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ON CERTAIN SEQUENCES OF PLUS AND MINUS ONES

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1. Suppose throughout that c is a fixed positive integer, that

$$\alpha = 1 - c + \sqrt{(c^2 + 1)},$$

and that

$$\epsilon_n = (-1)^{[\alpha n]}, \quad S_n = \sum_{k=1}^{n} \epsilon_k, \quad T_n = \sum_{k=1}^{n} S_k \quad \text{for } n = 1, 2, \ldots,$$

where $[x]$ is defined to be the largest integer not exceeding x. The following expansions of α and $\alpha/2$ as simple continued fractions are easily verified:

$$\alpha = \langle 1, 2c, 2c, \ldots \rangle$$

$$\alpha/2 = \begin{cases} \langle 0, 1, 2, 2, \ldots \rangle & \text{if } c = 1 \\ \langle 0, 1, 1, c-1, 1, 1, c-1, \ldots \rangle & \text{if } c > 1. \end{cases}$$

In a recent issue of the *American Mathematical Monthly* [*83*, 1976, No. 7, p. 573] H. Ruderman posed the problem of proving the convergence of the series $\sum_{n=1}^{\infty} \epsilon_n/n$ in the special case $\alpha = \sqrt{2}$, and asked for an estimate of its sum. To prove convergence we note that, by Abel's partial summation formula,

$$(1) \quad \sum_{k=1}^{n} \frac{\epsilon_k}{k} = \sum_{k=1}^{n} \frac{S_k}{k(k+1)} + \frac{S_n}{n+1}.$$

Furthermore we have $S_n = 2e_n - n$, where e_n is the number of positive integers $k \leqq n$ for which $[\alpha k]$ is even or, equivalently, for which the fractional part of $\alpha k/2$ is in the interval $(0, 1/2)$. The familiar result that the sequence $(\alpha n/2)$ is uniformly distributed modulo 1 when α is irrational, yields only that $e_n/n \to 1/2$ as $n \to \infty$, and hence that $S_n = o(n)$; but this is insufficient to establish the convergence of $\sum_{n=1}^{\infty} \epsilon_n/n$. A better estimate of S_n is obtained, however, from a known result on the *discrepancy* of the sequence $(\alpha n/2)$ [**3**, Theorem 3.4, p. 125] which yields

$$(2) \quad |S_n| = 2n \left| \frac{e_n}{n} - \frac{1}{2} \right| \leqq 6 + 2M_c \log n, \quad n \geqq 1,$$

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where

$$M_c = \begin{cases} \dfrac{1}{\log((1+\sqrt{5})/2)} + \dfrac{2}{\log 3} & \text{if } c = 1 \\ \dfrac{1}{\log((1+\sqrt{5})/2)} + \dfrac{c-1}{\log c} & \text{if } c > 1. \end{cases}$$

This estimate together with (1) shows that the series in question is convergent. Our primary object in this paper is to establish the following estimates for the sequence (S_n) and the sequence of its mean values (T_n/n):

(3) $|S_n| < A_c + B_c \log(n + (1/2c)), \quad n \geqq 1,$

where

$$A_c = \frac{\log 2(c - 1 + \sqrt{(c^2+1)})\sqrt{(c^2+1)}}{2\log(c + \sqrt{(c^2+1)})}$$

$$B_c = \frac{1}{2\log(c + \sqrt{(c^2+1)})};$$

and

(4) $0 \leqq -\dfrac{T_n}{n} \leqq \dfrac{1}{2}\left(1 + \dfrac{1}{n}\right), \quad n \geqq 1$

(See Theorems 1 and 2.) These estimates are best possible in a certain sense. (See (5) and the comments following the proof of Theorem 2.) Though (2) is derived from deep results concerning the uniform distribution of the sequence $(\alpha n/2)$, $2M_c$, the coefficient of the term involving $\log n$, tends to infinity as $c \to \infty$, whereas B_c, the corresponding coefficient in (3), tends monotonically to zero as $c \to \infty$. Evidently (4) cannot be derived from either (2) or (3).

In the final section we compare the effectiveness of the estimates (2) and (4) in determining the sum of the series $\sum_{n=1}^{\infty} \epsilon_n/n$. We also show that certain standard summability methods which sum the series $\sum_{n=1}^{\infty} (-1)^n$ fail to sum the series $\sum_{n=1}^{\infty} \epsilon_n$.

