E. Freitag and M. Oura
Nagoya Math. J.
Vol. 161 (2001), 69–83

A THETA RELATION IN GENUS 4

EBERHARD FREITAG AND MANABU OURA[1]

Abstract. The 2^g theta constants of second kind of genus g generate a graded ring of dimension $g(g+1)/2$. In the case $g \geq 3$ there must exist algebraic relations. In genus $g = 3$ it is known that there is one defining relation. In this paper we give a relation in the case $g = 4$. It is of degree 24 and has the remarkable property that it is invariant under the full Siegel modular group and whose Φ-image is not zero. Our relation is obtained as a linear combination of code polynomials of the 9 self-dual doubly-even codes of length 24.

Introduction

The theta constants of genus g of second kind are the 2^g functions

$$f_a(Z) := \sum_{n \in \mathbb{Z}^g} \exp 2\pi \mathrm{i} Z[n + a/2].$$

They generate a ring of dimension $g(g+1)/2+1$. In the cases $g \leq 2$ they are algebraically independent, in the case $g = 3$ there is a defining relation in degree 16, which is related to the so-called Schottky relation [Ru3]. In this paper we describe a similar relation in genus 4. Its degree is 24. This relation has two remarkable properties. It is invariant under the full Siegel modular group and it is mapped to a non trivial relation under Siegel Φ-operator.

It is a natural question, whether our relation is a consequence of Riemann's theta relations. We could not decide this. One reason is that the Riemann theta relations are quartic relations in the theta constants of first kind

$$\vartheta[m] = \sum_{n \in \mathbb{Z}^g} \exp \pi \mathrm{i}(Z[n + a/2] + b'(n + a/2)), \quad m = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^{2g}.$$

The functions $f_a f_b$ generate the same vetor space as the squares $\vartheta[m]^2$. We learnt from B. van Geemen and R. Salvati-Manni how to obtain relations

Received April 24, 1999.
1991 Mathematics Subject Classification: 11T71 94B05 11F46 11F27.
[1]The second author's work was supported in part by a grant from the Japan Society for the Promotion of Science.

between the f_a from the Riemann relations using elimination tricks. But it seems very hard to obtain the distinguished relation which we describe below.

Our relation gives rise to a certain Siegel cusp form of weight 12 and genus 5 with respect to the full Siegel modular group.

We use the theory of codes. There are 9 isomorphy classes of binary self-dual doubly-even codes in $\mathbb{F}_2^{24}$. They are associated to 9 of the 24 Niemeier lattices. To any binary code C and any genus g there is associated a certain code polynomial, which is a polynomial in 2^g variables F_a, $a \in \mathbb{F}_2^g$. If one replaces F_a by f_a one obtains a function which is nothing else but the usual theta function of the lattice L which is related to the code C. Recall that L is the inverse image of $C \subset \mathbb{F}_2^n$ in $\mathbb{Z}^n$ with the scalar product $2\langle x, y\rangle = \sum x_i y_i$. Our relation will be constructed as a linear combination of the code polynomials of the nine mentioned codes.

With help of computers we determined completely the code polynomials of genus 4 of the nine "Niemeier codes". The tables are printed at the end of this paper.

Section 1 of the paper contains the results of the paper. In Section 2 some of the computational problems are explained. Tables can be found in Section 3.

We would like to thank R. Salvati-Manni, B. van Geemen and B. Runge for helpful discussions. This work was done during the second author's stay in Mannheim. He would like to thank Professor R. Weissauer for his hospitality.

§1. Siegel modular forms of genus four and codes

Let $\mathbb{F}_2$ be the field of two elements and $V = \mathbb{F}_2^g$. Sometimes we will identify the elements of $\mathbb{F}_2$ with the integers 0, 1. For $a, b \in V$ let $a \cdot b$ denote the standard scalar product. Set

$$T_g := \left(\frac{1+\mathrm{i}}{2}\right)^g \left((-1)^{a\cdot b}\right)_{a,b\in V}$$

and for any symmetric $g \times g$-matrix with integral coefficients

$$D_S := \operatorname{diag}\left(\mathrm{i}^{S[a]} \text{ with } a \in V\right),$$

