Canad. Math. Bull. Vol. **57** (3), 2014 pp. 477–484
http://dx.doi.org/10.4153/CMB-2014-022-7

On Set Theoretically and Cohomologically Complete Intersection Ideals

Majid Eghbali

Abstract. Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{a}$ be an ideal of R. The inequalities

$$\operatorname{ht}(\mathfrak{a}) \leq \operatorname{cd}(\mathfrak{a}, R) \leq \operatorname{ara}(\mathfrak{a}) \leq l(\mathfrak{a}) \leq \mu(\mathfrak{a})$$

are known. It is an interesting and long-standing problem to determine the cases giving equality. Thanks to the formal grade we give conditions in which the above inequalities become equalities.

1 Introduction

Throughout this note, R is a commutative Noetherian ring with identity and $\mathfrak{a}$ is an ideal of R. The smallest number of elements of R required to generate $\mathfrak{a}$ up to radical is called the *arithmetic rank*, $\operatorname{ara}(\mathfrak{a})$ of $\mathfrak{a}$. Another invariant related to the ideal $\mathfrak{a}$ is $\operatorname{cd}(\mathfrak{a}, R)$, the so-called *cohomological dimension* of $\mathfrak{a}$, defined as the maximum index for which the local cohomology module $H^i_{\mathfrak{a}}(R)$ does not vanish.

It is well known that $\operatorname{ht}(\mathfrak{a}) \leq \operatorname{cd}(\mathfrak{a}, R) \leq \operatorname{ara}(\mathfrak{a})$. If $\operatorname{ht}(\mathfrak{a}) = \operatorname{ara}(\mathfrak{a})$, then $\mathfrak{a}$ is called a set-theoretic complete intersection ideal. Determining set-theoretic complete intersection ideals is a classical and long-standing problem in commutative algebra and algebraic geometry. Many questions related to an ideal $\mathfrak{a}$ being a set-theoretic complete intersection are still open. See [15] for more information.

Recently, there have been many attempts to investigate the equality $\operatorname{cd}(\mathfrak{a}, R) = \operatorname{ara}(\mathfrak{a})$ (see *e.g.*, [2,3,14] and their references), for certain classes of squarefree monomial ideals, but the equality does not hold in general (*cf.* [23]). However, in many cases, this question is open and many researchers are still working on it.

Hellus and Schenzel [12] defined an ideal $\mathfrak{a}$ to be a cohomologically complete intersection if $\operatorname{ht}(\mathfrak{a}) = \operatorname{cd}(\mathfrak{a}, R)$. In the case where $(R, \mathfrak{m})$ is a Gorenstein local ring, they gave a characterization of cohomologically complete intersections for a certain class of ideals.

One more concept we will use is the analytic spread of an ideal. Let $(R, \mathfrak{m})$ be a local ring with infinite residue field. We denote by $l(\mathfrak{a})$ the Krull dimension of $\bigoplus_{n=0}^{\infty}(\mathfrak{a}^n/\mathfrak{a}^n\mathfrak{m})$, called the *analytic spread* of $\mathfrak{a}$. In general,

$$\operatorname{ht}(\mathfrak{a}) \leq \operatorname{cd}(\mathfrak{a}, R) \leq \operatorname{ara}(\mathfrak{a}) \leq l(\mathfrak{a}) \leq \mu(\mathfrak{a}), \tag{1.1}$$

Received by the editors March 26, 2013; revised March 24, 2014.
Published electronically April 28, 2014.
This research was in part supported by a grant from IPM (No. 91130026).
AMS subject classification: **13D45**, 13C14.
Keywords: set-theoretically and cohomologically complete intersection ideals, analytic spread, monomials, formal grade, depth of powers of ideals.

where $\mu(\mathfrak{a})$ is the minimal number of generators of $\mathfrak{a}$. Burch [7] proved what is now known as Burch's inequality that $l(\mathfrak{a}) \leq \dim R - (\min_n \operatorname{depth} R/\mathfrak{a}^n)$. It should be noted that the stability of depth $R/\mathfrak{a}^n$ was established by Brodmann (*cf.* [4]). The equality $l(\mathfrak{a}) = \dim R - (\min \operatorname{depth} R/\mathfrak{a}^n)$ has been studied from several points of view by many authors, and deep results have been obtained in recent years by the assumptions that the associated graded ring of $\mathfrak{a}$ is Cohen–Macaulay; see for instance [11, Proposition 3.3] or [21, Proposition 5.1] for detailed information.