2. In this section we investigate some remarkable patterns in the behaviour of the sequences (ϵ_n), (S_n) and (T_n).

We introduce some notation additional to that given in the previous section. Let

$$d = c^2 + 1, \quad \beta = c - 1 + \sqrt{d}, \quad S_0 = 0, \quad T_0 = 0.$$

Let p_k/q_k be the k-th convergent to the continued fraction expansion of α. The convergents satisfy the well-known recurrence relations

$$p_{-1} = p_0 = 1, \quad p_k = 2c\, p_{k-1} + p_{k-2} \quad \text{for } k \geqq 1$$
$$q_{-1} = 0, \quad q_0 = 1, \quad q_k = 2c\, q_{k-1} + q_{k-2} \quad \text{for } k \geqq 1.$$

Let

$$n_k = \frac{1}{2c}(p_k - 1), \quad k \geqq 0.$$

It is easily verified that n_k is an integer, that

$$\alpha\beta = 2c, \quad -1/2 < 1 - \alpha < 0, \quad \epsilon_n = (-1)^{[\beta n]}$$

and that, for $k \geqq 0$,

$$2p_k\sqrt{d} = \alpha(1+\beta)^{k+1} + \beta(1-\alpha)^{k+1},$$
$$2q_k\sqrt{d} = (1+\beta)^{k+1} - (1-\alpha)^{k+1},$$
$$p_k - \alpha q_k = (1-\alpha)^{k+1}, \quad n_k + q_k = n_{k+1}, \quad n_k + p_{k+1} = n_{k+2}.$$

The first lemma is concerned with some basic identities involving the sequences (ϵ_n), (S_n) and (T_n).

LEMMA 1. *The following identities hold for* $k \geqq 1$.

(a) $\epsilon_{p_k} = (-1)^{k+1}$.

(b) $\epsilon_j = (-1)^j$ *if* $1 \leqq j \leqq 2c$.

(c) $\epsilon_{jq_k} = (-1)^{j+k}$ *if* $1 \leqq j \leqq 2c^2 + 1$.

(d) $\epsilon_j = \epsilon_i$ *if* $j = p_r + i, 1 \leqq i < q_k$.

(e) $\epsilon_i + \epsilon_j = 0$ *if* $i + j = p_k, 1 \leqq i < j$.

(f) $\epsilon_j = \epsilon_i$ *if* $i + j = q_k, 1 \leqq i \leqq j$.

(g) $\epsilon_j = (-1)^r \epsilon_i$ *if* $j = rq_k + i, 1 \leqq i < q_k, 1 \leqq r \leqq 2c$.

(h) $S_j = S_i$ *if* $i + j = p_k - 1, 0 \leqq i \leqq j$.

(i) $S_{p_k} = (-1)^{k+1}, \quad S_{n_{2k-1}} = -k, \quad S_{n_{2k}} = k$.

(j) *For* $i + j = q_k - 1, 0 \leqq i \leqq j$, $S_i + S_j = \begin{cases} 0 & \textit{if } k \textit{ is even} \\ -1 & \textit{if } k \textit{ is odd.} \end{cases}$

(k) *For* $1 \leqq r \leqq 2c + 1$, $S_{rq_k} = \begin{cases} -1 & \textit{if } k \textit{ is even, } r \textit{ is odd} \\ 0 & \textit{otherwise.} \end{cases}$

(l) *For* $j = rq_k + i, 0 \leqq i < q_k, 1 \leqq r \leqq 2c$,

$$S_i = S_j \quad \textit{if } r \textit{ is even},$$

$$S_i + S_j = \begin{cases} 0 & \textit{if } k \textit{ is odd} \\ -1 & \textit{if } k \textit{ is even} \end{cases} \quad \textit{and } r \textit{ is odd.}$$

(m) *For* $j = rq_k + i, 0 \leqq i < q_k, 1 \leqq r \leqq 2c$,

$$T_j - T_i = T_{rq_k} \quad \textit{if } r \textit{ is even},$$

$$T_j + T_i - T_{rq_k} = \begin{cases} 0 & \textit{if } k \textit{ is odd} \\ -1 & \textit{if } k \textit{ is even} \end{cases} \quad \textit{and } r \textit{ is odd.}$$

