ZERO TRACTS OF BLASCHKE PRODUCTS

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1. Introduction. Let $\{a_n\}$ be a sequence of complex numbers such that

$$0 < |a_n| < 1 \qquad (n = 1, 2, 3, \ldots)$$

and

$$\sum_{n=1}^{\infty} (1 - |a_n|) < \infty.$$

Then $\{a_n\}$ is called a *Blaschke sequence*. For each Blaschke sequence $\{a_n\}$ a *Blaschke product* is defined as

$$B(z) = B(z, \{a_n\}) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z}.$$

Thus a Blaschke product $B(z, \{a_n\})$ is a function regular in the open unit disk $D = \{z: |z| < 1\}$ and having a zero at each point of the sequence $\{a_n\}$.

Let $\mathfrak{C}$ be the family of all continuous curves in D each of whose members is defined in the form

$$z = z(t), \qquad 0 < t < 1,$$

where $z(t)$ is a continuous function of t,

$$|z(t)| < 1, \quad \lim_{t \to 0+0} |z(t)| < 1, \quad \text{and} \quad \lim_{t \to 1-0} |z(t)| = 1.$$

Clearly, each member of $\mathfrak{C}$ has at least one limit point on the circumference $C = \{z: |z| = 1\}$. Now suppose that $B(z, \{a_n\})$ is a given Blaschke product. Then we define a *zero tract* of $B(z, \{a_n\})$ as a curve Γ belonging to $\mathfrak{C}$ such that

$$\lim_{t \to 1-0} B(z(t), \{a_n\})$$

exists and is zero.

It is well known that there exist Blaschke products that do not have any zero tracts. In particular, any Blaschke product for which $\{a_n\}$ is a finite set of points has this property. Moreover, the number of zero tracts is limited by the fact that for almost all values θ in $[0, 2\pi)$ a Blaschke product $B(z, \{a_n\})$ must tend to a limit of modulus 1 as z tends to $e^{i\theta}$ in any Stolz angle

$$\{z: |\arg(e^{i\theta} - z)| < \delta < \tfrac{1}{2}\pi, 0 < |e^{i\theta} - z| < \sigma\}.$$

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However, Frostman **(2)** has shown that the Blaschke product $B(z, \{a_n\})$, where

$$a_n = 1 - \frac{1}{(n+1)^2}$$

for each positive integer n, has the curve $\{z\colon z = t,\ 0 < t < 1\}$ as a zero tract. Other Blaschke products are known to have a much greater set of zero tracts of a more complicated nature; cf. **(3)**.

In this paper we concern ourselves with the family $\mathfrak{C}$ of continuous curves in D and pose the following question. Given any arbitrary curve Γ of $\mathfrak{C}$, is there a Blaschke product for which Γ is a zero tract?

If Γ has more than one limit point on C, the answer to this question is known. For Blaschke products are analytic and bounded in D and functions of this type do not tend to any limit along Γ unless they are constant in D. Thus in this case the answer to the above question is in the negative. On the other hand, we shall prove below that if Γ has only one limit point on C, then the answer is in the affirmative. Stated precisely, our result takes the following form:

THEOREM 1. *Let Γ be a curve defined by the equation*

$$z = z(t), \qquad 0 < t < 1,$$

where $z(t)$ is a continuous function of t,

$$|z(t)| < 1, \quad \lim_{t\to 0+0} |z(t)| < 1, \quad \textit{and} \quad \lim_{t\to 1-0} z(t) = e^{i\theta}.$$

Then there exists a Blaschke product $B(z)$ such that

$$\lim_{t\to 1-0} B(z(t))$$

exists and is zero.

The rest of this paper will be concerned with the proof of Theorem 1 and an extension which states that, if $\mathfrak{C}'$ is the subset of $\mathfrak{C}$ consisting of those curves Γ which have just one limit point on C, then for each countable subset $\{\Gamma_n\}$ of $\mathfrak{C}'$ there is a Blaschke product $B(z)$ which tends to zero, as $|z| \to 1$ along any given member of $\{\Gamma_n\}$.

