0(X) \otimes A \otimes \mathbf{K})^+)$ since $u(x) - I_n = 0$ for all x in X with $|x| > 2$. For x in B, we have

$$\begin{aligned} u(x)q_{p_0}(x)u(x)^{-1} &= u_1(x)^{-1}u_0(x)q_{p_0}(x)u_0(x)^{-1}u_1(x) \\ &= q_{p_1}(x). \end{aligned}$$

And for x in $X - B$, we have

$$u(x) = w(x) \oplus v(x)$$

for some $w(x)$ in $U_k((A \otimes \mathbf{K})^+)$ and some $v(x)$ in $U_{n-k}((A \otimes \mathbf{K})^+)$; hence for x in $X - B$ we also have

$$\begin{aligned} u(x)q_{p_0}(x)u(x)^{-1} &= (w(x) \oplus v(x))(I_k \oplus O_{n-k})(w(x)^{-1} \oplus v(x)^{-1} \\ &= I_k \oplus O_{n-k} = q_{p_1}(x). \end{aligned}$$

Thus we get

$$uq_{p_0}u^{-1} = q_{p_1}$$

with u in $V_n((C_0(X) \otimes A \otimes \mathbf{K})^+)$. Hence q_{p_0} is unitarily equivalent over $(C_0(X) \otimes A \otimes \mathbf{K})^+$ to q_{p_1}.

Since q_p is unitarily equivalent to p for any standard projection p over $(C_0(X) \otimes A \otimes \mathbf{K})^+$, it follows that p_0 and p_1 are unitarily equivalent.

So we have proved that any two stably equivalent standard projections over $(C_0(X) \otimes A \otimes \mathbf{K})^+$ are unitarily equivalent. Therefore by Proposition 4.3, this property holds for all projections over $(C_0(X) \otimes A \otimes \mathbf{K})^+$, and the theorem is proved.

Remark. (1) More generally, we can prove that the cancellation law holds for projections of dimension greater than zero over $(A \otimes \mathbf{K})^+$ for any C^*-algebra A [**27**], by a different argument from that above (possibly an easier one), but the main idea of the above proof is essential to the discussion in the next few sections, concerning the cancellation problem for the C^*-algebras of nilpotent Lie groups.

(2) The above theorem shows that, although the cancellation law does not hold for finitely generated projective modules over $C(S^d) = C_0(\mathbf{R}^d)^+$ with $d \geqq 5$, it does hold for those over $(C_0(\mathbf{R}^d) \otimes \mathbf{K})^+$, the unitized stabilization of $C_0(\mathbf{R}^d)$, for any d.

(3) Note that in case $A = \mathbf{K}$, we may replace $w|_{S'}$ by $w|_S$ and replace $K_1(C_0(S') \otimes \mathbf{K})$ by $K_1(C(S) \otimes \mathbf{K})$ in the definition of T, since the homomorphism from $K_1(C_0(S') \otimes \mathbf{K})$ to

$$K_1(C(S) \otimes \mathbf{K}) \simeq K_1(C(S)) \simeq K_1(C_0(S')) \simeq K_1(C_0(S') \otimes \mathbf{K})$$

induced by the inclusion is an isomorphism. Thus, later on (especially in Sections 5 and 6), whenever we are in this situation, namely with $A = \mathbf{K}$, we shall take T to be with values in

$$K^1(S) \simeq K_1(C(S) \otimes \mathbf{K})$$

instead of with values in $K_1(C_0(S') \otimes \mathbf{K})$.

(4) The cancellation law holds for projections of dimension $\geqq \operatorname{tsr}(C(S^d))/n$ over $(C_0(\mathbf{R}^d) \otimes M_n(\mathbf{C}))^+$.

Now let us go back to the case of general A (not necessarily stable), and show that the map T is surjective. More specifically, let us construct a projection p_z in $S\mathrm{Proj}_\infty((C_0(X) \otimes A)^+)$ for each z in $K_1((C_0(S') \otimes A)^+)$, such that $T(p_z) = z$, and such that $\dim(p_z)$ is the lowest possible dimension of the projections p satisfying $T(p) = z$. When A is stable, we have

$$\dim(p_z) = 1 \quad \text{for all } z \text{ in } K_1((C_0(S') \otimes A)^+).$$

Let z in $K_1((C_0(S') \otimes A)^+)$ be realized by w in $V_k((C_0(S') \otimes A)^+)$ with k as small as possible; in other words, z can not be realized by any element in $V_{k-1}((C_0(S') \otimes A)^+)$. When A is stable, we have $k = 1$ by Proposition 4.4.

Let u_t be a path in $V_{2k}((C_0(S') \otimes A)^+)$ connecting $w \oplus w^{-1}$ with I_{2k}. We define

$$u(x) = \begin{cases} I_{2k} & \text{if } |x| = 0, \\ u_{|x|}(x/|x|) & \text{if } 0 < |x| < 1, \\ w(x/|x|) \oplus w(x/|x|)^{-1} & \text{if } |x| \geqq 1. \end{cases}$$

Then u is an element of $V_{2k}((C_b(X) \otimes A)^+)$ such that

$$u(e_1(x)) = I_{2k} \quad \text{for all } x \text{ in } X.$$

Let

$$p_z'(x) = u(x)^{-1}(I_k \oplus O_k)u(x) \quad \text{for all } x \text{ in } X.$$

Then

$$\begin{aligned} Q_A(p_z'(x)) &= Q_A(u(x)^{-1})(I_k \oplus O_k)(Q_A(u(x))) \\ &= I_{2k}(I_k \oplus O_k)I_{2k} = I_k \oplus O_k \end{aligned}$$

for all x in X, and $p_z'(x) = I_k \oplus O_k$ for all x in $X - B$.

Clearly we can find (a unique) p_z in $S\mathrm{Proj}_{2k}((C_0(X) \otimes A)^+)$ such that $q_{p_z} = p_z'$, simply by reversing the process that we used to define q; more precisely, we define

$$p_z(x) = p_z'(x/(1 + |x|)).$$

We have

$$\dim(p_z) = k \quad \text{and} \quad T(p_z) = [w] = z.$$

Thus we have proved the surjectivity of T.

4.6. PROPOSITION. *The map T' sending $[p] - [p']$ in $K_0((C_0(X) \otimes A)^+)$ to*

$$(T([p]) - T([p']), \dim(p) - \dim(p'))$$

in $K_1(C_0(S') \otimes A) \oplus \mathbf{Z}$ is an isomorphism.

Proof. The map sending $[p] - [p']$ in $K_0((C_0(X) \otimes A)^+)$ to $\dim(p) - \dim(p')$ in $\mathbf{Z} \simeq K_0(\mathbf{C})$ is exactly the homomorphism

$$K_0(Q_{C_0(X)\otimes A}).$$

So T' is a group homomorphism.

Let z be in $K_1(C_0(S') \otimes A)$ and r be in $\mathbf{Z}$. Then with $k = \dim(p_z)$, we have

$$\begin{aligned} &T'([p_z] + (r - k)[I_1]) \\ &= (T([p_z]) + (r - k)T([I_1]), \dim(p_z) + (r - k)\dim(I_1)) \\ &= (z + (r - k) \cdot 0, k + r - k) = (z, r). \end{aligned}$$

So T' is surjective.