where

$$S[a] := \sum_{i,j} s_{ij} a_i a_j \quad (\in \mathbb{Z}).$$

The group generated by the matrices T_g and D_S is

$$H_g := \langle T_g, D_S \rangle.$$

After a choice of an ordering of V this is a subgroup of $\mathrm{GL}(2^g, \mathbb{C})$. It is known that H_g is a finite group. We introduce for every $a \in V$ a variable F_a and consider the ring of invariants

$$R_g := \mathbb{C}[(F_a)_{a \in V}]^{H_g}.$$

This ring is connected with the ring $A(\Gamma_g)$ of Siegel modular forms. The link comes from the "theta constants of second kind":

$$f_a(Z) := \sum_{n \in \mathbb{Z}^g} \exp 2\pi \mathrm{i} Z[n + a/2].$$

Here Z varies in the Siegel half plane of genus g which consists of all symmetric $g \times g$-matrices with positive definite imaginary part. From the classical theta transformation formula one knows that the evaluation $F_a \mapsto f_a$ defines a homomorphism

$$R_g \longrightarrow A_g.$$

Here

$$A_g = A(\Gamma_g) = \sum_{k=0}^{\infty} [\Gamma_g, k],$$

where $[\Gamma_g, k]$ denotes the vector space of Siegel modular forms of weight k (with respect to the full Siegel modular group $\Gamma_g = \mathrm{Sp}(g, \mathbb{Z})$). We use the weight convention that a form of weight k transforms as

$$f\big((AZ + B)(CZ + D)^{-1}\big) = \det(CZ + D)^k f(Z).$$

The weight of the f_a have to be counted as $1/2$.

A variant of "Igusa's fundamental lemma" states that the subring $A_g^{(2)} \subset A_g$ of modular forms of even weights equals the normalization of the image of the ring R_g. From this point of view the determination of the ring of Siegel modular forms depends on the determination of the kernel of $R_g \to A_g$. Along these lines the cases $g \leq 3$ have been treated successfully (reproducing and enlarging known results) [Ru1], [Ru2].

This paper is an approach to attack the case $g = 4$. What we get is a simple element of the kernel. The second author determined the dimension formula for the ring R_4 [Ou]. The dimension of the space of Siegel modular forms in genus four is known in some cases:

weight k	4	6	8	10	12	14	16
$\dim(R_4)_k$	1	1	2	3	7	7	19
$\dim[\Gamma_4, k]$	1	1	2	?	6	?	?

This table shows that there must be a relation in weight 12, i.e. there must exist an invariant polynomial of degree 24 in the 16 variables F_a which is not identically zero but which vanishes if one replaces $F_a \mapsto f_a$. We use codes to determine this polynomial. Recall that a (binary) code C is nothing else but a subvector space $C \subset \mathbb{F}_2^n$. The code ploynomial of genus g associated to such a code is

$$P(C) = P^{(g)}(C) := \sum_{\alpha_1,\dots,\alpha_g \in C} \prod_{a \in \mathbb{F}_2^g} F_a^{a(\alpha_1,\dots,\alpha_g)},$$

where $a(\alpha_1, \dots, \alpha_g)$ denotes the number of all $i \in \{1, \dots, n\}$ such that $a = (\alpha_{1i}, \dots, \alpha_{gi})$. The *weight* of an element $\alpha \in C$ is the number of digits 1 in α. A code is called doubly-even if all weights are divisible by four. A code is called self-dual if it agrees with its orthogonal complement with respect to $a \cdot b$.

It is well-known [Du], [Gl], [Ru1], that a H_g-invariant polynomial $P \in R_g$ is a linear combination of code polynomials of self-dual doubly-even codes if and only if its degree is divisible by 8. In our case ($g = 4$ and $k = 12$) we have to consider codes in $\mathbb{F}_2^{24}$. It is known that up to isomorphism precisely 9 self-dual doubly-even codes exist. We denote representatives by $C_1, \dots, C_9$. The first seven denote the indecomposable ones. We refer to the list [PS]. The 9 types are characterized by their root subcodes which are direct products of the codes a_n, d_n, $n \geq 4$, e_7, e_8. The correspondence is as follows:

C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
d_{12}^2	$d_{10}e_7^2$	d_8^3	d_6^4	d_{24}	d_4^6	a_1^{24}	$d_{16}e_8$	e_8^3

First we consider the 9 Siegel modular forms. We know that the dimension of the space $[\Gamma_4, 12]$ is 6.