(n) *For* $1 \leqq r \leqq 2c + 1$, $T_{rq_k} = \begin{cases} 0 & \textit{if } r \textit{ is even} \\ -q_k/2 & \textit{if } r \textit{ is odd} \end{cases}\ k \textit{ is odd}$; $\begin{cases} -rq_k/2 & \textit{if } r \textit{ is even} \\ -(r-1)q_k/2 - 1 & \textit{if } r \textit{ is odd} \end{cases}\ k \textit{ is even.}$

Proof. (a) We have $\beta p_k = 2cq_k + \beta(1-\alpha)^{k+1}$ and thus, since $0 < \beta(\alpha - 1)^{k+1} < 1$ and $-\frac{1}{2} < 1 - \alpha < 0$,

$$[\beta p_k] = \begin{cases} 2cq_k & \text{if } k \text{ is odd} \\ 2cq_k - 1 & \text{if } k \text{ is even;} \end{cases}$$

and this implies that $\epsilon_{p_k} = (-1)^{[\beta p_k]} = (-1)^{k+1}$.

(b) For $1 \leqq j \leqq 2c$, we have $[\alpha j] = [j/(c+\sqrt{d})] + j$ which implies that $\epsilon_j = (-1)^j$.

(c) Starting from the identity $\alpha j q_k = jp_k - j(1-\alpha)^{k+1}$ we get, since $0 < j(\alpha-1)^{k+1} \leqq (2c^2+1)/(c+\sqrt{d})^2 < 1$, that

$$[\alpha j q_k] = \begin{cases} jp_k - 1 & \text{if } k \text{ is odd} \\ jp_k & \text{if } k \text{ is even.} \end{cases}$$

Further p_k is odd and hence $\epsilon_{jp_k} = (-1)^{j+k}$.

(d) For $j = p_k + i$, $1 \leqq i < q_k$, we have

$$\begin{aligned} \beta j &= \alpha\beta q_k + \beta(1-\alpha)^{k+1} + \alpha i + 2(c-1)i \\ &= 2cq_k + 2(c-1)i + \beta(1-\alpha)^{k+1} + \delta + a, \quad \text{where } a = [\alpha i]. \end{aligned}$$

Since $i < q_k$, it follows that $\delta = \alpha i - a \geqq |\alpha q_{k-1} - p_{k-1}| = (\alpha - 1)^k$ by standard theory. (See e.g. [**4**, p. 167, Theorem 7.13].) Further $0 < \beta(\alpha - 1) < 2c(\sqrt{d} - c) < 1$ and so $\delta > \beta(\alpha - 1)^{k+1}$. Likewise we obtain $1 - \delta = 1 + a - \alpha i > \beta(1-\alpha)^{k+1}$ and so

$$[\beta j] = 2cq_k + 2(c-1)i + [\alpha i],$$

from which it follows that $\epsilon_j = \epsilon_i$.

(e) Let $i + j = p_k$, $1 \leqq i < p_k/2$. Since $1 < \alpha < 3/2$, we have

$$\frac{p_k}{2} < \frac{\alpha q_k}{2} + \frac{\alpha - 1}{2} = q_k - q_k\left(1 - \frac{\alpha}{2}\right) + \frac{\alpha}{2} - \frac{1}{2} \leqq q_k + \alpha - \frac{3}{2} < q_k,$$

and by the same argument as in the proof of (d) it follows that $[\beta j] = 2cq_k - 2(c-1)i - 1 - [\alpha i]$ and thus that $\epsilon_j = -\epsilon_i$.

(f) For $j = q_k - 1$, $1 \leqq i < q_k$, we have $\alpha_j = p_k - (1-\alpha)^{k+1} - a - \delta$, where $a = [\alpha i]$; and as above $\delta > (\alpha - 1)^{k+1}$ and $1 - \delta > (\alpha - 1)^{k+1}$. Hence $[\alpha j] = p_k - 1 - [\alpha i]$, and so, since $p_k - 1$ is even, $\epsilon_j = \epsilon_i$.

(g) For $j = rq_k + i$, $1 \leqq i < q_k$, $1 \leqq r \leqq 2c$, we have

$$\alpha j = rp_k + r(1-\alpha)^{k+1} + a + \delta,$$

where $a = [\alpha i]$; and as before $\delta \geqq (\alpha - 1)^k > 2c(\alpha - 1)^{k+1} \geqq r(\alpha - 1)^{k+1}$ and $1 - \delta > r(\alpha - 1)^{k+1}$. Thus $[\alpha j] = rp_k + [\alpha i]$ and so $\epsilon_j = (-1)^r \epsilon_i$, since rp_k has the same parity as r.