The outline of this paper is as follows. In Section 2, we give a slight generalization of a result of Cowsik and Nori in order to turn some of the inequalities in (1.1) into equalities (*cf.* Theorem 2.8). According to the results given in Section 2, one can see that the equality $\operatorname{ht}(\mathfrak{a}) = \operatorname{cd}(\mathfrak{a}, R)$ has a critical role in clarifying the structure of $\mathfrak{a}$. In Sections 3 and 4 we have focused our attention on this equality.

2 Formal Grade and Depth

Throughout this section, $(R, \mathfrak{m})$ is a commutative Noetherian local ring. Let $\mathfrak{a}$ be an ideal of R and M be an R-module. For an integer i, let $H^i_{\mathfrak{a}}(M)$ denote the i-th local cohomology module of M. We have the isomorphism of $H^i_{\mathfrak{a}}(M)$ to $\varinjlim_n \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$ for every $i \in \mathbb{Z}$; see [5] for more details.

Consider the family of local cohomology modules $\{H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)\}_{n\in\mathbb{N}}$. For every n there is a natural homomorphism $H^i_{\mathfrak{m}}(M/\mathfrak{a}^{n+1}M) \to H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ such that the family forms a projective system. The projective limit $\mathfrak{F}^i_{\mathfrak{a}}(M) := \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ is called the i-th formal local cohomology of M with respect to $\mathfrak{a}$ (*cf.* [18]; see also [1] and [10] for more information).

For an ideal $\mathfrak{a}$ of R , the formal grade, $\operatorname{fgrade}(\mathfrak{a}, M)$, is defined as the index of the minimal nonvanishing formal cohomology module, *i.e.*,

$$\operatorname{fgrade}(\mathfrak{a}, M) = \inf\big\{\, i \in \mathbb{Z} : \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) \neq 0 \big\}.$$

The formal grade plays an important role throughout this note. We first recall a few remarks.

Remark 2.1 Let $\mathfrak{a}$ denote an ideal of a local ring $(R, \mathfrak{m})$. Let M be a finitely generated R-module.

(a) $\operatorname{fgrade}(\mathfrak{a}, M) \leq \dim \widehat{R}/(\mathfrak{a}\widehat{R}, \mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Ass} \widehat{M}$ (*cf.* [18, Theorem 4.12]).
(b) In the case where M is a Cohen–Macaulay module, $\operatorname{fgrade}(\mathfrak{a}, M) = \dim M - \operatorname{cd}(\mathfrak{a}, M)$ (*cf.* [1, Corollary 4.2]).

A key point in the proof of the main results in this section is the following proposition.

Proposition 2.2 *Let $\mathfrak{a}$ be an ideal of a d-dimensional local ring $(R, \mathfrak{m})$. Then the inequality*

$$\min_n \operatorname{depth} R/\mathfrak{a}^n \leq \operatorname{fgrade}(\mathfrak{a}, R)$$

holds.

https://doi.org/10.4153/CMB-2014-022-7 Published online by Cambridge University Press

Proof Put $\min_n \operatorname{depth} R/\mathfrak{a}^n := t$, then for each integer n, $H^i_{\mathfrak{m}}(R/\mathfrak{a}^n) = 0$ for all $i < t$. This implies that $\varprojlim H^i_{\mathfrak{m}}(R/\mathfrak{a}^n) = 0$ for all $i < t$. Then the definition of the formal grade implies that $\min_n \operatorname{depth} R/\mathfrak{a}^n \leq \operatorname{fgrade}(\mathfrak{a}, R)$. ■

The above inequality may be strict, as the next example demonstrates.

Example 2.3 Let k be a field and $R = k[|x, y, z|]$ denote the formal power series ring in three variables over k. Put $\mathfrak{a} := (x, y) \cap (y, z) \cap (x, z)$. One can easily see that $\operatorname{depth} R/\mathfrak{a} = 1$. However, $\mathfrak{a}^2 = (x, y)^2 \cap (y, z)^2 \cap (x, z)^2 \cap (x^2, y^2, z^2)$, and consequently $\operatorname{depth} R/\mathfrak{a}^2 = 0$.

On the other hand, $\varprojlim_n H^0_{\mathfrak{m}}(R/\mathfrak{a}^n) = 0$. Then we have $\operatorname{fgrade}(\mathfrak{a}, R) = 1$.