2. Preliminaries. For the proof of Theorem 1 we may, without any essential loss of generality, take the only limit point of Γ on C to be the point $z = 1$. Then using the standard polar representation of the complex plane, we define a function $f(\theta)$ as follows. Let

$$(1)\quad f(\theta) = \begin{cases} \max(\sup\{|z|\colon \arg z \geqslant \theta;\ z \in \Gamma\},\ 1-\theta) & \text{when } 0 < \theta < \tfrac{1}{4}\pi, \\ 1 & \text{when } \theta = 0, \\ \max(\sup\{|z|\colon \arg z \leqslant \theta;\ z \in \Gamma\},\ 1+\theta) & \text{when } -\tfrac{1}{4}\pi < \theta < 0, \end{cases}$$

where in accordance with the standard notation we take the supremum of the empty set to be $-\infty$. As an immediate consequence of this definition we have

$$0 < f(\theta) < 1 \qquad \text{when } 0 < |\theta| < \tfrac{1}{4}\pi.$$

Now let a region A be defined by

$$A = \bigcup_{|\theta|<\frac{1}{4}\pi} \{z : z = re^{i\theta};\ \ 0 \leqslant r \leqslant f(\theta)\}. \tag{2}$$

Then there exists a fixed real number t_0 such that $0 < t_0 < 1$ and for which the set

$$\Gamma_0 = \{z\colon z = z(t);\, t_0 \leqslant t < 1\}$$

is a subset of A. We define sets $L(\theta)$ such that

$$L(\theta) = \begin{cases} \{z\colon \arg z = \theta, f(\theta+0) \leqslant |z| \leqslant f(\theta-0)\} & \text{when } 0 < \theta < \frac{1}{4}\pi, \\ \{z\colon \arg z = \theta, f(\theta-0) \leqslant |z| \leqslant f(\theta+0)\} & \text{when } -\frac{1}{4}\pi < \theta < 0, \end{cases} \tag{3}$$

and we form the sets

$$\Gamma_1 = \bigcup_{0<\theta<\frac{1}{4}\pi} L(\theta), \qquad \Gamma_2 = \bigcup_{-\frac{1}{4}\pi<\theta<0} L(\theta).$$

It follows from the definitions that Γ_1 and Γ_2 are in the closed upper and lower half-planes respectively and that $\Gamma_1 \cup \{1\} \cup \Gamma_2$ includes that part of the boundary of A for which $|\theta| < \frac{1}{4}\pi$.

We shall prove Theorem 1 by constructing Blaschke products which tend to zero along each of Γ_1 and Γ_2 respectively. It will suffice to make the construction for Γ_1 and then deduce the corresponding result for Γ_2 by analogy. Thus, although Lemmas 1, 2, and 3 refer to Γ_1, these lemmas have obvious counterparts which can be applied to Γ_2.

3. Two lemmas.

In this section we describe the nature of Γ_1 by means of two lemmas.

LEMMA 1. *Let c be any non-zero complex number belonging to Γ_1 and ρ a real number such that $0 < \rho < |1 - c|$. Then if $C(c, \rho) = \{z\colon |z - c| = \rho\}$, there exists one and only one complex number k belonging to $C(c, \rho) \cap \Gamma_1$ such that* $\arg k \leqslant \arg c$ *and* $|k| \geqslant |c|$.

Proof. Let

$$\phi = \inf\{\theta\colon z = re^{i\theta} \in \Gamma_1;\, z \in K(c, \rho)\} \tag{4}$$

where $K(c, \rho) = \{z\colon |z - c| \leqslant \rho\}$. Then the point $f(\phi + 0)e^{i\phi}$ belongs to $K(c, \rho)$ since it is a limit point of points contained in the compact set $K(c, \rho)$. We also remark here that if $f(\phi + 0)e^{i\phi}$ lies on the boundary of $K(c, \rho)$, there would appear to be two possible positions for this point. However, since $f(\theta)$ is a monotonic decreasing function of θ, $|f(\phi + 0)| \geqslant c$ and a simple geometric argument shows that $f(\phi + 0)e^{i\phi}$ must be located at that position which is farther from the origin.

On the other hand, if $0 < \theta < \phi$ and $z = re^{i\theta} \in \Gamma_1$, then $z \notin K(c, \rho)$. Therefore $f(\phi - 0)e^{i\phi}$, being a limit point of points outside $K(c, \rho)$, is itself outside $K(c, \rho)$ or on its boundary. Thus $L(\phi)$ has one and only one point of contact with $C(c, \rho)$ and hence $C(c, \rho) \cap \Gamma_1$ contains at least one point k such that $\arg k \leqslant \arg c$ and $|k| \geqslant |c|$.

Finally we must show that k is unique. The definition (4) of ϕ and the proof above shows that any point k_1, other than k, which satisfies the requirements of Lemma 1 must also satisfy the inequalities

$$\arg c \geqslant \arg k_1 > \arg k. \tag{5}$$

But, since $k_1 \in C(c, \rho)$ and $|k_1| \geqslant |c|$, the inequality (5) implies that $|k_1| > |k|$. Further, since $k_1 \in \Gamma_1$, the inequality (5) implies that $|k_1| \leqslant |k|$. Hence we have a contradiction and the proof of Lemma 1 is complete.