(h) By (e) we have, for $i + 1 + j = p_k$, $0 \leqq i < j$, that

$$S_j = S_i + \sum_{\nu=i+1}^{j} \epsilon_\nu = S_i,$$

since $j - 1$ is even.

(i) By (h), $S_{p_k - 1} = 0$ and so, by (a), $S_{p_k} = \epsilon_{p_k} = (-1)^{k+1}$. Next, since $n_i + p_{i+1} = n_{i+2}$ and $n_i < q_i < q_{i+1}$, we have, by (d), that

$$S_{n_{i+2}} - S_{p_{i+1}} = \sum_{\nu=p_{i+1}+1}^{n_{i+2}} \epsilon_\nu = S_{n_i}$$

and so $S_{n_{i+2}} - S_{n_i} = S_{p_{i+1}} = (-1)^i$ for $i \geqq 0$. Hence

$$S_{n_{2k}} = \sum_{i=1}^{k} (S_{n_{2i}} - S_{n_{2i-2}}) = k$$

and

$$S_{n_{2k-1}} = \sum_{i=1}^{k-1} (S_{n_{2i+1}} - S_{n_{2i-1}}) + S_{n_1} = -(k-1) - 1 = -k,$$

since $n_1 = 1$ and $S_1 = \epsilon_1 = -1$.

(j) By (f) we have, for $i + 1 + j = q_k$, $0 \leqq i \leqq j$, that $S_i = S_{q_k - 1} - S_j$; and hence that $S_i + S_j = S_{q_k} - \epsilon_{q_k} = S_{q_k} + (-1)^k$, by (c). Next, since $n_k + q_k = n_{k+1}$ and $n_k < q_k$, it follows, by (g) with $r = 1$, that $S_{n_{k+1}} - S_{q_k} = -S_{n_k}$ and so, by (i),

$$S_{q_k} = S_{n_k} + S_{n_{k+1}} = \begin{cases} 0 & \text{when } k \text{ is odd} \\ -1 & \text{when } k \text{ is even.} \end{cases}$$

Hence

$$S_{q_k} + (-1)^k = \begin{cases} 0 & \text{when } k \text{ is even} \\ -1 & \text{when } k \text{ is odd,} \end{cases}$$

and this completes the proof.

(k) Let $j = rq_k + i$, $0 \leqq i < q_k$, $1 \leqq r \leqq 2c$. Applying (g) we get $S_j - S_{rq_k} = (-1)^r S_i$. Taking $i = q_k - 1$ we get $j = (r+1)q_k - 1$ and hence

$$\begin{aligned} S_{rq_k} = S_{(r+1)q_k - 1} - (-1)^r S_{q_k - 1} &= S_{(r+1)q_k} - (-1)^r S_{q_k} - \epsilon_{(r+1)q_k} + (-1)^r \epsilon_{q_k} \\ &= S_{(r+1)q_k} - (-1)^r S_{q_k}, \quad \text{by (c).} \end{aligned}$$

Consequently

$$\begin{aligned} S_{(r+1)q_k} = \sum_{i=1}^{r} (S_{(i+1)q_k} - S_{iq_k}) + S_{q_k} &= S_{q_k} \sum_{i=1}^{r} (-1)^i + S_{q_k} \\ &= \begin{cases} 0 & \text{when } r \text{ is odd} \\ S_{q_k} & \text{when } r \text{ is even.} \end{cases} \end{aligned}$$

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In the proof of (j) it was shown that

$$S_{q_k} = \begin{cases} 0 & \text{when } k \text{ is odd} \\ -1 & \text{when } k \text{ is even.} \end{cases}$$

It follows that, for $1 \leqq r \leqq 2c + 1$,

$$S_{rq_k} = \begin{cases} -1 & \text{when } k \text{ is even and } r \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

(l) Applying (k) to the formula $S_j - (-1)^r S_i = S_{rq_k}$, which was established in the preceding proof, we get the desired identities.

(m) This is an immediate consequence of (l).

(n) Since $q_k - 1$ has the same parity as k, an application of (j) yields

$$T_{q_k-1} = \begin{cases} 0 & \text{when } k \text{ is even} \\ -q_k/2 & \text{when } k \text{ is odd;} \end{cases}$$

and hence, by (k),

$$T_{q_k} = \begin{cases} -1 & \text{when } k \text{ is even} \\ -q_k/2 & \text{when } k \text{ is odd.} \end{cases}$$

Next, we consider two cases.