In view of the above results we state the next definition.

Definition 2.4 Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$. We define a nonnegative integer $\operatorname{dg}(\mathfrak{a})$ to measure the distance between $\operatorname{fgrade}(\mathfrak{a}, R)$ and the lower bound of $\operatorname{depth} R/\mathfrak{a}^n$, $n \in \mathbb{N}$, *i.e.,*

$$\operatorname{dg}(\mathfrak{a}) := \operatorname{fgrade}(\mathfrak{a}, R) - \min_n \operatorname{depth} R/\mathfrak{a}^n.$$

It should be noted that the stability of $\operatorname{depth} R/\mathfrak{a}^n$ was established by Brodmann (*cf.* [4]).

Inspired by Remark 2.1, $\operatorname{fgrade}(\mathfrak{a}, R) \leq \dim(\widehat{R}/\mathfrak{a}\widehat{R} + \mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Ass} \widehat{R}$. It can be a suitable upper bound to control the formal grade of $\mathfrak{a}$ and $\min_n \operatorname{depth} R/\mathfrak{a}^n$ as well. It is clear that if $\operatorname{Rad}(\mathfrak{a}\widehat{R} + \mathfrak{p}) = \mathfrak{m}\widehat{R}$ for some $\mathfrak{p} \in \operatorname{Ass} \widehat{R}$, then $\min_n \operatorname{depth} R/\mathfrak{a}^n = \operatorname{fgrade}(\mathfrak{a}, R) = 0$ and consequently $\operatorname{dg}(\mathfrak{a}) = 0$.

Example 2.5 Let $R = k[[x, y, z]]/(xy, xz)$ and $\mathfrak{a} := (x, y)$. One can see that $\operatorname{fgrade}(\mathfrak{a}, R) = 0$, and consequently $\operatorname{dg}(\mathfrak{a}) = 0$.

Proposition 2.6 *Let $\mathfrak{a}$ be an ideal of a Cohen–Macaulay local ring $(R, \mathfrak{m})$.*

(i) *If $\operatorname{dg}(\mathfrak{a}) = 0$, then the following are equivalent:*
   (a) $\operatorname{ht}(\mathfrak{a}) = \operatorname{cd}(\mathfrak{a}, R)$*;*
   (b) $\mathfrak{a}$ *is a set-theoretic complete intersection ideal.*

(ii) *Suppose that $\operatorname{dg}(\mathfrak{a}) = 1$. Then $l(\mathfrak{a}) \neq \dim R - \min_n \operatorname{depth} R/\mathfrak{a}^n$ if and only if $\operatorname{cd}(\mathfrak{a}, R) = \operatorname{ara}(\mathfrak{a}) = l(\mathfrak{a})$.*

Proof (i) In the case where $\operatorname{dg}(\mathfrak{a}) = 0$, the inequalities

$$\operatorname{ht}(\mathfrak{a}) \leq \operatorname{cd}(\mathfrak{a}, R) \leq l(\mathfrak{a}) \leq \dim R - \operatorname{fgrade}(\mathfrak{a}, R)$$

hold. Moreover, if R is a Cohen-Macaulay ring, then in light of Remark 2.1(b) and (1.1), one has

$$\begin{aligned}\operatorname{ht}(\mathfrak{a}) \leq \operatorname{cd}(\mathfrak{a}, R) \leq l(\mathfrak{a}) &\leq \dim R - \min_n \operatorname{depth} R/\mathfrak{a}^n \\ &= \dim R - \operatorname{fgrade}(\mathfrak{a}, R) = \operatorname{cd}(\mathfrak{a}, R).\end{aligned}$$

https://doi.org/10.4153/CMB-2014-022-7 Published online by Cambridge University Press

By virtue of (1.1) and in conjunction with the above equalities, statements (a) and (b) are equivalent.

(ii) Assume that $l(\mathfrak{a}) \neq \dim R - \min_n \operatorname{depth} R/\mathfrak{a}^n$. Then, by assumption, we have

$$\begin{aligned}\operatorname{cd}(\mathfrak{a}, R) &\leq \operatorname{ara}(\mathfrak{a}) \leq l(\mathfrak{a}) < \dim R - \min \operatorname{depth} R/\mathfrak{a}^n \\ &= \dim R - \operatorname{fgrade}(\mathfrak{a}, R) + 1 = \operatorname{cd}(\mathfrak{a}, R) + 1.\end{aligned}$$

Now the claim is clear.