If $T'([p_0] - [p_1]) = (0, 0)$, then

$$\dim(p_0) = \dim(p_1) \quad \text{and} \quad T(p_0) = T(p_1).$$

If $\dim(p_0) = 0$ then $p_0 = 0 = p_1$. So we may assume that

$$k = \dim(p_0) = \dim(p_1) > 0.$$

https://doi.org/10.4153/CJM-1987-018-7 Published online by Cambridge University Press

For $i = 0, 1$, let n be in $\mathbf{N}$ and u_i be in $V_n((C_b(X) \otimes A)^+)$ with

$$u_i(e_1(x)) = I_n \quad \text{for all } x,$$

so that p_i is in $S\mathrm{Proj}_n((C_0(X) \otimes A)^+)$ and

$$u_i q_{p_i} u_i^{-1} = I_k \oplus O_{n-k}.$$

There are w_i in $V_k((C_b(X - B) \otimes A)^+)$ and v_i in $V_{n-k}((C_b(X - B) \otimes A)^+)$ such that

$$u_i(x) = w_i(x) \oplus v_i(x) \quad \text{for all } x \text{ in } X - B.$$

Then

$$[w_0|_{S'}] = T(p_0) = T(p_1) = [w_1|_{S'}] \quad \text{in } K_1(C_0(S') \otimes A).$$

So $w_0|_{S'} \oplus I_N$ and $w_1|_{S'} \oplus I_N$ are connected by a path in $V_{k+N}((C_0(S') \otimes A)^+)$ for some large N. Thus by the same argument as used in the proof of 4.5, we can construct u in $V_{k+N}((C_0(X) \otimes A)^+)$ such that

$$u(q_{p_0} \oplus I_N)u^{-1} = q_{p_1} \oplus I_N.$$

Hence $q_{p_0} \oplus I_N$ and $q_{p_1} \oplus I_N$ are unitarily equivalent. So p_0 and p_1 are stably equivalent, since q_p is unitarily equivalent to p for any standard projection p. Thus $[p_0] = [p_1]$ and hence T' is injective.

By 4.5 and the existence of one-dimensional p_z over $(C_0(\mathbf{R}^d) \otimes \mathbf{K})^+$ for all z in $K_1(C(S^{d-1}) \otimes \mathbf{K})$, we get the following classification theorem.

4.7. THEOREM. *The semigroup of unitary equivalence classes of projections over* $(C_0(\mathbf{R}^d) \otimes \mathbf{K})^+$ *is*

$$\{[p_z \oplus I_{n-1}] : z \text{ in } K_1(C(S^{d-1}) \otimes \mathbf{K}), n \text{ in } \mathbf{N}\} \cup \{0\},$$

which is isomorphic to $(\mathbf{Z} \oplus \mathbf{N}) \cup \{0\}$ *if d is even and to* $\mathbf{N} \cup \{0\}$ *if d is odd.*

5. The cancellation property for projections over $C^*(G)^+$ with G in Γ. In this section, we shall prove an analogue of 1.3, which says that the cancellation law holds for projections of dimension $\geqq [(d - 1)/2] + 1$ over $C(S^d)$, for certain "non-commutative spheres", namely $C^*(G)^+$ with G in Γ.

5.1. LEMMA. *Let J be a closed ideal of A and denote by π the quotient map from A to A/J. If $K_0(\pi)$ from $K_0(A)$ to $K_0(A/J)$ is the zero homomorphism and if the cancellation law holds for projections of dimension $\geqq k$ over $(A/J)^+$, then every projection of dimension $\geqq k$ over A^+ is unitarily equivalent over A^+ to some projection in* $\mathrm{Proj}_\infty(J^+)$.

Proof. Let p be in $\mathrm{Proj}_n(A^+)$ of dimension m. Then $[p] - [I_m]$ is in

$$K_0(A) = \text{Ker}(K_0(Q_A))$$

where Q_A is the quotient map from A^+ to **C**. So

$$0 = K_0(\pi)([p] - [I_m]) = [\pi(p)] - [\pi(I_m)] = [\pi(p)] - [I_m]$$

in $K_0(A/J)$ since $K_0(\pi)$ is the zero map. Thus $\pi(p)$ and I_m are stably equivalent projections over $(A/J)^+$.

Now if $m \geqq k$, then since the cancellation law holds for projections of dimension $\geqq k$ over $(A/J)^+$, we get $\pi(p)$ actually unitarily equivalent to I_m, that is, for some large M in **N**, $\pi(p) \oplus O_{M-n}$ and $I_m \oplus O_{M-m}$ are connected by a path in $\text{Proj}_M((A/J)^+)$. Hence there is some u in $U_M((A/J)^+)^\circ$ such that

$$u(\pi(p) \oplus O_{M-n})u^{-1} = I_m \oplus O_{M-m}.$$

Since u is in the connected component of the identity in $U_M((A/J)^+)$, we can find v in $U_M(A^+)^\circ$ such that $\pi(v) = u$ and hence

$$\pi(v(p \oplus O_{M-n})v^{-1}) = I_m \oplus O_{M-m}.$$

Since $v(p \oplus O_{M-n})v^{-1}$ is connected to $p \oplus O_{M-n}$ in $\text{Proj}_M(A^+)$, hence in $\text{Proj}_\infty(A^+)$, we only need to show that $v(p \oplus O_{M-n})v^{-1}$ is in $\text{Proj}_M(J^+)$ considered as a subset of $\text{Proj}_M(A^+)$. But this is obvious from the fact that

$$\pi(v(p \oplus O_{M-n})v^{-1}) = I_m \oplus O_{M-m}.$$

5.2. LEMMA. *The only projection of dimension* 0 *over* $C^*(G)^+$ *for* G *in* Γ *is the zero projection.*

Proof. We shall prove this by induction on $\dim(G)$.

(1) If $\dim(G) = 1$, then $G = \mathbf{R}$ and $C^*(G) = C_0(\mathbf{R})$. If p is a projection of dimension 0 over $C^*(\mathbf{R})^+$, say in $\text{Proj}_n(C_0(\mathbf{R})^+)$, then p is in

$$\text{Proj}_n(C_0(\mathbf{R})) = \{q \mid q \text{ is a } \text{Proj}_n(\mathbf{C})\text{-valued continuous function on } \mathbf{R} \text{ which vanishes at infinity}\} = \{0\}$$

(any non-zero projection in $\text{Proj}_n(\mathbf{C})$ has norm 1).

(2) Suppose that the theorem holds for G in Γ with $\dim(G) = d$. Let G in Γ be of dimension $d + 1$. If G is abelian then

$$G = \mathbf{R}^{d+1} \quad \text{and} \quad C^*(G) = C_0(\mathbf{R}^{d+1}),$$

so using exactly the same argument as in (1), we get the conclusion of the theorem. If G is non-abelian, then

$$G = \mathbf{R}^d \times_\alpha \mathbf{R}$$

for some non-trivial α and

$$C^*(G) = C_0(\mathbf{R}^d) \times_{\hat{\alpha}} \mathbf{R}.$$

So by Theorems 2.7 and 2.10, there is an F in Γ of dimension d and a short exact sequence

$$0 \to D = D_+ \oplus D_- \xrightarrow{\iota} C^*(G) \xrightarrow{\pi} C^*(F) \to 0$$

with

$$D_+ \simeq D_- \simeq C_0(\mathbf{R}^{d-1}) \otimes \mathbf{K}.$$

Let p be a projection of dimension 0 in $\mathrm{Proj}_n(C^*(G)^+)$. Then $\pi(p)$ is a projection of dimension 0 in $\mathrm{Proj}_n(C^*(F)^+)$, and hence by the induction hypothesis, we get $\pi(p) = 0$. Thus p is in $\mathrm{Proj}_n(D^+)$. As observed in Section 4, the only projection of dimension 0 over D^+ is the zero projection, so $p = 0$.