PROPOSITION 1.1. *The Siegel modular forms $S_i = S(C_i)$ of genus 4 which are attached to the nine codes C_i generate the six dimensional space*

$[\Gamma_4, 12]$. *A basis is given by* S_1, S_2, S_3, S_4, S_6, S_7. *The expressions of the three remaining terms with respect to this basis is*

$$\begin{aligned} S_5 &= 66S_1 - 495S_3 + 880S_4 - 594S_6 + 144S_7, \\ 3S_8 &= 17S_1 + 20S_2 - 145S_3 + 208S_4 - 125S_6 + 28S_7, \\ S_9 &= 3S_1 + 20S_2 - 75S_3 + 96S_4 - 55S_6 + 12S_7. \end{aligned}$$

Each of the codes C_i corresponds to one of the 24 Niemeier lattices and the modular form S_i is the theta series of this Niemeier lattice. The proof of Propisition 1.1 uses the knowledge of some Fourier coefficients. Recall that the theta series $\vartheta(L; Z)$ of genus g of a definite lattice L admits a Fourier expansion whose Fourier coefficients are given by the number of isometric embeddings of an arbitrary lattice M of rank $\leq g$ into L. In case of the 24 Niemeier lattices L and the irreducible root lattices M of rank ≤ 12 these numbers can be found in [BFW]. For our purpose it is sufficient to take the six root lattices A_0, A_1, A_2, A_3, A_4, D_4 (and the 9 Niemeier code lattices) to seperate the theta series. In [BFW] they carry the same name as the corresponding codes but using capital letters. For example the Neimeier lattice corresponding to the code e_8 is the lattice E_8.

As we know from the results [Ou] the space generated by the code polynomials has dimension 7. To get a basis we computed the coefficients of the monomial

$$F_0^9 \prod_{a \neq 0} F_a$$

in the code polynomial $P_i = P(C_i)$. This monomial (to be precise an equivalent one) carries number 127 in our tables.

PROPOSITION 1.2. *The coefficients corresponding to the monomial* $F_0^9 \prod_{a \neq 0} F_a$ *of the nine code polynomials* $P_i = P(C_i)$ *are*

$$0, \quad 70, \quad 24, \quad 72, \quad 0, \quad 148, \quad 253, \quad 80, \quad 168$$

multiplied with $2^{10}3^3 5 \cdot 7$.

From Propositions 1.1 and 1.2 we obtain

THEOREM 1.3. *The code polynomial*

$$R := 3P_1 + 20P_2 - 75P_3 + 96P_4 - 55P_6 + 12P_7 - P_9$$

does not vanish but the corresponding Siegel modular form is zero.

We can consider the same linear combination of code polynomials in genus 5. Computing one further Fourier coefficient one sees that the corresponding Siegel modular form is not 0.

THEOREM 1.4. *The linear combination R considered in genus five defines a non-vanishing Siegel cusp form of weight* 12 *and genus* 5 *with respect to the full Siegel modular group.*

For sake of completeness we give the relations for the 9 code polynomials in genus 4:

PROPOSITION 1.5. *A basis for the H_4-invariant code polynomials is given by P_1, P_2, P_3, P_4, P_6, P_7, P_8. The expressions for the two remaining are*

$$P_5 = 66P_1 - 495P_3 + 880P_4 - 594P_6 + 144P_7,$$
$$P_9 = -14P_1 + 70P_3 - 112P_4 + 70P_6 - 16P_7 + 3P_8.$$

There is a homomorphism

$$\Phi : R_g \longrightarrow R_{g-1}, \quad F_{(a_1,a_2,a_3,a_4)} \longmapsto \begin{cases} 0 & \text{if } a_4 = 0, \\ F_{(a_1,a_2,a_3)} & \text{otherwise.} \end{cases}$$

This induces the Siegel Φ-operator on the level of modular forms. We apply this operator to the relation R (see Theorem 1.3). We get a relation in genus three. In this case there is a defining relation of degree 16 [Ru1]. It is the code polynomial

$$P^{(3)}(e_8^2) - P^{(3)}(d_{16}).$$

The upper index 3 indicates that we consider code polynomials of genus three. Recall that we denote by e_8 resp. d_{16} the unique irreducible self-dual doubly-even code in dimension 8 resp. 16. So $R|\Phi$ must be a product of this polynomial and an invariant polynomial of degree 8. There is only one up to a constant factor, namely $P^{(3)}(e_8)$. We obtain