For the reverse implication, assume that $l(\mathfrak{a}) = \dim R - \min \operatorname{depth} R/\mathfrak{a}^n$. If this is the case, then $l(\mathfrak{a}) = \dim R - \operatorname{fgrade}(\mathfrak{a}, R) + 1 = \operatorname{cd}(\mathfrak{a}, R) + 1$, which is a contradiction. ■

Example 2.7 Let $R = k[[x_1, x_2, x_3, x_4]]$ be the formal power series ring over a field k in four variables and let $\mathfrak{a} = (x_1, x_2) \cap (x_3, x_4)$. Clearly one can see that $\dim R/\mathfrak{a} = 2$, $\operatorname{fgrade}(\mathfrak{a}, R) = 1$, and by virtue of [19, Lemma 2] $\min_n \operatorname{depth} R/\mathfrak{a}^n = 1$, *i.e.*, $\operatorname{dg}(\mathfrak{a}) = 0$. On the other hand, $\operatorname{ht}(\mathfrak{a}) = 2$ and $\operatorname{cd}(\mathfrak{a}, R) = 3$. By a Mayer–Vietoris sequence, one can see that $H^3_{\mathfrak{a}}(R) \neq 0$, that is $\operatorname{ara}(\mathfrak{a}) = 3 = l(\mathfrak{a})$.

For a prime ideal $\mathfrak{p}$ of R, the n-th symbolic power of $\mathfrak{p}$ is denoted by $\mathfrak{p}^{(n)} = \mathfrak{p}^n R_{\mathfrak{p}} \cap R$. The following theorem, gives conditions at which the required equality (1.1) is provided.

Theorem 2.8 *Let $\mathfrak{p}$ be a prime ideal of a Cohen–Macaulay local ring $(R, \mathfrak{m})$ with* $\operatorname{fgrade}(\mathfrak{p}, R) \leq 1$.

(i) *If $\mathfrak{p}^{(n)} = \mathfrak{p}^n$, for all n, then* $l(\mathfrak{p}) = \operatorname{cd}(\mathfrak{p}, R) = \dim R - 1$.

(ii) *If* $l(\mathfrak{p}) = \dim R - 1$ *and* $\operatorname{ht}(\mathfrak{p}) = \operatorname{cd}(\mathfrak{p}, R)$, *then $\mathfrak{p}$ is a set-theoretic complete intersection.*

Proof (i) As $\mathfrak{p}^{(n)} = \mathfrak{p}^n$, for all n, all of prime divisors of $\mathfrak{p}^n$ are minimal for all n, that is $\operatorname{depth} R/\mathfrak{p}^n > 0$. On the other hand, Proposition 2.2 implies that $\min \operatorname{depth} R/\mathfrak{p}^n = \operatorname{fgrade}(\mathfrak{p}, R) = 1$, so the claim follows. To this end, note that

$$\operatorname{cd}(\mathfrak{p}, R) \leq l(\mathfrak{p}) \leq \dim R - 1 = \dim R - \operatorname{fgrade}(\mathfrak{p}, R) = \operatorname{cd}(\mathfrak{p}, R).$$

(ii) As $\operatorname{fgrade}(\mathfrak{p}, R) \leq 1$ and R is a Cohen–Macaulay local ring, we have $\operatorname{cd}(\mathfrak{p}, R) = \dim R - \operatorname{fgrade}(\mathfrak{p}, R) \geq \dim R - 1$. Hence,

$$\dim R - 1 \leq \operatorname{cd}(\mathfrak{p}, R) = \operatorname{ht}(\mathfrak{p}) \leq l(\mathfrak{p}) = \dim R - 1.$$

It follows that $\mathfrak{p}$ is a set-theoretic complete intersection ideal. ■

Since $\operatorname{fgrade}(\mathfrak{p}, R) \leq \dim R/\mathfrak{p}$, one can get the following corollary of Theorem 2.8.

Corollary 2.9 *Let $\mathfrak{p}$ be a one-dimensional prime ideal of a Cohen-Macaulay local ring $(R, \mathfrak{m})$. Then* (i) *implies* (ii), *and* (ii) *implies* (iii).

(i) $\mathfrak{p}^{(n)} = \mathfrak{p}^n$, *for all n.*

(ii) $l(\mathfrak{p}) = \dim R - 1$.