Let G be a non-abelian group in the class Γ. Recalling the results of Section 2 about $C^*(G)$, we shall set (i) X to be the disjoint union of X_+ and X_-, where X_+ and X_- are $\mathbf{R}^{d-1}$, (ii) $D = D_+ \oplus D_-$ where

$$D_i = C_0(X_i) \otimes \mathbf{K} \quad \text{for } i = +, -,$$

(iii) $E = E_+ \oplus E_-$ where

$$E_i = D_i/C_0(B_i) \otimes \mathbf{K} = C_0(X_i - B_i) \otimes \mathbf{K}$$

for $i = +, -$ and $B_i =$ the unit open ball in X_i,

(iv) $\quad L = C^*(G)/(C_0(B_+) \oplus C_0(B_-)) \otimes \mathbf{K}$

which is contained in

$$(C_b(X_+ - B_+) \oplus C_b(X_- - B_-)) \otimes \mathbf{K},$$

and (v) $T = (T_+, T_-)$ to be the map from the semigroup

$$\pi_0(\mathrm{Proj}_\infty((C_0(X) \otimes \mathbf{K})^+))$$

to the group

$$K^1(S) = K^1(S_+) \oplus K^1(S_-),$$

as constructed in Section 4, where S is the union of unit spheres S_+ and S_- in X_+ and X_- respectively. With these notations, we have the two exact sequences

$$0 \to D \xrightarrow{\iota} C^*(G) \xrightarrow{\pi} C^*(F) \to 0,$$

$$0 \to E \xrightarrow{\mu} L \xrightarrow{\rho} C^*(F) \to 0.$$

From now on, whenever we deal with non-abelian G in Γ, we shall use the above conventions without explanation.

Let $G = \mathbf{R}^d \times_\alpha \mathbf{R}$ be non-abelian in Γ. Then

$$\begin{aligned} K_*(D) &= K_*(D_+) \oplus K_*(D_-) \\ &= K_*(C_0(\mathbf{R}^{d-1})) \oplus K_*(C_0(\mathbf{R}^{d-1})) \\ &= K_{*+d-1}(\mathbf{C}) \oplus K_{*+d-1}(\mathbf{C}) \end{aligned}$$

by the stability of K-theory and the Bott periodicity theorem [**30**, **22**] and

$$K_*(C^*(G)) = K_*(C_0(\mathbf{R}^{d+1})) = K_{*+d+1}(\mathbf{C})$$

while

$$K_*(C^*(F)) = K_*(C_0(\mathbf{R}^{d})) = K_{*+d}(\mathbf{C})$$

by Connes's Thom isomorphism theorem [**4**] and the Bott periodicity theorem.

If d is even, we get the 6-term exact sequence

$$(5.1)\qquad \begin{tikzcd} K_0(D) = 0 \arrow[r, "K_0(\iota)"] & K_0(C^*(G)) = 0 \arrow[r, "K_0(\pi)"] & K_0(C^*(F)) = \mathbf{Z} \arrow[d, "\delta_0"] \\ K_1(C^*(F)) = 0 \arrow[u, "\delta_1"] \arrow[r, "K_1(\pi)"'] & K_1(C^*(G)) = \mathbf{Z} \arrow[r, "K_1(\iota)"'] & K_1(D) = \mathbf{Z} \oplus \mathbf{Z} \end{tikzcd}$$

while if d is odd, we get the 6-term exact sequence

$$(5.2)\qquad \begin{tikzcd} K_0(D) = \mathbf{Z} \oplus \mathbf{Z} \arrow[r, "K_0(\iota)"] & K_0(C^*(G)) = \mathbf{Z} \arrow[r, "K_0(\pi)"] & K_0(C^*(F)) = 0 \arrow[d, "\delta_0"] \\ K_1(C^*(F)) = \mathbf{Z} \arrow[u, "\delta_1"] \arrow[r, "K_1(\pi)"'] & K_1(C^*(G)) = 0 \arrow[r, "K_1(\iota)"'] & K_1(D) = 0, \end{tikzcd}$$

where δ_0 and δ_1 are the exponential map and the index map, respectively.

We claim that under suitable identification of D_+ and D_- with $C_0(\mathbf{R}^{d-1}) \otimes \mathbf{K}$ as in 2.12, the map $K_1(\iota)$ sends (n, m) in $\mathbf{Z} \oplus \mathbf{Z} \simeq K_1(D)$ to $n - m$ in $\mathbf{Z} \simeq K_1(C^*(G))$ if $\dim(G)$ is odd, and the map $K_0(\iota)$ sends (n, m) in $\mathbf{Z} \oplus \mathbf{Z} \simeq K_0(D)$ to $n - m$ in $\mathbf{Z} \simeq K_0(C^*(G))$ if $\dim(G)$ is even.

In fact, by the diagram in 2.12, we get that

$$\begin{tikzcd} K_*(C_0(\mathbf{R}^{d-1}) \otimes \mathbf{K}) \oplus K_*(C_0(\mathbf{R}^{d-1}) \otimes \mathbf{K}) \simeq K_*(D) \arrow[r, "K_*(\iota)"] \arrow[d, "f"] & K_*(C^*(G)) \arrow[d, "K_*(h)"] \\ K_*(C_0(\mathbf{R}^{d-1}) \otimes \mathbf{K}) \oplus K_*(C_0(\mathbf{R}^{d-1}) \otimes \mathbf{K}) \simeq K_*(D) \arrow[r, "K_*(\iota)"] & K_*(C^*(G)) \end{tikzcd}$$

commutes, where f sends (n, m) to (m, n) for m and n in $K_*(C_0(\mathbf{R}^{d-1}) \otimes \mathbf{K})$. So (m, n) is in $\operatorname{Ker}(K_*(\iota))$ if and only if (n, m) is in $\operatorname{Ker}(K_*(\iota))$. Clearly if $\mathrm{Ker}(g) \simeq \mathbf{Z}$ for a homomorphism g from $\mathbf{Z} \oplus \mathbf{Z}$ onto $\mathbf{Z}$ and $\mathrm{Ker}(g)$ is symmetric in $\mathbf{Z} \oplus \mathbf{Z}$, then

$$\mathrm{Ker}(g) = \{ (n, n) \mid n \text{ is in } \mathbf{Z}\}$$

(since the generator of $\mathrm{Ker}(g)$ must be symmetric, say (m, m), and m must be 1 or -1 for $(\mathbf{Z} \oplus \mathbf{Z})/\mathrm{Ker}(g) \simeq \mathbf{Z}$ has no torsion element). Thus we know that the kernel of $K_{\dim(G)}(\iota)$ from

$$K_{\dim(G)}(D) \simeq \mathbf{Z} \oplus \mathbf{Z}$$

to

$$K_{\dim(G)}(C^*(G)) = \mathbf{Z}$$

is $\{ (n, n) \mid n \text{ is in } \mathbf{Z}\}$. So $K_{\dim(G)}(\iota)$ sends (n, m) in

$$K_{\dim(G)}(D) \simeq \mathbf{Z} \oplus \mathbf{Z}$$

to $n - m$ (or $m - n$) in

$$\mathbf{Z} \simeq K_{\dim(G)}(C^*(G)).$$

Since $K_*(C^*(G))$ is a free abelian group for all G in Γ, by the universal coefficient formula [**23**, **24**] we get