PROPOSITION 1.6. *The image of the relation R (see Theorem* 1.3) *under the Φ-operator is*

$$R|\Phi = \left(P^{(3)}(e_8^2) - P^{(3)}(d_{16})\right) \cdot P^{(3)}(e_8).$$

Only the constant has to be verified. To do this we use the coefficients of the monomial

$$F^{16}_{(0,0,0,0)} \prod_{a_4=0} F_a.$$

An equivalent one carries number 40 in our table. The coefficients of the nine C_i in our ordering are

$$0, \quad 336, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 1344, \quad 4032.$$

Hence the coefficient of R is 2688. This is also the coefficient of $R|\Phi$ with respect to the degree-3-monomial

$$F^{16}_{(0,0,0)} \prod_{a\in\mathbb{F}_2^3} F_a.$$

On the other hand the coefficient of $\big(P^{(3)}(e_8^2) - P^{(3)}(d_{16})\big) \cdot P^{(3)}(e_8)$ is also 2688 (see [Ru3]).

§2. Admissible monomials

The number of monomials of degree 24 in 16 variables is 25 140 840 660. But not all of them can occur in a code polynomial of a self-dual doubly-even code.

DEFINITION 2.1. A monomial (α_a) is called *admissible* if

$$\sum_{a\in\mathbb{F}_2^g} \alpha_a \equiv 0 \bmod 8 \quad \text{and} \quad \sum_{a\in\mathbb{F}_2^g} \alpha_a S[a] \equiv 0 \bmod 4$$

for all integral symmetric $g \times g$-matrices S.

Here we use the usual notation $S[a] = \sum_{i,j} s_{ij} a_i a_j$. It is of course sufficient to take for S a system of generators, for example all elements in the diagonal or above the diagonal are zero besides one which is 1. It is easy to prove (see [Ru3]) that in the code polynomial of self-dual doubly-even codes only admissible monomials appear.

There is a certain group $\mathrm{AGL}(g)$ which acts on the set of admissible monomials, namely the semidirect product

$$\mathrm{AGL}(g) = \mathrm{GL}(g, \mathbb{F}_2) \cdot \mathbb{F}_2^g.$$

https://doi.org/10.1017/S0027763000022133 Published online by Cambridge University Press

The group AGL(g) acts on $\mathbb{F}_2^g$ as the group of affine transformations,

$$x \longmapsto Ux + a, \quad (U, a) \in \mathrm{AGL}(g).$$

The order of AGL(4) is

$$\sharp\,\mathrm{AGL}(4) = 2^4(2^4-1)(2^4-2)(2^4-2^2)(2^4-2^3) = 322\,560 = 2^{10}3^2 5 \cdot 7.$$

The group AGL(g) is a subgroup of H_g. Hence our code polynomials are invariant under AGL(g). Our goal is to determine a complete system of representatives of admissible monomials of weight $wt(a) = \sum_a \alpha_a = 24$.

Therefore it is sufficient to consider a system of representatives of all admissible monomials under AGL(g). We determined a system of representatives in the case $g = 4$.

PROPOSITION 2.2. *There exist* 160 *admissible monomials of weight* 24*, up to* AGL(4).

An explicit list is given in Table I below.

We give some explanations how this table has been computed. For the computations we used the computer algebra system MAGMA. The admissible monomials can be considered as elements of $\mathbb{Z}^{16}$. The group AGL(4) acts on this domain permuting the components. A monomial $(\alpha_a) \in \mathbb{Z}^{16}$ is called admissible mod N, if

$$\sum_{a \in \mathbb{F}_2^g} \alpha_a \equiv 0 \bmod (8, N) \quad \text{and} \quad \sum_{a \in \mathbb{F}_2^g} \alpha_a S[a] \equiv 0 \bmod (4, N)$$

for all integral symmetric $g \times g$-matrices S. This condition depends only on α mod N, hence this condition can be defined for elements $\alpha \in (\mathbb{Z}/N\mathbb{Z})^{16}$. Let $\beta \in \mathbb{Z}^{16}$ be the reduced representative mod N of α, i.e. $0 \le \beta_i \le N$ for all i. We use the notation

$$wt(\alpha, N) := \sum_{i=1}^{16} \beta_i.$$

We consider the set

$$\mathcal{A}(N) := \{\alpha \in (\mathbb{Z}/N\mathbb{Z})^{16}\ ;\ wt(\alpha, N) \le 24\}.$$

https://doi.org/10.1017/S0027763000022133 Published online by Cambridge University Press

The idea is to compute step by step a system of representatives of the sets $\mathcal{A}(2)$, $\mathcal{A}(4)$, $\mathcal{A}(8)$, $\mathcal{A}(16)$.