(iii) *$\mathfrak{p}$ is a set-theoretic complete intersection.*

https://doi.org/10.4153/CMB-2014-022-7 Published online by Cambridge University Press

It should be noted that with some extra assumptions, Cowsik and Nori [9, Proposition 3] have shown that the conditions in Corollary 2.9 are equivalent for $\mathfrak{p}$ generated by an R-sequence.

3 Case One: The Ring of Positive Characteristic

Let p be a prime number and R a commutative Noetherian ring of characteristic p. The Frobenius endomorphism of R is the map $\varphi\colon R \to R$, where $\varphi(r) = r^p$. Let $\mathfrak{a} = (x_1, \dots, x_n)$ be an ideal of R. Then $\mathfrak{a}^{[p^e]}$ is the e-th Frobenius power of $\mathfrak{a}$, defined by $\mathfrak{a}^{[p^e]} = (x_1^{p^e}, \dots, x_n^{p^e})R$. Then $\mathfrak{a}^{np^e} \subseteq \mathfrak{a}^{[p^e]} \subseteq \mathfrak{a}^{p^e}$; *i.e.*, $\mathfrak{a}^{[p^e]}$ and $\mathfrak{a}^{p^e}$ have the same radical (*cf.* [6]).

Peskine and Szpiro [17, Chap. 3, Proposition 4.1] proved that for a regular local ring R of characteristic $p > 0$ and an ideal $\mathfrak{a}$ of R, if $R/\mathfrak{a}$ is a Cohen–Macaulay ring, then $\operatorname{ht}(\mathfrak{a}) = \operatorname{cd}(\mathfrak{a}, R)$. Below (see Proposition 3.2), we give a generalization of their result.

Remark 3.1 Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$. Then the following inequality holds:

$$\operatorname{depth} R/\mathfrak{a} \le \operatorname{fgrade}(\mathfrak{a}, R) \le \dim R/\mathfrak{a}.$$

Proof It is known that $\operatorname{fgrade}(\mathfrak{a}, R) \le \dim R/\mathfrak{a}$ (*cf.* Section 2). By what we have seen above, depth of $R/\mathfrak{a}$ is the same as the depth of every iteration of it. Put $l := \operatorname{depth} R/\mathfrak{a} = \operatorname{depth} R/\mathfrak{a}^{[p^e]}$ for each integer e. Since $H^i_{\mathfrak{m}}(R/\mathfrak{a}^{[p^e]})$ is zero for all $i < l$, so is

$$\varprojlim H^i_{\mathfrak{m}}(R/\mathfrak{a}^{p^e}) = \varprojlim H^i_{\mathfrak{m}}(R/\mathfrak{a}^{[p^e]})$$

(*cf.* [18, Lemma 3.8]). Hence, $l \le \operatorname{fgrade}(\mathfrak{a}, R)$. Therefore we get the desired inequality. ■

Note that in the case where R is a Cohen–Macaulay local ring (not necessarily of positive characteristic), then in the light of Remark 2.1(b), the following statement holds:

$$\operatorname{ht}(\mathfrak{a}) = \operatorname{cd}(\mathfrak{a}, R) \text{ if and only if } \operatorname{fgrade}(\mathfrak{a}, R) = \dim R/\mathfrak{a}.$$

Proposition 3.2 *Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$. Then the following statements are equivalent:*

(i) *$R/\mathfrak{a}$ is a Cohen–Macaulay ring;*

(ii) *$\operatorname{ht}(\mathfrak{a}) = \operatorname{cd}(\mathfrak{a}, R)$ and $H^s_{\mathfrak{m}}(R/\mathfrak{a}^{[p^{e+1}]}) \to H^s_{\mathfrak{m}}(R/\mathfrak{a}^{[p^e]})$ is epimorphism for each integer e, where $s := \operatorname{depth} R/\mathfrak{a}$.*

Proof (i) $\Rightarrow$ (ii) As $R/\mathfrak{a}$ is a Cohen–Macaulay ring, by assumption every iteration of $R/\mathfrak{a}$ is again a Cohen–Macaulay ring. Hence, $H^i_{\mathfrak{m}}(R/\mathfrak{a}^{[p^e]})$ is zero for all $i < \dim R/\mathfrak{a}$, then so is

$$\varprojlim H^i_{\mathfrak{m}}(R/\mathfrak{a}^{[p^e]}) \cong \varprojlim H^i_{\mathfrak{m}}(R/\mathfrak{a}^{p^e})$$

for all $i < \dim R/\mathfrak{a}$ (*cf.* [18, Lemma 3.8]).