$$\begin{aligned} KK^1(C^*(F), C_0(X)) &\simeq \mathrm{Ext}(C^*(F), C_0(X)) \\ &\simeq \mathrm{Hom}(K_0(C^*(F)), K_1(D)) \\ &\oplus \mathrm{Hom}(K_1(C^*(F)), K_0(D)), \end{aligned}$$

where the last identification γ is implemented by sending an extension

$$0 \to D \to C \to C^*(F) \to 0$$

to (δ_0, δ_1) where δ_0 and δ_1 are the exponential map and the index map for the extension. Since

$$\begin{aligned} \mathrm{Hom}(K_{\dim(G)+1}(C^*(F)), K_{\dim(G)}(D)) &\simeq \mathrm{Hom}(\mathbf{Z}, \mathbf{Z} \oplus \mathbf{Z}) \\ &\simeq \mathbf{Z} \oplus \mathbf{Z} \end{aligned}$$

and

$$\mathrm{Hom}(K_{\dim(G)}(C^*(F)), K_{\dim(G)+1}(D)) \simeq \mathrm{Hom}(0, 0) \simeq 0,$$

we get

$$KK^1(C^*(F), C_0(X)) \simeq \mathbf{Z} \oplus \mathbf{Z}$$

for any non-abelian G in Γ, and the extension

$$0 \to D = C_0(X) \otimes \mathbf{K} \to C^*(G) \to C^*(F) \to 0$$

corresponds to the generator $(1, 1)$ (or $(-1, -1)$) of $\mathrm{Ker}(K_{\dim(G)}(\iota))$.

Thus we get

5.3. THEOREM. *For any non-abelian G in Γ, the extension*

$$0 \to D \simeq C_0(X) \otimes \mathbf{K} \to C^*(G) \to C^*(F) \to 0$$

gotten in 2.7 *corresponds to the element* (1, 1) *in*

$$\mathbf{Z} \oplus \mathbf{Z} \simeq KK^1(C^*(F), C_0(X))$$

(*under a suitable identification of* $C_0(X_+) \otimes \mathbf{K}$ *and* $C_0(X_-) \otimes \mathbf{K}$ *with* $C_0(\mathbf{R}^{\dim(G)-2}) \otimes \mathbf{K}$). *In particular, this extension does not split.*

Remark. (1) This generalizes a result of Kasparov's, the non-splitting of the sequence

$$0 \to (C_0(\mathbf{R}) \oplus C_0(\mathbf{R})) \otimes \mathbf{K} \to C^*(H) \to C_0(\mathbf{R}^2) \to 0$$

for the 3-dimensional Heisenberg Lie group H [**13**, **23**, **31**].

(2) By Corollary 7.5 of [**25**], we have $C^*(G)^+$ KK-equivalent to $C(S^n)$ for any n-dimensional simply connected solvable Lie group G, since their K-groups are the same. This says, among other things, that

$$KK(C^*(G)^+, A) \simeq KK(C(S^n), A) \quad \text{and}$$

$$KK(A, C^*(G)^+) \simeq KK(A, C(S^n))$$

for any (separable, nuclear) C^*-algebra A. So $C^*(G)^+$ is not much different from the algebra of continuous functions on a sphere as far as KK-theory is concerned and we may think of $C^*(G)^+$ (with G simply connected solvable) as a "non-commutative sphere" from this point of view.

Now we are ready to prove the main theorem of this paper.

5.4. THEOREM. *For any G in Γ, the cancellation law holds for projections of dimension* $\geqq C_G$ *over* $C^*(G)^+$ *where*

$$C_G = \max\{\operatorname{cansr}(C(S^{r(G)})), \operatorname{sK_1sr}(C(S^{r(G)}))\}.$$

In other words, we have

$$\operatorname{cansr}(C^*(G)^+) \leqq C_G.$$

Proof. By Lemma 5.2, we only need to consider projections of dimension greater than 0 over $C^*(G)^+$.

We consider the abelian case first.

If G is abelian, say $G = \mathbf{R}^d$, then $r(G) = d$ and $C^*(G) \simeq C_0(\mathbf{R}^d)$, so

$$C^*(G)^+ \simeq C_0(\mathbf{R}^d)^+ \simeq C(S^d).$$

The theorem follows from the definition of cansr.

Now we shall prove the theorem by induction on $\dim(G)$.

If $\dim(G) = 1$, then G equals $\mathbf{R}$ and the theorem holds.

Suppose that the theorem holds for all G in Γ with $\dim(G) = d$.

Let $\dim(G) = d + 1$. Since the theorem has been shown to hold in the abelian case, we may assume that G is non-abelian. Thus $G = \mathbf{R}^d \times_\alpha \mathbf{R}$ for some non-trivial α, and by 2.7 and 2.10, we have the exact sequence

$$0 \to D \overset{\iota}{\to} C^*(G) \overset{\pi}{\to} C^*(F) \to 0$$

for some F in Γ with

$$r(F) = r(G) \quad \text{and} \quad \dim(F) = \dim(G) - 1 = d.$$

We shall use the notations introduced right after Lemma 5.2.

By the induction hypothesis, the cancellation law holds for projections of dimension $\geqq C_F = C_G$ over $C^*(F)^+$, since $r(G) = r(F)$.

(1) If d is even, we have an exact sequence

$$0 \to K_0(D) \simeq 0 \xrightarrow{K_0(\iota)} K_0(C^*(G)) \simeq 0 \xrightarrow{K_0(\pi)} K_0(C^*(F)) \simeq \mathbf{Z}$$

as in the diagram (5.1). Clearly $K_0(\pi)$ is the zero homomorphism, so by Lemma 5.1, every projection of dimension $\geqq C_G$ over $C^*(G)^+$ is unitarily equivalent over $C^*(G)^+$ to some projection in $\iota(\mathrm{Proj}_\infty(D^+))$. So in order to consider the cancellation law for projections of dimension $\geqq C_G$ over $C^*(G)^+$, we only need to consider projections of dimenion $\geqq C_G$ in $\iota(\mathrm{Proj}_\infty(D^+))$. Let p and q be in $\mathrm{Proj}_\infty(D^+)$ such that $\iota(p)$ and $\iota(q)$ are stably equivalent in $\mathrm{Proj}_\infty(C^*(G)^+)$. Then

$$(K_0(\iota))([p]) = [\iota(p)] = [\iota(q)] = (K_0(\iota))([q])$$

in $K_0(C^*(G)^+)$. But $K_0(\iota)$ from

$$K_0(D^+) \simeq \mathbf{Z}$$

to

$$K_0(C^*(G)^+) \simeq \mathbf{Z}$$

is the identity map since

$$K_0(D) \simeq 0 \simeq K_0(C^*(G)).$$

Thus $[p] = [q]$ in $K_0(D^+)$, i.e., p and q are stably equivalent over D^+. By 4.5, we get p and q unitarily equivalent over D^+, hence over $C^*(G)^+$. Thus the cancellation law holds for projections of dimension $\geqq C_G$ over $C^*(G)^+$ if $\dim(G)$ is odd.