Case 1: k is odd. By (m) we have, for $j = rq_k + i$, $0 \leqq i < q_k$, $1 \leqq r \leqq 2c$, that $T_j = T_i(-1)^r + T_{rq_k}$. Taking $i = q_k - 1$ we get $j = (r + 1)q_k - 1$ and hence $T_{(r+1)q_k-1} = T_{q_k-1}(-1)^r + T_{rq_k}$; and so $T_{(r+1)q_k} - T_{rq_k} = T_{q_k-1}(-1)^r$, since $S_{(r+1)q_k} = 0$ by (k). Thus

$$T_{(r+1)q_k} = \begin{cases} T_{q_k} = -q_k/2 & \text{when } r \text{ is even} \\ T_{q_k} - T_{q_{k-1}} = 0 & \text{when } r \text{ is odd.} \end{cases}$$

Case 2: k is even. Again by (m) we have, for $j = rq_k + i$, $0 \leqq i < q_k$, $1 \leqq r \leqq 2c$, that

$$T_j = \begin{cases} -T_i - i + T_{rq_k} & \text{when } r \text{ is odd} \\ T_i + T_{rq_k} & \text{when } r \text{ is even.} \end{cases}$$

Hence, since $T_{q_k-1} = 0$, we conclude as in the preceding case, that

$$T_{(r+1)q_k} - T_{rq_k} = \begin{cases} S_{(r+1)q_k} - q_k + 1 = -q_k + 1 & \text{when } r \text{ is odd} \\ S_{(r+1)q_k} = -1 & \text{when } r \text{ is even.} \end{cases}$$

It follows that

$$T_{(r+1)q_k} = \begin{cases} -\dfrac{r+1}{2} q_k & \text{when } r \text{ is odd} \\ -\dfrac{r}{2} q_k - 1 & \text{when } r \text{ is even.} \end{cases}$$

This completes the proof of (n).

Our next lemma shows that n_{2k-1} is the first value of n for which S_n attains the value $-k$.

LEMMA 2. *If* $k \geqq 1$ *and* $n < n_{2k-1}$, *then* $|S_n| < k$.

Proof. We proceed by induction with respect to k. The proposition that $|S_i| < k - 1$ for $i < n_{2k-3}$ holds for $k = 2$. Assume it to be true for a given $k \geqq 2$. Supposing first that $k \geqq 3$ we proceed from the induction hypothesis as follows. Since $q_{2k-4} \leqq n_{2k-3}$, we have, for $j = rq_{2k-4} + i$, $0 \leqq i < q_{2k-4}$, $1 \leqq r \leqq 2c$, by (l), that

$$S_j = \begin{cases} S_i & \text{when } r \text{ is even} \\ -S_i - 1 & \text{when } r \text{ is odd.} \end{cases}$$

Also, by (k), $S_i = -1$ for $i = (2c + 1)q_{2k-4}$. Thus

$$-k < S_i < k - 1 \quad \text{for } i \leqq (2c + 1)q_{2k-4}.$$

Further $q_{2k-3} < (2c + 1)q_{2k-4}$ and $n_{2k-3} + q_{2k-3} = n_{2k-2}$ and so, by (l), we have, for $j = q_{2k-3} + i$, $0 \leqq i < n_{2k-3}$, that

$$|S_j| = |S_i| < k - 1,$$

since $n_{2k-3} < q_{2k-3}$. Therefore

$$-k < S_i < k - 1 \quad \text{for } i < n_{2k-2}.$$

But, by (b), the final inequalities also hold for $k = 2$, since $n_2 = 2c + 1$. In what follows we suppose $k \geqq 2$. By (l) again, we have, for $j = rq_{2k-3} + i$, $0 \leqq i < q_{2k-3}$, $1 \leqq r \leqq 2c$, that $|S_j| = |S_i|$ and so, since $q_{2k-3} < n_{2k-2}$,

$$-k < S_i < k \quad \text{for } i < (2c + 1)q_{2k-3}.$$

Finally, the relations $q_{2k-2} < (2c + 1)q_{2k-3}$ and $n_{2k-2} + q_{2k-2} = n_{2k-1}$ imply, by (l), that $-S_j = S_i + 1$ for $j = q_{2k-2} + i$, $0 \leqq i < n_{2k-2}$. Hence

$$-k < S_j < k - 1 \quad \text{for } q_{2k-2} < j < n_{2k-1}.$$

Since $q_{2k-2} < (2c + 1)q_{2k-3}$, we have established that

$$|S_i| < k \quad \text{for all } i < n_{2k-1}.$$

This completes the proof.