https://doi.org/10.4153/CMB-2014-022-7 Published online by Cambridge University Press

By virtue of [18, Remark 3.6], one can see that $H_{\mathfrak{a}}^{\dim R-i}(R)=0$ for all $\dim R-i>\operatorname{ht}(\mathfrak{a})$, *i.e.,* $\operatorname{ht}(\mathfrak{a})=\operatorname{cd}(\mathfrak{a},R)$. The second part of the claim follows by Hartshorne's non-vanishing Theorem, since $\operatorname{depth} R/\mathfrak{a}=\dim R/\mathfrak{a}$.

(ii) $\Rightarrow$ (i) Assume that $\operatorname{ht}(\mathfrak{a})=\operatorname{cd}(\mathfrak{a},R)$. Then $\operatorname{fgrade}(\mathfrak{a},R)=\dim R/\mathfrak{a}$. If we can prove that $\operatorname{depth} R/\mathfrak{a}\geq \operatorname{fgrade}(\mathfrak{a},R)$, we are done. Consider the epimorphism of nonzero R-modules for each e:

$$H_{\mathfrak{m}}^{s}(R/\mathfrak{a}^{[p^{e+1}]})\to H_{\mathfrak{m}}^{s}(R/\mathfrak{a}^{[p^{e}]})\to 0.$$

Hence, [22, Lemma 3.5.3] implies that $\operatorname{fgrade}(\mathfrak{a},R)\leq \operatorname{depth} R/\mathfrak{a}$. This completes the proof. ■

4 Case Two: The Polynomial Ring

Throughout this section, assume that $R=k[x_1,\dots,x_n]$ is a polynomial ring in n variables $x_1,\dots,x_n$ over a field k. Let $S:=k[x_1,\dots,x_n]_{(x_1,\dots,x_n)}$ be the local ring and let I be a square free monomial ideal of S.

Proposition 4.1 *Let S and I be as above. Then the following are true.*

(i)
$$H_I^i(S)=0\Longleftrightarrow \varprojlim_t H_{\mathfrak{m}}^{n-i}(S/I^t)=0\Longleftrightarrow H_{\mathfrak{m}}^{n-i}(S/I)=0,$$
for a given integer i. In particular, S/I is a Cohen–Macaulay ring if and only if $\varprojlim_t H_{\mathfrak{m}}^{j}(S/I^t)=0$ *for all* $j<n-\operatorname{ht} I$.

(ii) *If* $\operatorname{ht}(\mathfrak{a})=\operatorname{cd}(\mathfrak{a},R)$, *then* $R/\mathfrak{a}$ *is a Cohen–Macaulay ring, provided that* $\mathfrak{a}$ *is a squarefree monomial ideal of R.*

Proof

(i) By virtue of [18, Remark 3.6], $H_I^i(S)=0$ if and only if $\varprojlim_t H_{\mathfrak{m}}^{n-i}(S/I^t)=0$. On the other hand, by virtue of [20, Corollary 4.2], we have $H_I^i(S)=0$ if and only if $H_{\mathfrak{m}}^{n-i}(S/I)=0$. The second assertion follows easily from the first one.

(ii) Without loss of generality, we may assume that R is a local ring with the graded maximal ideal $\mathfrak{m}=(x_1,\dots,x_n)$. Now the claim follows by part (i). ■

The next result provides as a consequence an upper bound for the depth S/I^l for each $l\geq 1$. Moreover, the second part of the next result was proved by Lyubeznik [16].

Corollary 4.2 *Let R, S and I be as above.*

(i) $\operatorname{depth} S/I=\operatorname{fgrade}(I,S)$ *holds.*

(ii) *Assume that* $\mathfrak{a}$ *is a squarefree monomial ideal in R. Then* $\operatorname{pd}_R R/\mathfrak{a}=\operatorname{cd}(\mathfrak{a},R)$.

Proof Assume that $\operatorname{fgrade}(I,S):=t$. Then for all $i<t$ we have $\varprojlim_t H_{\mathfrak{m}}^{i}(S/I^t)=0$ if and only if $H_{\mathfrak{m}}^{i}(S/I)=0$ (*cf.* Proposition 4.1). Hence, $t\leq \operatorname{depth} S/I$. On the other hand, assume that $\operatorname{depth} S/I:=s$. Again using Proposition 4.1 we have $s\leq \operatorname{fgrade}(I,S)$, as desired.