LEMMA 2. *Let $z_0 = r_0 e^{i\theta_0}$ be any non-zero point belonging to Γ_1. Then the set*

$$\gamma_1 = \{z\colon |z| \geqslant |z_0|, \arg z \leqslant \arg z_0, z \in \Gamma_1\}$$

is the image of the interval $[0, |1 - z_0|)$ by a continuous mapping $z = z(s)$ where $|z(s)| < 1$ and

$$\lim_{s \to |1-z_0|-0} z(s) = 1.$$

Proof. For any number s in $[0, |1 - z_0|)$ let $z(s)$ denote that point on γ_1 which satisfies $|z_0 - z(s)| = s$, $\arg z(s) \leqslant \arg z_0$, $|z(s)| \geqslant |z_0|$. The point $z(s)$ exists and is unique by Lemma 1. Then $z = z(s)$ maps the interval $[0, |1 - z_0|)$ onto γ_1 and satisfies the relations $|z(s)| < 1$ and

$$\lim_{s \to |1-z_0|-0} z(s) = 1.$$

Now $|z(s)|$ is a monotonic increasing function of s and, since $|z(s)| = r$ always has a solution for each r in $[|z_0|, 1)$, it follows that $|z(s)|$ is continuous on $[0, |1 - z_0|)$. Similarly $\arg z(s)$ is also continuous on $[0, |1 - z_0|)$ and the continuity of $z(s)$ itself follows immediately. This completes the proof of Lemma 2.

4. Proof of Theorem 1. Before proving Theorem 1, we find that it is convenient to prove a special case of the theorem which is stated below as Lemma 3. First, however, we recall a known result (3) which will be required for the proof of the lemma.

THEOREM A. *Let $\{a_n\}$ be a Blaschke sequence which contains a subsequence $\{\alpha_m\}$ tending to $e^{i\theta}$ in such a manner that*

$$\lim_{m \to \infty} \frac{|\alpha_m - \alpha_{m+1}|}{1 - |\alpha_m|} = 0$$

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and none of the closed disks

$$K_m = \{z\colon |z - \alpha_m| \leqslant |\alpha_m - \alpha_{m-1}|\}, \qquad m = 1, 2, 3, \ldots,$$

intersects C. Then the corresponding Blaschke product $B(z, \{a_n\})$ *tends to* 0 *as* z *tends to* $e^{i\theta}$, z *being confined to the set* $\bigcup_{m=1}^{\infty} K_m$.

LEMMA 3. *There exists a Blaschke sequence* $\{a_n\}$ *lying on* Γ_1 *such that the corresponding Blaschke product* $B(z, \{a_n\})$ *tends to* 0 *as* z *tends to* 1 *along* Γ_1.

Proof. The sequence $\{a_n\}$ is obtained by a construction based on induction.

Let γ be a fixed real number in (0, 1) and let a_1 be an arbitrary point of Γ_1. Then if a_n has been determined, we select a_{n+1} to be the complex number that lies on Γ_1 and on

$$C_n = \{z\colon |z - a_n| = (1 - |a_n|)(\arg a_n)^\gamma\} \tag{6}$$

and satisfies the inequalities $\arg a_{n+1} \leqslant \arg a_n$ and $|a_{n+1}| \geqslant |a_n|$. Lemma 1 shows that a_{n+1} is uniquely defined. It is immediate that $\{a_n\}$ is an infinite sequence of points converging to the point 1 and satisfying $0 < |a_n| < 1$ for all positive integers n. Since $0 < \arg a_n < \frac{1}{4}\pi$, it is clear that none of the circles defined by (6) intersects the circle C. We must show further that $\{a_n\}$ is a Blaschke sequence in order to justify an application of Theorem A.

By definition of $\{a_n\}$ we have that

$$|a_{n+1} - a_n| = (1 - |a_n|)(\arg a_n)^\gamma$$

when $n \geqslant 1$. Let $a_n = r_n e^{i\theta_n}$ for each n. Then we have

$$\begin{aligned}(1 - r_n)\theta_n^{\gamma} &= \{(r_{n+1} - r_n)^2 + 4r_n r_{n+1} \sin^2 \tfrac{1}{2}(\theta_n - \theta_{n+1})\}^{\frac{1}{2}} \\ &\leqslant \{(r_{n+1} - r_n)^2 + (\theta_n - \theta_{n+1})^2\}^{\frac{1}{2}} \\ &\leqslant (r_{n+1} - r_n) + (\theta_n - \theta_{n+1}).\end{aligned}$$