Similar considerations show that n_{2k} is the first value of n for which S_n attains the value k.

THEOREM 1. *If* $n \geqq 1$, *then*

$$|S_n| < \frac{\log \frac{2}{\alpha}(2cn + 1)\sqrt{d}}{2 \log (1 + \beta)}.$$

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Proof. Suppose there exists an integer $n \geqq 1$ such that

$$|S_n| \geqq \frac{\log \dfrac{2}{\alpha}(2cn+1)\sqrt{d}}{2\log(1+\beta)}.$$

Then, putting $k = |S_n|$, we get $2k \log(1+\beta) \geqq \log(2(2cn+1)\sqrt{d}/\alpha)$ and so

$$n \leqq \frac{1}{2c}\left(\frac{\alpha}{2\sqrt{d}}(1+\beta)^{2k} - 1\right) < \frac{1}{2c}(p_{2k-1} - 1) = n_{2k-1},$$

which contradicts Lemma 2.

Using Lemma 1 (i) and Theorem 1, we can easily verify that

$$(5) \quad \limsup_{n\to\infty} \frac{2S_n \log(1+\beta)}{\log \dfrac{2}{\alpha}(2cn+1)\sqrt{d}} = 1, \quad \liminf_{n\to\infty} \frac{2S_n \log(1+\beta)}{\log \dfrac{2}{\alpha}(2cn+1)\sqrt{d}} = -1,$$

thereby showing that Theorem 1 is best possible in the sense indicated.

Note that $\alpha \to 1$ as $c \to \infty$, and thus the sequence $((-1)^n)$ is in a sense the limiting case of the sequence $((-1)^{[\alpha n]})$. Moreover, for fixed n, the bound in (3) for $|S_n|$ tends to 1 as $c \to \infty$, and this is the least upper bound for $|S_n'|$, where $S_n' = \sum_{k=1}^n (-1)^k$. Putting $T_n' = \sum_{k=1}^n S_k'$ we observe that $n/2 \leqq -T_n' \leqq (n+1)/2$. The following theorem shows that a surprisingly similar estimate holds for T_n.

THEOREM 2. *If* $n \geqq 1$, *then* $0 \leqq -T_n \leqq \frac{1}{2}(n+1)$.

Proof. We prove by induction with respect to k that

$$0 \leqq -T_i \leqq \tfrac{1}{2}(i+1) \quad \text{for } i < q_k,\ k \geqq 1.$$

Since $q_1 = 2c$, we have, by Lemma 1 (b), that $S_\nu = \sum_{j=1}^\nu (-1)^j$ for $\nu \leqq 2c$, which implies that $0 \leqq -T_i \leqq (i+1)/2$ for $i < q_1$. Now suppose that $k \geqq 1$ and that $0 \leqq -T_i \leqq (i+1)/2$ for $i < q_k$. Let $j = rq_k + i$, $0 \leqq i < q_k$, $1 \leqq r \leqq 2c$. We consider three cases, applying Lemma 1 (m) and (n) in each case.

Case 1: r is even. Then $T_j - T_i = T_{rq_k} \geqq -rq_k/2 = (i-j)/2$ and so

$$T_j + j/2 \geqq T_i + i/2 \geqq -1/2 \quad \text{and} \quad T_j \leqq T_i \leqq 0.$$

Case 2: r is odd and k is even. Then

$$T_j + \frac{j}{2} = T_{rq_k} - \frac{i}{2} + \frac{j-i}{2} - T_i > T_{rq_k} - \frac{q_k}{2} + \frac{rq_k}{2}$$
$$= T_{rq_k} + \frac{(r-1)q_k}{2} = -1,$$

and so, since T_j is an integer,

$$T_j + j/2 \geqq -1/2.$$

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Also,

$$T_j = T_{rq_k} - T_i - i \leqq -\frac{(r-1)q_k}{2} - 1 + \frac{i+1}{2} - i \leqq -\frac{1}{2}.$$

Case 3: r and k are odd. Then

$$T_j + j/2 = T_{rq_k} - T_i + j/2 > -q_k/2 + rq_k/2 \geqq 0$$

and

$$T_j = T_{rq_k} - T_i \leqq -q_k/2 + (i+1)/2 \leqq -q_k/2 + q_k/2 = 0.$$

Hence we have in every case that $0 \leqq -T_j \leqq (j+1)/2$ for $j < (2c+1)q_k$. Since $q_{k+1} < (2c+1)q_k$, the proof is complete.