In order to prove the second part, note that both $\operatorname{pd}_R R/\mathfrak{a}$ and $\operatorname{cd}(\mathfrak{a},R)$ are finite. Since $\operatorname{pd} R/\mathfrak{a}=\operatorname{pd} R_{\mathfrak{m}}/\mathfrak{a}R_{\mathfrak{m}}$ and $\operatorname{cd}(\mathfrak{a},R)=\operatorname{cd}(\mathfrak{a}R_{\mathfrak{m}},R_{\mathfrak{m}})$, with $\mathfrak{m}=(x_1,\dots,x_n)$,

https://doi.org/10.4153/CMB-2014-022-7 Published online by Cambridge University Press

then without loss of generality, we may assume that R is a local ring with the homogeneous maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. Now, by the Auslander–Buchsbaum formula and the first part, one can get the claim. To this end note that

$$\operatorname{pd}_R R/\mathfrak{a} = \operatorname{depth} R - \operatorname{depth} R/\mathfrak{a} = \dim R - \operatorname{fgrade}(\mathfrak{a}, R) = \operatorname{cd}(\mathfrak{a}, R). \qquad \blacksquare$$

In the light of Corollary 4.2, it is noteworthy to mention that for a squarefree monomial ideal I, we have

$$\operatorname{depth} S/I^l \leq \operatorname{fgrade}(I, S)$$

for all positive integers l. Notice that $\operatorname{depth} S/I^l \leq \operatorname{depth} S/I$ for all positive integers l; see for example [13].

Corollary 4.3 *Let $R = k[x_1, \dots, x_n]$ be a polynomial ring in n variables $x_1, \dots, x_n$ over a field k and $\mathfrak{a}$ be a squarefree monomial ideal of R. Then the following are equivalent:*

(i) $H^i_{\mathfrak{a}}(R) = 0$ *for all* $i \neq \operatorname{ht} \mathfrak{a}$, i.e., $\mathfrak{a}$ *is cohomologically a complete intersection ideal;*
(ii) $R/\mathfrak{a}$ *is a Cohen–Macaulay ring.*

Proof Since each of the modules in question is graded, the issue of vanishing is unchanged under localization at the homogeneous maximal ideal of R. Hence, the claim follows by Proposition 4.1. ■

Let $\overline{x_1}, \dots, \overline{x_n}$ be the image of the regular sequence $x_1, \dots, x_n$ in S. Let $k, l \leq n$ be arbitrary integers. For all $i = 1, \dots, k$ set $I_i := (\overline{x_{i_1}}, \dots, \overline{x_{i_{r_i}}})$, where the elements $\overline{x_{i_j}}$, $1 \leq j \leq r_i \leq l$ are from the set $\{\overline{x_1}, \dots, \overline{x_n}\}$ and a squarefree monomial ideal I is as follows

$$I = I_1 \cap I_2 \cap \cdots \cap I_k,$$

where the set of basis elements of the I_i are disjoint.

Proposition 4.4 *Let I be as above. Then* $\operatorname{cd}(I, S) = \sum_{i=1}^k r_i - k + 1$, *and in particular,* $\operatorname{dg}(I) = 0$.

Proof By virtue of [19, Lemma 2], $\operatorname{depth} S/I = \operatorname{depth} S - \sum_{i=1}^k r_i + k - 1$. As $\operatorname{depth} S/I = \operatorname{fgrade}(I, S) = \dim S - \operatorname{cd}(I, S)$ (*cf.* Corollary 4.2 and Remark 2.1), the claim is clear. ■

Acknowledgments I would like to thank Professors Markus Brodmann, Jürgen Herzog, and Moty Katzman for useful discussions on $\min_n \operatorname{depth} R/\mathfrak{a}^n$ during the preparation of this paper. My thanks are due to Professor Peter Schenzel for comments on Proposition 4.1. I am also grateful to the reviewer for suggesting several improvements to the manuscript.