Now the definition (1) implies that $1 - |a_n| \leqslant \arg a_n$ when $a_n \in \Gamma_1$. Hence from the last displayed inequalities we have that

$$1 - r_n \leqslant \frac{r_{n+1} - r_n}{(1 - r_n)^\gamma} + \frac{\theta_n - \theta_{n+1}}{\theta_n^{\gamma}}. \tag{7}$$

Since $1 - r_n = 1 - |a_n|$ and $\theta_n = \arg a_n$ are both non-increasing functions of n, it follows by consideration of Riemann sums that

$$\sum_{n=1}^{N} \frac{r_{n+1} - r_n}{(1 - r_n)^\gamma} < \int_{r_1}^{1} \frac{dt}{(1 - t)^\gamma} \quad \text{and} \quad \sum_{n=1}^{N} \frac{\theta_n - \theta_{n+1}}{\theta_n^{\gamma}} < \int_0^{\theta_1} \frac{dt}{t^\gamma},$$

for all positive integral values of N. Since both of the right-hand integrals exist, we deduce from (7) that

$$\sum_{n=1}^{\infty} (1 - |a_n|)$$

converges and that $\{a_n\}$ is a Blaschke sequence.

Finally we note that Γ_1 is contained in

$$\bigcup_{n=1}^{\infty} K^{(n)} \quad \text{where} \quad K^{(n)} = \{z: |z - a_n| \leqslant |a_n - a_{n+1}|\}$$

for each positive integer n and that, by definition of a_{n+1},

$$\lim_{n\to\infty} \frac{|a_n - a_{n+1}|}{1 - |a_n|} = \lim_{n\to\infty} |\arg a_n|^{\gamma} = 0.$$

Hence Lemma 3 follows immediately from Theorem A.

In proving Theorem 1, we consider the curve Γ_2 defined in §2 and its reflection

$$\Gamma' = \{z: \bar{z} \in \Gamma_2\}.$$

By applying the proof of Lemma 3 to Γ', we deduce the existence of a Blaschke product $B(z, \{b_n\})$ which tends to zero along Γ'. Hence $B(z, \{\bar{b}_n\})$ is a Blaschke product which tends to zero along Γ_2. But we have constructed $B(z, \{a_n\})$ to tend to zero along Γ_1 and, since all Blaschke products are bounded in D, it follows that

$$B_1(z) = B(z, \{a_n\}) \cdot B(z, \{\bar{b}_n\})$$

is a Blaschke product which tends to zero along Γ_1 and Γ_2.

By Lemma 2, Γ_1 and Γ_2 are continuous curves. Hence an extension of a Theorem of Lindelöf (**1**, p. 460) now asserts that $B_1(z)$ tends to zero uniformly as z tends to 1 in the region A which is defined by (2) and its union with $\Gamma_1 \cup \{1\} \cup \Gamma_2$. Since, by definition, $\Gamma_0 \subset A$ and

$$\Gamma = \{z: z = z(t), 0 < t < t_0\} \cup \Gamma_0,$$

it follows that the function $B_1(z)$ tends to zero as z tends to 1 along Γ. This completes the proof of Theorem 1.

5. Generalizations of Theorem 1. In conclusion it seems to be worth while to mention two possible ways in which Theorem 1 may be generalized.

Firstly we observe that the argument pursued above may be applied not only to prove Theorem 1 for a continuous set Γ but also for any set which is included in D and has one and only one limit point on C.

Secondly we note that an application of the methods of Somadasa (**3**) yields the following generalization of Theorem 1.

THEOREM 2. *Let $\{\Gamma_n\}$ be any sequence of continuous curves contained in D each of which is defined by an equation*

$$z = z_n(t), \qquad 0 < t < 1,$$

where, for each n, $z_n(t)$ is a continuous function of t and Γ_n has one and only one limit point on C. Then there exists a Blaschke product for which each of the curves Γ_n is a zero tract.

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We sketch the proof of Theorem 2 as follows. By Theorem 1, each curve Γ_m corresponds to a Blaschke product $B(z, \{a_n(m)\})$ which tends to zero as $|z| \to 1$ on Γ_m. Let $N(m)$ be chosen so that

$$\sum_{n=N(m)}^{\infty} (1 - |a_n(m)|) < 2^{-m}.$$

Then

$$\bigcup_{m=1}^{\infty} \{a_{n+N(m)}(m)\} = \{\alpha_n\}$$

is a Blaschke sequence and it can be proved **(3)** that $B(z, \{\alpha_n\})$ is a Blaschke product with the required properties.

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