The first step is to determine the admissible monomials mod 2. The number of all possibilities $2^{16} = 65\,536$ is small enough to compute by a silly computer program. We have 3 representatives. In the next step one considers, for each $\alpha \in (\mathbb{Z}/2\mathbb{Z})^{16}$ of the three, the possible expansions $\beta \in (\mathbb{Z}/4\mathbb{Z})^{16}$ to admissibles mod 4 with the property $wt(\beta, 4) \leq 24$. We take a system of representatives under the stabilizer of α in AGL(4). Using this method we found that $\mathcal{A}(4)$ has 22 representatives and similarly that $\mathcal{A}(8)$, $\mathcal{A}(16)$ have 109, 156 representatives respectively. In the last step one expands to $\mathcal{A}(24)$ which gives the final result.

The computation of the coefficients of the monomials is more involved. For each member C of the nine codes one needs a system of representatives $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of code words with respect to the action of the automorphism group of C. One restricts to tuples such that the corresponding monomial belongs to the list of the 160 representatives. One starts with a list of representatives of code words α_1, computes the stabilizer of α_1 and determines a system of representatives α_2 with respect to this stabilizer, and so on. Again we used the MAGMA system which provides programs to handle codes and their automorphism groups. The result of the computations is in Table II.

The programs and tables can be found on the WEB-page

http://www.rzusers.uni-heidelberg.de/~t91

We would like to mention one further result of the calculations:

Remark 2.3. The total number of monomials with non-zero coefficients in the code polynomial R (see Theorem 1.3) is 2 004 480.

§3. Tables

The following table gives a system of representatives of admissible monomials. The tuple $(\alpha_1, \ldots, \alpha_{16})$ stands for the monomial $\prod F_i^{\alpha_i}$. We use the binary convention, i.e. the characteristic $a = (a_1, a_2, a_3, a_4)$ is encoded as digit $i = 8a_1 + 4a_2 + 2a_3 + a_4$. The number behind the monomial gives the length of the orbit of the monomial with respect to the group AGL(4).

https://doi.org/10.1017/S0027763000022133 Published online by Cambridge University Press