References

[1] M. Asgharzadeh and K. Divaani-Aazar, *Finiteness properties of formal local cohomology modules and Cohen-Macaulayness.* Comm. Algebra **39**(2011), no. 3, 1082–1103. http://dx.doi.org/10.1080/00927871003610312

[2] M. Barile, *On the number of equations defining certain varieties.* Manuscripta Math. **91**(1996), 483–494. http://dx.doi.org/10.1007/BF02567968

[3] ———, *A note on monomial ideals.* Arch. Math. **87**(2006), no. 6, 516–521. http://dx.doi.org/10.1007/s00013-006-1834-3

[4] M. Brodmann, *The asymptotic nature of the analytic spread.* Math. Proc. Cambridge Philos. Soc. **86**(1979), no. 1, 35–39. http://dx.doi.org/10.1017/S030500410000061X

[5] M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications.* Cambridge Studies in Advanced Mathematics, 60, Cambridge University Press, Cambridge, 1998.

[6] W. Bruns and J. Herzog, *Cohen-Macaulay rings.* Cambridge Studies in Advanced Mathematics, 39, Cambridge University Press, Cambridge, 1993.

[7] L. Burch, *Codimension and analytic spread.* Proc. Cambridge Philos. Soc. **72**(1972), 369–373. http://dx.doi.org/10.1017/S0305004100047198

[8] R. C. Cowsik, *Symbolic powers and numbers of defining equations.* In: Algebra and its applications (New Delhi, 1981), Lecture Notes in Pure and Applied Math., 91, Dekker, New York, 1984, pp. 13–14.

[9] R. C. Cowsik and M. V. Nori, *On the fibers of blowing up.* J. Indian Math. Soc. (N.S.) **40**(1976), no. 1–4, 217–222.

[10] M. Eghbali, *On Artinianness of formal local cohomology, colocalization and coassociated primes.* Math. Scand. **113**(2013), no. 1, 5–19.

[11] D. Eisenbud and C. Huneke, *Cohen-Macaulay Rees algebras and their specialization.* J. Algebra **81**(1983), no. 1, 202–224. http://dx.doi.org/10.1016/0021-8693(83)90216-8

[12] M. Hellus and P. Schenzel, *On cohomologically complete intersections.* J. Algebra **320**(2008), no. 10, 3733–3748. http://dx.doi.org/10.1016/j.jalgebra.2008.09.006

[13] J. Herzog, Y. Takayama, and N. Terai, *On the radical of a monomial ideal.* Arch. Math. **85**(2005), no. 5, 397–408. http://dx.doi.org/10.1007/s00013-005-1385-z

[14] K. Kimura, N. Terai, and K. I. Yoshida, *Arithmetical rank of squarefree monomial ideals of small arithmetic degree.* J. Algebraic Combin. **29**(2009), no. 3, 389–404. http://dx.doi.org/10.1007/s10801-008-0142-3

[15] G. Lyubeznik, *A survey of problems and results on the number of defining equations.* In: Commutative algebra (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ., 15, Springer, New York, 1989, pp. 375–390.

[16] ———, *On the local cohomology modules $H^i_{\mathfrak{a}}(R)$ for ideals $\mathfrak{a}$ generated by monomials in an R-sequence.* In: Complete intersections (Acireale, 1983), Lecture Notes in Math., 1092, Springer, Berlin, 1984.

[17] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck.* Inst. Hautes Études Sci. Publ. Math. **42**(1973),47–119.

[18] P. Schenzel, *On formal local cohomology and connectedness.* J. Algebra **315**(2007), no. 2, 894–923. http://dx.doi.org/10.1016/j.jalgebra.2007.06.015

[19] P. Schenzel and W. Vogel, *On set-theoretic intersections.* J. Algebra **48**(1977), no. 2, 401–408. http://dx.doi.org/10.1016/0021-8693(77)90317-9

[20] A. Singh and U. Walter, *Local cohomology and pure morphisms.* Illinois J. Math. **51**(2007), no. 1, 287–298.

[21] N. V. Trung and S. Ikeda, *When is the Rees algebra Cohen-Macaulay?* Comm. Algebra **17**(1989), no. 12, 2893–2922. http://dx.doi.org/10.1080/00927878908823885

[22] C. A. Weibel, *An introduction to homological algebra.* Cambridge Studies in Advanced Mathematics, 38, Cambridge University Press, Cambridge, 1994.

[23] Z. Yan, *An étale analog of the Goresky-Macpherson formula for subspace arrangements.* J. Pure Appl. Algebra **146**(2000), no. 3, 305–318. http://dx.doi.org/10.1016/S0022-4049(98)00128-5

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran-Iran
e-mail: m.eghbali@yahoo.com

https://doi.org/10.4153/CMB-2014-022-7 Published online by Cambridge University Press