(2) If d is odd, we have an exact sequence

$$0 \to K_1(C^*(F)) \simeq \mathbf{Z} \xrightarrow{\delta_1} K_0(D) \simeq \mathbf{Z} \oplus \mathbf{Z} \xrightarrow{K_0(\iota)} K_0(C^*(G))$$
$$\simeq \mathbf{Z} \xrightarrow{K_0(\pi)} K_0(C^*(F)) \simeq 0$$

as in the diagram (5.2). Clearly $K_0(\pi)$ is the zero homomorphism, so by 5.1

and 4.4, every projection of dimension $\geqq C_G$ over $C^*(G)^+$ is unitarily equivalent over $C^*(G)^+$ to some projection in $\iota(S\mathrm{Proj}_\infty(D^+))$.

By 4.6, we have an explicit isomorphism T' identifying $K_0(D^+)$ with

$$\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} = K_0(\mathbf{C}) \oplus K^1(S_+) \oplus K^1(S_-).$$

Note that the homomorphism $K_0(\iota)$ sends (k, n, m) in

$$K_0(D^+) \simeq K_0(\mathbf{C}) \oplus \mathbf{Z} \oplus \mathbf{Z}$$

to $(k, n - m)$ in

$$K_0(C^*(G)^+) \simeq K_0(\mathbf{C}) \oplus K_0(C^*(G)) \simeq K_0(\mathbf{C}) \oplus \mathbf{Z},$$

as we saw just before Theorem 5.3.

Since the cancellation law holds for all projections over D^+ by 4.5, every projection p over D^+ is characterized up to unitary equivalence over D^+ by

$$\begin{aligned}[p] &= (\dim(p), T_+(p), T_-(p)) \text{ in } K_0(D^+) \\ &= K_0(\mathbf{C}) \oplus K^1(S_+) \oplus K^1(S_-) \simeq \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}.\end{aligned}$$

For such p, we have

$$K_0(\iota)([p]) = (\dim(p), T_+(p) - T_-(p))$$

in

$$\mathbf{Z} \oplus \mathbf{Z} \simeq K_0(\mathbf{C}) \oplus K_0(C^*(G)) \simeq K_0(C^*(G)^+).$$

It remains to show that, for any two standard projections p and p' of dimension $\geqq C_G$ over D^+, if $\iota(p)$ and $\iota(p')$ are stably equivalent over $C^*(G)^+$, then $\iota(p)$ and $\iota(p')$ are unitarily equivalent over $C^*(G)^+$.

For any such p and p', we have

$$K_0(\iota)([p]) = [\iota(p)] = [\iota(p')] = K_0(\iota)([p'])$$

in $K_0(C^*(G)^+)$; hence

$$\dim(p) = \dim(p') \quad \text{and} \quad T_+(p) - T_-(p) = T_+(p') - T_-(p')$$

in $\mathbf{Z} \simeq K_0(C^*(G))$. Let

$$k = \dim(p) = \dim(p') \geqq C_G.$$

Suppose that we can find u'' in $V_n(C^*(G)^+)$ for arbitrarily large n such that

$$u''(x) = w''(x) \oplus v''(x) \quad \text{for all } x \text{ in } X - B,$$

where w'' is in $V_k(L^+)$ and v'' is in $V_{n-k}(L^+)$, and

$$[w''|_S] = (T_+(p) - T_+(p'), T_-(p) - T_-(p'))$$

in $\mathbf{Z} \oplus \mathbf{Z} \simeq K^1(S)$. Let n be large enough that p and p' are in $S\mathrm{Proj}_n(C^*(G)^+)$. Since

$$\begin{aligned}(u''q_pu''^{-1})(x) &= (w''(x) \oplus v''(x))(I_k \oplus O_{n-k})\\ &\quad (w''(x)^{-1} \oplus v''(x)^{-1})\\ &= I_k \oplus O_{n-k}\end{aligned}$$

for all x in $X - B$, we can find some p'' in $S\mathrm{Proj}_n(D^+)$ such that

$$q_{p''} = u''q_pu''^{-1}.$$

Note that p'' is unitarily equivalent to

$$q_{p''} = u''q_pu''^{-1},$$

hence unitarily equivalent over $C^*(G)^+$ to q_p and hence to p. Let u be in $V_n((C_b(X) \otimes \mathbf{K})^+)$ such that

$$uq_pu^{-1} = I_k \oplus O_{n-k} \quad \text{and} \quad u|_{X-B} = w \oplus v$$

for some w in $V_k((C_b(X - B) \otimes \mathbf{K})^+)$ and v in $V_{n-k}(C_b(X - B) \otimes \mathbf{K})^+)$. Then $[w|_S] = T(p)$. Since

$$\begin{aligned}&(uu''^{-1})q_{p''}(uu''^{-1})^{-1} = uq_pu^{-1} = I_k \oplus O_{n-k} \quad \text{and}\\ &uu''^{-1}|_{X-B} = ww''^{-1} \oplus vv''^{-1},\end{aligned}$$

we get

$$\begin{aligned}T(p'') &= [(ww''^{-1})|_S] = [w|_S] - [w''|_S] = T(p) - [w''|_S]\\ &= (T_+(p) - (T_+(p) - T_+(p')),\\ &\qquad\qquad T_-(p) - (T_-(p) - T_-(p')))\\ &= (T_+(p'), T_-(p')) = T(p')\end{aligned}$$

in $\mathbf{Z} \oplus \mathbf{Z} \simeq K^1(S)$. Since

$$\dim(p'') = \dim(p) = k = \dim(p') \quad \text{and} \quad T(p'') = T(p'),$$

by Proposition 4.6 we get $[p''] = [p']$ in $K_0(D^+)$. So by Theorem 4.5, we get p'' unitarily equivalent over D^+ to p'. Thus p is unitarily equivalent over $C^*(G)^+$ to p'.

Thus, in order to prove the theorem, we need only to find such a u with

$$[w|_S] = (T_+(p) - T_+(p'), T_-(p) - T_-(p')).$$

But $T_+(p) - T_+(p') = T_-(p) - T_-(p')$. So it remains to show that, for all m in $\mathbf{Z}$, there are some n in $\mathbf{N}$ and u in $V_n(C^*(G)^+)$ such that $u(x) = w(x) \oplus v(x)$ for all x in $X - B$ and some w in $V_k(L^+)$ and v in $V_{n-k}(L^+)$, with $[w|_S] = (m, m)$ in $\mathbf{Z} \oplus \mathbf{Z} \simeq K^1(S)$. (Note that n can be arbitrarily large by replacing u by $u \oplus I_N$ for large N.) But it is sufficient to prove this for $m = 1$ since if $u(x) = w(x) \oplus v(x)$ for x in $X - B$ and $[w|_S] = (1, 1)$, then

$$u^m(x) = w^m(x) \oplus v^m(x) \quad \text{for } x \text{ in } X - B \text{ and}$$

$$[w^m|_S] = [(w|_S)^m] = m[w|_S] = (m, m) \quad \text{in } K^1(S).$$

Let p in $S\mathrm{Proj}_n(D^+)$ be a representative (e.g. $p_{(1,1)} \oplus I_{k-1}$ as constructed in Section 4) of $(k, 1, 1)$ in $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \simeq K_0(D^+)$, i.e., $T'(p) = (k, 1, 1)$ as in 4.6. Then since

$$[\iota(p)] = K_0(\iota)([p]) = (k, 0) = [I_k] \text{ in } K_0(C^*(G)^+),$$

so $\iota(p)$, and hence $\iota(q_p)$, are stably equivalent over $C^*(G)^+$ to I_k. Hence there are some N in $\mathbf{N}$ and some u' in $V_{N+n}(C^*(G)^+)$ such that

$$u'(I_N \oplus \iota(q_p))(u')^{-1} = I_N \oplus I_k \oplus O_{n-k}.$$

(We may need to enlarge n at this point.) So for x in $X - B$, we have $u'(x) = w'(x) \oplus v'(x)$ for some w' in $V_{N+k}(L^+)$ and v' in $V_{n-k}(L^+)$, since

$$(I_N \oplus \iota(q_p))(x) = I_N \oplus I_k \oplus O_{n-k} \quad \text{for } x \text{ in } X - B.$$