Table I

1	(0,0,0,2,0,0,0,2,0,6,4,0,4,6,0,0)	40320	51	(0,0,0,2,0,4,0,6,0,2,0,0,4,2,0,4)	80640
2	(1,2,0,1,6,1,1,0,1,2,0,1,6,1,1,0)	2520	52	(1,4,0,1,8,1,1,0,1,0,0,1,0,1,1,4)	5040
3	(0,8,0,2,0,0,0,2,0,2,0,0,0,6,0,4)	20160	53	(3,0,0,1,0,3,1,0,1,6,0,1,0,3,3,2)	13440
4	(0,0,0,2,8,0,0,10,0,2,0,0,0,2,0,0)	6720	54	(1,4,0,3,0,1,1,2,1,2,0,1,0,3,1,4)	10080
5	(0,0,0,0,2,0,2,0,2,14,0,0,2,0,0,2)	2688	55	(0,0,4,0,4,0,0,0,8,0,0,0,4,4,0,0)	13440
6	(1,0,0,1,4,1,1,4,3,0,0,3,0,3,3,0)	5040	56	(1,0,8,9,0,1,1,0,1,0,0,1,0,1,1,0)	1920
7	(1,0,2,3,0,1,3,2,3,2,0,1,2,3,1,0)	840	57	(0,0,8,0,12,0,0,4,0,0,0,0,0,0,0,0)	3360
8	(0,2,0,2,4,2,2,4,2,0,2,0,0,2,2,0)	20160	58	(1,0,0,7,0,1,1,2,1,2,0,1,0,7,1,0)	3360
9	(0,0,0,2,0,0,0,10,0,6,0,0,0,6,0,0)	1680	59	(1,0,2,1,2,3,1,2,1,2,0,1,2,1,3,2)	3360
10	(0,0,0,0,2,0,2,4,2,2,4,0,2,4,0,2)	26880	60	(0,0,0,0,0,0,0,0,8,0,0,0,0,16,0,0)	240
11	(1,0,0,3,0,1,1,2,1,2,0,1,0,11,1,0)	6720	61	(0,4,0,2,0,0,0,6,0,6,0,0,0,6,0,0)	6720
12	(1,0,0,1,0,1,1,0,11,0,0,3,0,3,3,0)	1680	62	(0,0,0,0,4,0,0,4,0,0,0,0,0,0,16,0)	1680
13	(3,0,2,1,0,1,1,0,1,2,0,3,2,3,3,2)	13440	63	(0,0,0,2,0,0,0,2,0,2,0,0,0,14,0,4)	6720
14	(1,2,0,1,2,1,1,4,1,2,4,1,2,1,1,0)	2520	64	(0,0,2,2,0,0,6,2,6,2,0,0,2,2,0,0)	840
15	(3,0,0,1,0,7,1,0,1,2,0,1,0,3,3,2)	26880	65	(0,4,0,6,0,4,0,2,0,2,0,0,0,2,0,4)	6720
16	(0,0,2,2,0,0,2,2,2,2,0,4,2,2,4,0)	840	66	(0,0,4,2,0,0,4,2,4,2,4,0,0,2,0,0)	13440
17	(1,0,4,1,0,1,5,4,1,4,0,1,0,1,1,0)	13440	67	(0,0,0,10,0,0,0,10,0,2,0,0,0,2,0,0)	840
18	(1,4,0,1,0,1,1,0,1,4,4,1,4,1,1,0)	1680	68	(9,0,0,1,0,1,1,0,9,0,0,1,0,1,1,0)	840
19	(3,2,0,1,0,1,3,2,3,0,2,1,2,1,3,0)	5040	69	(0,0,0,2,0,0,4,6,0,2,0,0,4,6,0,0)	5040
20	(1,0,0,1,0,9,1,0,3,0,0,3,0,3,3,0)	1680	70	(4,0,4,0,2,0,2,0,2,2,0,4,2,0,0,2)	26880
21	(5,0,0,3,0,1,1,2,5,2,0,1,0,3,1,0)	40320	71	(0,4,4,2,0,0,0,2,8,2,0,0,0,2,0,0)	40320
22	(1,0,0,1,6,1,3,0,3,2,0,1,2,1,1,2)	26880	72	(0,4,0,0,6,0,6,0,2,2,0,0,2,0,0,2)	40320
23	(0,0,0,2,0,0,0,2,0,2,0,0,0,18,0,0)	560	73	(1,2,0,1,6,1,1,4,1,2,0,1,2,1,1,0)	6720
24	(3,0,0,1,0,3,1,0,5,2,0,1,0,3,3,2)	26880	74	(1,0,0,1,0,1,1,0,3,0,0,7,0,7,3,0)	2520
25	(2,2,2,0,2,2,0,2,2,2,0,2,0,4,0,2)	2688	75	(1,2,0,1,2,1,1,0,9,2,0,1,2,1,1,0)	3360