It follows from Lemma 1 (k), (m), and (n) that if $n = 2q_{2k} + 1$, then $T_n = T_{2q_{2k}} + T_1 = -q_{2k} - 1 = -(n+1)/2$, whereas if $n = 2q_{2k-1} - 1$, then $T_n = T_{q_{2k-1}} - T_{q_{2k-1}-1} = S_{q_{2k-1}} = 0$. This shows that the inequalities in Theorem 2 are sharp.

3. In this section we show how the preceding estimates can be used to determine the sum σ of the series $\sum_{n=1}^{\infty} \epsilon_n/n$. In addition we contrast the behaviour of the series $\sum_{n=1}^{\infty} \epsilon_n$ with that of $\sum_{n=1}^{\infty} (-1)^n$ with regard to summability by certain standard methods.

The problem of estimating the sum of the series $\sum_{n=1}^{\infty} \epsilon_n/n$ reduces to knowing how close its n-th partial sum σ_n is to σ. Applications of Abel's partial summation formula yield

$$\sigma - \sigma_n + \frac{S_n}{n+1} = \sum_{k=n+1}^{\infty} \frac{S_k}{k(k+1)} = \rho_n,$$

say, and

$$\sigma - \sigma_n + \frac{S_n}{n+1} + \frac{T_n}{(n+1)(n+2)} = 2\sum_{k=n+1}^{\infty} \frac{T_k}{k(k+1)(k+2)} = \tau_n,$$

say. It follows from (2) that

$$|\rho_n| < \frac{6 + 2M_c(1 + \log n)}{n},$$

and from (4) that

$$0 < -\tau_n < \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2}\right).$$

Consider now the special case $\alpha = \sqrt{2}$ $(c = 1)$. We find that $M_1 < 3.9$. For $n = q_{19} = 15994428$, we have, by Lemma 1 (k) and (n), that $S_n = 0$ and $T_n = -n/2$; and a computer yielded $\sigma_n \doteq -0.5154184551$. Using the above estimate for ρ_n we get

$$-0.515428 < \sigma < -0.515409,$$

and using the estimate for τ_n we get

$$-0.5154186 < \sigma < -0.5154184.$$

It is familiar that the series $\sum_{n=1}^{\infty} (-1)^n$ is summable to $-1/2$ by the Cesàro method C_1 and consequently by the Abel method A. It is also summable to $-1/2$ by the Borel method B. We shall show, on the other hand, that the series $\sum_{n=1}^{\infty} \epsilon_n$ is not summable by any of the above standard methods. Let $U_n = \sum_{k=1}^{\infty} T_k$. Then, by Lemma 1 (m), we have $T_j = T_{2q_k} + T_i$ for $j = 2q_k + i$, $1 \leqq i < q_k$, and so

$$U_{3q_k-1} - U_{2q_k} - U_{q_k-1} = (q_k - 1)T_{2q_k}.$$

It follows, by Lemma 1 (n), that

$$U_{3n-1} - U_{2n} - U_{n-1} = \begin{cases} 0 & \text{when } n = q_{2\nu-1} \\ -n(n-1) & \text{when } n = q_{2\nu}. \end{cases}$$

If we now suppose that U_n/n^2 tends to a finite limit l as $n \to \infty$, we get the contradictory conclusions that $9l - 4l - l = 0$ and $9l - 4l - l = -1$. Hence the sequence (U_n/n^2) is not convergent and, equivalently, the sequence (T_n/n) is not limitable C_1. Now it is known (see e.g. [**1**, p. 214]) that if $\sum_{n=1}^{\infty} \epsilon_n$ is summable A, then (T_n/n) is limitable A and hence, by a familiar tauberian theorem, that (T_n/n) is limitable C_1, since $T_n/n \leqq 0$ [**2**, p. 154, Theorem 93]. Thus $\sum_{n=1}^{\infty} \epsilon_n$ is not summable A and, a fortiori, not summable C_1. Another familiar tauberian theorem [**2**, p. 210, Theorem 147] now shows that the series in question cannot be summable B, for if it were, the order relation $\epsilon_n = O(1)$ would imply it to be summable C_ρ for every $\rho > 1$.

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