Clearly

$$\dim(I_N \oplus p) = N + k \quad \text{and} \quad q_{I_N \oplus p} = I_N \oplus q_p,$$

so we have

$$[w'|_S] = T(I_N \oplus p) = T(I_N) + T(p) = 0 + T(p) = (1, 1)$$

in $K_1(C(S) \otimes \mathbf{K}) \simeq K^1(S)$. But since w' is in $V_{N+k}(L^+)$, not in $V_k(L^+)$, u' is not what we are looking for. But by considering the connected stable rank of L^+, we may reduce the size of w' in some sense. In fact, by 3.25,

$$\mathrm{sK_1sr}(L^+) \leqq \mathrm{sK_1sr}(C^*(F)^+).$$

But either F is non-abelian and hence

$$\mathrm{sK_1sr}(C^*(F)^+) \leqq 1$$

by 3.24, or $F = \mathbf{R}^{r(G)}$ and hence

$$\mathrm{sK_1sr}(C^*(F)^+) = \mathrm{sK_1sr}(C(S^{r(G)})).$$

In any case, $k \geqq \mathrm{sK_1sr}(L^+)$. Thus by the definition of $\mathrm{sK_1sr}$, we get that w' is connected to $w \oplus I_N$ by a path in $V_{N+k}(L^+)$ for some w in $V_k(L^+)$. Hence $w'|_S$ is connected to $w|_S \oplus I_N$ by a path in $V_{N+k}((C(S) \otimes \mathbf{K})^+)$, so $[w'|_S] = [w|_S]$ in $K^1(S)$.

Now we are going to construct u in $V_n(C^*(G)^+)$ from w. By enlarging n, we may assume that $n \geqq 2k$. Since $w \oplus w^{-1}$ is connected to I_{2k} by the path

$$\begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w^{-1} \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

(t in $[0, \pi/2]$), we get $(w \oplus w^{-1} \oplus I_{n-2k})|_S$ connected to I_n by a path u_t in $V_n((C(S) \otimes \mathbf{K})^+)$, with

$$u_0 = I_n \quad \text{and} \quad u_1 = (w \oplus w^{-1} \oplus I_{n-2k})|_S.$$

Let

$$u(x) = \begin{cases} I_n & \text{if } |x| = 0, \\ u_{|x|}(x/|x|) & \text{if } 0 < |x| \leqq 1, \\ w(x) \oplus w(x)^{-1} \oplus I_{n-2k} & \text{if } |x| > 1. \end{cases}$$

Then u is an element of $V_n(C^*(G)^+)$, since $w \oplus w^{-1} \oplus I_{n-2k}$ is in $V_n(L^+)$ and u coincides with $w \oplus w^{-1} \oplus I_{n-2k}$ on $X - B$.

Since w (or $w^{-1} \oplus I_{n-2k}$) is in $V_k(L^+)$ (or $V_{n-k}(L^+)$) with

$$[w|_S] = [w'|_S] = (1, 1) \quad \text{in } K^1(S), \text{ and}$$

$$u(x) = w(x) \oplus (w^{-1} \oplus I_{n-2k})(x) \quad \text{for } x \text{ in } X - B,$$

the u constructed above is what we are looking for.

Let $G = \mathbf{R}^d \times_\alpha \mathbf{R}$ be non-abelian in Γ with d odd. From the proof of 5.4, we can get an explicit description of a generator of $K_1(L)$. Before we do that, we shall compute $K_*(L)$ first. Note that

$$E = C_0(X - B) \otimes \mathbf{K} \simeq C_0(S \times [1, \infty)) \otimes \mathbf{K}$$
$$\simeq C_0([1, \infty)) \otimes C(S) \otimes \mathbf{K}$$

is a contractible C^*-algebra. So $K_0(E) = K_1(E) = 0$. Thus from the following 6-term exact sequence

$$\begin{tikzcd} K_0(E) = 0 \arrow[r] & K_0(L) \arrow[r] & K_0(C^*(F)) = K_1(\mathbf{C}) = 0 \arrow[d] \\ K_1(C^*(F)) = \mathbf{Z} \arrow[u, leftarrow] & K_1(L) \arrow[l] & K_1(E) = 0 \arrow[l] \end{tikzcd}$$

we get $K_0(L) = 0$ and $K_1(L) = \mathbf{Z}$.

5.5. COROLLARY. *If $G = \mathbf{R}^d \times_\alpha \mathbf{R}$ is non-abelian in Γ with d odd, and if F is related to G as in 2.7, then a generator of $K_1(L) = \mathbf{Z}$ is given by a w in $U_{C_G}(L^+)$ such that $[w|_S] = (1, 1)$ in $K^1(S) = \mathbf{Z} \oplus \mathbf{Z}$.*

Proof. The existence of such a w has been proved in the proof of 5.4. If u is a generator of $\mathbf{Z} \simeq K_1(L)$ and u is in $U_n(L^+)$, then

$$[w] = m[u] = [u^m] \quad \text{in } K_1(L) \text{ for some } m \text{ in } \mathbf{Z}.$$

Thus $w \oplus I_{N-C_G}$ and $u^m \oplus I_{N-n}$ are connected by a path u_t in $U_N(L^+)$ for some large N; hence $w|_S \oplus I_{N-C_G}$ is connected to $(u|_S)^m \oplus I_{N-n}$ by the path $u_t|_S$ in $U_N((C(S) \otimes \mathbf{K})^+)$. So

$$(1, 1) = [w|_S] = [(u|_S)^m] = m[u|_S] \quad \text{in } K^1(S) \simeq \mathbf{Z} \oplus \mathbf{Z}.$$

Hence $m = 1$ or -1, so $[w] = [u]$ or $-[u]$, which is certainly a generator of $K_1(L)$.

The following corollary generalizes 1.3.

5.6. COROLLARY. (a) *The cancellation law holds for all projections over* $C^*(G)^+$ *with* G *in* Γ *and* $r(G) = 1, 2,$ *or* 4.

(b) *The cancellation law holds for projections of dimension* $\geqq 2$ *over* $C^*(G)^+$ *with* G *in* Γ *and* $r(G) = 3$.

(c) *For any* G *in* Γ, *the cancellation law holds for projections of* dim $\geqq M_G$ *over* $C^*(G)^+$, *where*

$$\begin{aligned} M_G &= \max\{\mathrm{tsr}(C_0(\mathbf{R}^{r(G)-1})), \mathrm{csr}(C_0(\mathbf{R}^{r(G)})) - 1\} \\ &= [(r(G) - 1)/2] + 1. \end{aligned}$$

Proof. (a) Apply 5.4, 5.2 and 3.22.

(b) Apply Theorem 5.4 and 3.22.