26	(1,0,0,3,4,1,1,6,1,2,0,1,0,3,1,0)	40320	76	(0,4,4,2,0,0,0,2,0,2,0,0,0,10,0,0)	26880
27	(0,2,0,2,0,2,2,4,2,0,2,0,4,2,2,0)	3360	77	(1,4,0,1,4,1,1,0,1,4,0,1,4,1,1,0)	420
28	(0,0,0,0,0,0,0,0,24,0,0,0,0,0,0,0)	16	78	(1,0,0,1,0,1,1,0,9,0,0,5,0,1,5,0)	5040
29	(0,0,2,2,8,0,2,2,2,2,0,0,2,2,0,0)	240	79	(0,2,2,2,0,2,2,2,2,2,0,2,2,2,0,2)	140
30	(1,0,0,3,0,1,1,2,9,2,0,1,0,3,1,0)	20160	80	(0,0,0,2,0,0,0,2,8,2,0,0,0,6,0,4)	53760
31	(1,0,0,5,4,1,1,0,9,0,0,1,0,1,1,0)	13440	81	(0,0,8,8,4,0,0,4,0,0,0,0,0,0,0,0)	10080
32	(1,0,0,5,0,5,1,0,3,0,0,3,0,3,3,0)	2520	82	(1,0,0,1,0,1,5,0,7,0,0,3,0,3,3,0)	6720
33	(5,6,0,1,2,1,1,0,1,2,0,1,2,1,1,0)	13440	83	(0,0,0,0,12,0,0,12,0,0,0,0,0,0,0,0)	120
34	(0,4,0,2,0,0,0,2,8,2,0,0,0,2,0,4)	20160	84	(1,0,0,1,2,1,3,0,7,2,0,1,2,1,1,2)	13440
35	(1,2,0,1,10,1,1,0,1,2,0,1,2,1,1,0)	1680	85	(4,0,0,0,2,4,2,0,2,6,0,0,2,0,0,2)	40320
36	(0,2,0,2,0,2,2,0,2,0,2,4,0,6,2,0)	40320	86	(1,0,0,13,4,1,1,0,1,0,0,1,0,1,1,0)	1920
37	(0,0,0,2,4,0,0,2,4,2,0,0,4,2,4,0)	10080	87	(0,2,0,2,8,2,2,0,2,0,2,0,0,2,2,0)	6720
38	(1,0,0,1,8,1,1,0,1,0,0,5,0,1,5,0)	6720	88	(4,0,0,0,0,4,0,0,0,4,0,0,0,4,4,4)	448
39	(0,0,2,2,4,0,2,6,2,2,0,0,2,2,0,0)	1920	89	(0,0,4,4,0,4,0,0,0,0,4,0,0,0,4,4)	6720
40	(1,0,0,1,0,1,1,0,1,0,0,1,0,1,17,0)	240	90	(0,0,4,0,12,0,0,0,0,0,0,0,4,4,0,0)	6720
41	(0,4,0,2,0,8,0,2,0,2,0,0,0,2,0,4)	5040	91	(1,0,0,3,0,1,1,2,1,2,0,5,0,7,1,0)	40320
42	(1,4,0,1,0,1,1,0,3,0,0,3,0,3,3,4)	6720	92	(4,4,0,2,0,0,4,6,0,2,0,0,0,2,0,0)	8960
43	(1,1,3,3,1,1,1,1,1,1,1,1,1,1,3,3)	140	93	(1,8,0,5,4,1,1,0,1,0,0,1,0,1,1,0)	13440
44	(1,0,0,1,8,1,1,0,3,0,0,3,0,3,3,0)	3360	94	(3,2,2,3,0,3,3,0,3,0,0,3,0,1,1,0)	3360
45	(0,6,0,2,0,2,2,0,2,0,2,0,0,2,6,0)	20160	95	(1,4,0,1,0,1,1,0,9,0,0,1,0,1,1,4)	6720
46	(1,0,0,1,8,1,1,8,1,0,0,1,0,1,1,0)	840	96	(1,0,0,1,0,1,1,0,1,0,0,1,0,1,1,16)	240
47	(0,0,0,2,0,0,0,2,8,6,0,0,0,6,0,0)	10080	97	(0,4,0,2,0,0,0,2,0,10,0,0,0,2,0,4)	10080
48	(1,4,0,1,2,1,3,0,3,2,0,1,2,1,1,2)	26880	98	(0,0,0,2,0,0,0,2,8,2,0,0,0,2,8,0)	6720
49	(1,2,4,5,2,1,1,0,1,2,0,1,2,1,1,0)	13440	99	(0,0,0,2,0,0,0,2,0,2,0,0,0,6,0,12)	6720
50	(1,0,0,3,0,1,1,6,1,2,0,1,0,3,5,0)	40320	100	(1,0,0,3,0,1,5,2,5,2,0,1,0,3,1,0)	10080