(c) We shall first show that

$$\max[\mathrm{tsr}(C(S^{d-1})), \mathrm{csr}(C(S^d)) - 1\} = [(d - 1)/2] + 1$$

for all d in N, to justify the last equality in the statement. By Theorem 3.4 and example (1) following Theorem 3.8, we have

(i) $\quad \mathrm{tsr}(C(S^0)) = 1 \quad \text{and} \quad \mathrm{csr}(C(S^1)) = 2,$

(ii) $\quad \mathrm{tsr}(C(S^{d-1})) = [(d - 1)/2] + 1 \quad \text{and}$

$$\mathrm{csr}(C(S^d)) \leqq [(d + 1)/2] + 1 = [(d - 1)/2] + 2 \quad \text{if } d \geqq 2.$$

The equality follows.

By 3.21(4), we have

$$\mathrm{sK_1sr}(C(S^{r(G)})) \leqq \mathrm{csr}(C(S^{r(G)})) - 1,$$

and by 3.21(2), we have

$$\mathrm{cansr}(C(S^{r(G)})) \leqq \mathrm{tsr}(C(S^{r(G)-1})).$$

Thus $C_G \leqq M_G$ for all G in Γ, and the statement follows from 5.4.

Remark. (1) In Section 6, we shall show that the cancellation law fails for projections of dimension one over $C^*(G)^+$ for some G in Γ with $r(G) = 3$.

(2) By 2.11 and 5.6(a), we can find infinitely many G in Γ with $\dim(G)$ arbitrarily high such that the cancellation law holds for all projections over $C^*(G)^+$ (cf. 1.3(2)).

By 4.7, the proof of 5.4, and 5.6(a), we get

5.7. THEOREM. *The semigroup of unitary equivalence classes of projections over* $C^*(G)^+$, *for a non-abelian group* G *in* Γ *with* $r(G) = 1, 2$, *or* 4, *is isomorphic to* $\{0\} \cup (\mathbf{Z} \oplus \mathbf{N})$ *if* $\dim(G)$ *is even and to* $\{0\} \cup \mathbf{N}$ *if* $\dim(G)$ *is odd.*

6. Cancellation law for the case of simply connected nilpotent Lie groups of dimension no greater than four. In this section, we shall solve completely the cancellation problem over $C^*(G)^+$ for all simply connected nilpotent Lie groups of dimension $\leqq 4$.

As we have seen in Section 1, the groups $G = \mathbf{R}^d \times_\alpha \mathbf{R}$ in Γ are classified (up to isomorphism) by the (strictly upper triangular) Jordan canonical form of the generator of the action α, that is,

$$\frac{d}{dt}\alpha\Big|_{t=0} \quad \text{in } M_d(\mathbf{R}).$$

Since we can easily enumerate all possible strictly upper triangular Jordan canonical forms (up to permutations of columns and rows) of size no more than three, namely

$$(0), \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{and} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

the groups of dimension no more than 4 in Γ are classified into seven non-isomorphic groups, namely, $\mathbf{R}$, $\mathbf{R}^2$, $\mathbf{R}^3$, H (the 3-dimensional Heisenberg Lie group), $\mathbf{R}^4$, $H \times \mathbf{R}$ and $\mathbf{R}^3 \times_\delta \mathbf{R}$, where

$$\delta(t)(x, y, z) = (x, tx + y, (t^2/2)x + ty + z).$$

All of these are in Γ, and computation shows that

$$r(\mathbf{R}) = 1, \quad r(\mathbf{R}^2) = r(H) = r(\mathbf{R}^3 \times_\delta \mathbf{R}) = 2,$$
$$r(\mathbf{R}^3) = r(H \times \mathbf{R}) = 3 \quad \text{and} \quad r(\mathbf{R}^4) = 4.$$

So by 5.6(a), the cancellation law holds for all projections over $C^*(G)^+$ if $G = \mathbf{R}, \mathbf{R}^2, H, \mathbf{R}^3 \times_\delta \mathbf{R}$, or $\mathbf{R}^4$. By 1.3(2) it holds if $G = \mathbf{R}^3$.

By 5.6(b), the cancellation law holds for projections of dimension $\geqq 2$ over $C^*(H \times \mathbf{R})^+$. So it remains to classify all projections of dimension one over $C^*(H \times \mathbf{R})^+$ up to unitary equivalence.

By computation, we have the exact sequence

$$0 \to D = (C_0(\mathbf{R}^2) \oplus C_0(\mathbf{R}^2)) \otimes \mathbf{K} \xrightarrow{\iota} C^*(H \times \mathbf{R})$$
$$\xrightarrow{\pi} C^*(\mathbf{R}^3) = C_0(\mathbf{R}^3) \to 0$$

as in 2.7 for $G = H \times \mathbf{R}$. Since

$$K_0(C_0(\mathbf{R}^3)) = 0 \quad \text{and} \quad \operatorname{cansr}(C_0(\mathbf{R}^3)) = 0,$$

we know that every projection over $C^*(H \times \mathbf{R})^+$ is unitarily equivalent over $C^*(H \times \mathbf{R})^+$ to some projection over D^+ by Lemma 5.1.

In the following, we shall use the same notations as used in Section 5 for this special case.

As we have seen in Section 4, projections p over D^+ are determined by

$$T'(p) = (\dim(p), T_+(p), T_-(p)) \quad \text{in } \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \simeq K_0(D^+)$$

up to unitary equivalence (over D^+), and we have a projection $p_{(n,m)}$ in $\mathrm{Proj}_2(D^+)$ of dimension one (since $\mathbf{K}$ is stable) over D^+ such that $T(p_{(n,m)}) = (n, m)$ for each (n, m) in

$$\mathbf{Z} \oplus \mathbf{Z} \simeq K^1(S) \simeq K_1(C(S) \otimes \mathbf{K}),$$

where S is the disjoint union of the unit circles in those two $\mathbf{R}^2$'s. So the set

$$\{p_{(n,m)} | (n, m) \text{ is in } K^1(S) \simeq \mathbf{Z} \oplus \mathbf{Z}\}$$

is a complete set of representatives for the unitary equivalence classes of 1-dimensional projections over D^+.

Now we want to show that $p_{(n,m)}$ is unitarily equivalent over $C^*(H \times \mathbf{R})^+$ to $p_{(n',m')}$ if and only if $(n, m) = (n', m')$.

Indeed, if $p_{(n,m)}$ is unitarily equivalent over $C^*(H \times \mathbf{R})^+$ to $p_{(n',m')}$, then by the discussion before 5.3, we have

$$(1, n - m) = K_0(\iota)([p_{(n,m)}]) = K_0(\iota)([p_{(n',m')}]) = (1, n' - m')$$

and we can find u in $U_N(C^*(H \times \mathbf{R})^+)$ for some large N such that

$$u(q_{(n,m)} \oplus O_{N-2})u^{-1} = q_{(n',m')} \oplus O_{N-2}$$

where $q_{(n,m)}$ is defined to be $q_{p_{(n,m)}}$. So

$$u(x) = w(x) \oplus v(x) \quad \text{for all } x \text{ in } X - B$$

for some v in $U_{N-1}(C^*(H \times \mathbf{R})^+/C_0(B) \otimes \mathbf{K})$ and w in $U_1(C^*(H \times \mathbf{R})^+/C_0(B) \otimes \mathbf{K})$, since

$$(q_{(n,m)} \oplus O_{N-2})(x) = (I_1 \oplus O_1) \oplus O_{N-2} = (q_{(n',m')} \oplus O_{N-2})(x)$$

for x in $X - B$. Let u' in $V_N((C_b(X) \otimes \mathbf{K})^+)$ be such that

$$u'(q_{(n',m')} \oplus O_{N-2})u'^{-1} = I_1 \oplus O_{N-1},$$

$$u'(x) = w'(x) \oplus v'(x) \quad \text{for } x \text{ in } X - B,$$

and

$$u'(e_1(x)) = I_N \quad \text{for all } x.$$

Then

$$[w'|_S] = T(p_{(n',m')}) = (n', m') \quad \text{in } K^1(S) = \mathbf{Z} \oplus \mathbf{Z}.$$

Since

$$(u'u)(q_{(n,m)} \oplus O_{N-2})(u'u)^{-1} = I_1 \oplus O_{N-1} \quad \text{and}$$
$$(u'u)|_{X-B} = (w'w) \oplus (v'v),$$

we get $[(w'w)|_S] = T(p_{(n,m)})$ by remark (3) following Theorem 4.5. So

$$(n', m') + [w|_S] = [w'|_S] + [w|_S]$$
$$= [(w'w)|_S] = T(p_{(n,m)}) = (n, m),$$

that is, $[w|_S] = (n - n', m - m')$ in $K^1(S) = \mathbf{Z} \oplus \mathbf{Z}$. (Note that $n - n' = m - m'$ since $n - m = n' - m'$.) Now by Corollary 5.5, we have

$$[w \oplus I_1] = n - n' = m - m' \quad \text{in } \mathbf{Z} \simeq K_1(L^+)$$

since $M_{H \times \mathbf{R}} = 2$.

If $(n, m) \neq (n', m')$, then

$$[w] = [w \oplus I_1] \neq 0 \quad \text{in } \mathbf{Z} \simeq K_1(L^+).$$

But by the fact that $K_*(E) = 0$ as explained before Corollary 5.5, we have the exact sequence

$$0 = K_1(E) \to K_1(L^+) \simeq \mathbf{Z} \xrightarrow{K_1(\rho)} K_1(C_0(\mathbf{R}^3)^+)$$
$$\simeq \mathbf{Z} \to K_0(E) = 0;$$

hence $K_1(\rho)$ is an isomorphism. Thus $\rho(w)$ in $U_1(C_0(\mathbf{R}^3)^+)$ represents a non-zero element of $K_1(C_0(\mathbf{R}^3)^+) \simeq \mathbf{Z}$. But

$$\pi_0(U_1(C_0(\mathbf{R}^3)^+)) = \pi_3(U_1(\mathbf{C})) = \pi_3(S^1) = 0,$$

so every element in $U_1(C_0(\mathbf{R}^3)^+)$ represents the zero of $K_1(C_0(\mathbf{R}^3)^+) \simeq \mathbf{Z}$. Thus we get a contradiction. So $(n, m) = (n', m')$ as claimed.

Thus $\{ p_{(n,m)} | n$ and m are in $\mathbf{Z}\}$ is indeed a complete set of inequivalent representatives for unitary equivalence classes of 1-dimensional projections over $C^*(H \times \mathbf{R})^+$.

By Lemma 5.2, the only 0-dimensional projection over $C^*(H \times \mathbf{R})^+$ is 0.

Since the cancellation law holds for projections of dimension $\geqq 2$ over $C^*(H \times \mathbf{R})^+$, and since

$$[\iota(p_{(n,0)} \oplus I_k)]$$
$$= (\dim(p_{(n,0)} \oplus I_k), T_+(p_{(n,0)} \oplus I_k) - T_-(p_{(n,0)} \oplus I_k))$$
$$= (1 + k, n - 0)$$
$$= (1 + k, n)$$

in $\mathbf{Z} \oplus \mathbf{Z} \simeq K_0(C^*(H \times \mathbf{R})^+)$ for any k in $\mathbf{N}$ and n in $\mathbf{Z}$, we get that

$$\{p_{(n,0)} \oplus I_k | n \text{ is in } \mathbf{Z}\}$$

is a complete set of representatives for the equivalence classes of $(k + 1)$-dimensional projections over $C^*(H \times \mathbf{R})^+$ for any k in $\mathbf{N}$.

Remark. For k in $\mathbf{N}$ and m, n in $\mathbf{Z}$, we have

$$\begin{aligned}[\iota(p_{(n,m)} \oplus I_k)] &= (1 + k, T_+(p_{(n,m)} \oplus I_k) - T_-(p_{(n,m)} \oplus I_k)) \\ &= (1 + k, n - m) \\ &= [\iota(p_{(n-m,0)} \oplus I_k)]\end{aligned}$$

in $K_0(C^*(H \times \mathbf{R})^+) \simeq \mathbf{Z} \oplus \mathbf{Z}$. Since

$$1 + k \geqq 2 \quad \text{and} \quad \operatorname{cansr}(C^*(H \times \mathbf{R})^+) \leqq 2,$$

we get $p_{(n,m)} \oplus I_k$ unitarily equivalent over $C^*(H \times \mathbf{R})^+$ to $p_{(n-m,0)} \oplus I_k$.

In $K_0(D^+) \simeq \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$, we have

$$\begin{aligned}&[(p_{(n,m)} \oplus I_k) \oplus (p_{(n',m')} \oplus I_{k'})] \\ &= [p_{(n,m)} \oplus I_k] + [p_{(n',m')} \oplus I_{k'}] \\ &= (1 + k, n, m) + (1 + k', n', m') \\ &= (2 + k + k', n + n', m + m') \\ &= [p_{(n+n',m+m')} \oplus I_{k+k'+1}]\end{aligned}$$

for all non-negative integers k and k' and all n, n', m, and m' in $\mathbf{Z}$. Since $k + k' + 1 \geqq 1$, we have $p_{(n+n',m+m')} \oplus I_{k+k'+1}$ unitarily equivalent over $C^*(H \times \mathbf{R})^+$ to $p_{(n+n'-m-m',0)} \oplus I_{k+k'+1}$ by the above remark.

Summarizing the above, we get

6.1. THEOREM. *Let G be a simply connected nilpotent Lie group of dimension $\leqq 4$ other than $H \times \mathbf{R}$. Then the cancellation law holds for all projections over $C^*(G)^+$.*

6.2. THEOREM. (1) $\operatorname{cansr}(C^*(H \times \mathbf{R})^+) = 2$, *that is, the cancellation law holds for projections of dimension $\geqq 2$ over $C^*(H \times \mathbf{R})^+$, but fails for projections of dimension one.*

(2) *The projections* 0, $p_{(n,m)}$, *and* $p_{(n,0)} \oplus I_k$, *with* n, m *in* $\mathbf{Z}$ *and* k *in* $\mathbf{N}$, *form a complete set of representatives for the unitary equivalence classes of projections over $C^*(H \times \mathbf{R})^+$, endowed with the following abelian semigroup structure*:

$$\begin{aligned}&p_{(n,m)} \oplus p_{(n',m')} = p_{(n+n'-m-m',0)} \oplus I_1, \\ &p_{(n,m)} \oplus (p_{(n',0)} \oplus I_k) = p_{(n+n'-m,0)} \oplus I_{k+1}, \\ &(p_{(n,0)} \oplus I_k) \oplus (p_{(n',0)} \oplus I_{k'}) = p_{(n+n',0)} \oplus I_{k+k'+1},\end{aligned}$$

and 0 *is the additive unit.*

Thus, the cancellation problem is completely solved for projections over $C^*(G)^+$ with G a simply connected nilpotent Lie group of dimension $\leqq 4$. By 5.7, the proof of 6.1, and 6.2, the classification of unitary equivalence classes of projections over such algebras is also complete.

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