101	(0,4,4,2,8,0,0,2,0,2,0,0,0,2,0,0)	26880
102	(0,0,0,0,2,0,2,0,10,6,0,0,2,0,0,2)	13440
103	(1,0,0,1,0,1,1,0,1,0,0,13,0,1,5,0)	1680
104	(1,0,0,3,0,1,1,2,5,2,0,1,4,3,1,0)	40320
105	(0,0,0,0,2,8,2,0,2,2,0,4,2,0,0,2)	40320
106	(4,4,0,0,0,0,12,4,0,0,0,0,0,0,0,0)	560
107	(3,0,0,3,0,3,3,0,3,0,0,3,0,3,3,0)	30
108	(0,0,4,2,0,0,4,2,4,2,0,0,4,2,0,0)	420
109	(1,0,0,3,0,1,1,2,1,10,0,1,0,3,1,0)	6720
110	(0,0,0,2,0,0,0,2,0,14,0,0,0,6,0,0)	1680
111	(2,2,2,0,2,2,0,2,6,2,0,2,0,0,0,2)	4480
112	(1,0,4,5,0,1,5,0,1,0,0,1,0,5,1,0)	13440
113	(0,4,0,0,6,0,2,0,2,2,0,0,2,0,4,2)	80640
114	(0,4,0,2,0,0,0,2,0,6,4,0,4,2,0,0)	26880
115	(4,0,0,2,0,0,0,2,0,2,0,0,4,2,4,4)	40320
116	(0,12,0,2,0,0,0,2,0,2,0,0,0,2,0,4)	5040
117	(1,0,0,7,0,1,1,6,1,2,0,1,0,3,1,0)	13440
118	(1,0,4,1,0,1,1,0,7,0,0,3,0,3,3,0)	13440
119	(0,6,0,2,0,2,2,0,2,0,6,0,0,2,2,0)	3360
120	(3,0,0,1,4,3,1,0,1,2,0,1,0,3,3,2)	40320
121	(1,0,0,3,0,1,1,6,1,6,0,1,0,3,1,0)	3360
122	(0,2,4,6,0,2,2,0,2,0,2,0,0,2,2,0)	13440
123	(0,0,4,2,0,0,4,6,0,6,0,0,0,2,0,0)	10080
124	(1,2,2,1,2,1,1,2,1,2,2,1,2,1,1,2)	30
125	(8,0,0,0,0,8,0,0,0,8,0,0,0,0,0,0)	560
126	(4,4,0,0,0,0,4,4,8,0,0,0,0,0,0,0)	1680
127	(1,1,1,1,1,1,1,1,9,1,1,1,1,1,1,1)	16
128	(0,0,4,0,4,0,0,0,0,0,8,0,4,4,0,0)	6720
129	(0,0,0,0,2,0,2,0,10,2,0,4,2,0,0,2)	26880
130	(1,0,0,3,0,1,1,2,1,2,4,1,0,3,1,4)	40320
131	(0,2,0,2,0,2,2,0,10,0,2,0,0,2,2,0)	6720
132	(0,0,0,0,2,0,2,0,2,6,0,8,2,0,0,2)	26880
133	(0,4,4,0,4,0,0,4,0,0,0,0,4,0,0,4)	840
134	(0,0,0,0,6,0,2,0,2,2,0,0,6,0,0,6)	8960
135	(1,2,0,1,2,1,1,0,1,2,0,5,2,5,1,0)	6720
136	(1,1,1,1,5,1,1,5,1,1,1,1,1,1,1,1)	120
137	(0,0,0,0,4,0,0,20,0,0,0,0,0,0,0,0)	240
138	(5,4,0,3,0,1,1,2,1,2,0,1,0,3,1,0)	80640
139	(1,0,0,3,8,1,1,2,1,2,0,1,0,3,1,0)	20160
140	(4,0,0,0,2,0,2,0,6,2,0,0,6,0,0,2)	26880
141	(1,0,0,1,2,1,3,0,3,2,0,5,2,1,1,2)	40320
142	(1,0,0,5,12,1,1,0,1,0,0,1,0,1,1,0)	1920
143	(4,4,0,2,0,0,0,2,0,2,0,0,0,2,4,4)	5040
144	(1,0,4,5,0,1,1,0,3,0,0,3,0,3,3,0)	13440
145	(0,0,0,2,0,0,0,2,0,10,0,0,0,6,0,4)	20160
146	(5,0,0,5,0,5,5,0,1,0,0,1,0,1,1,0)	420
147	(1,2,0,1,2,1,1,0,1,2,8,1,2,1,1,0)	1680
148	(8,0,0,8,4,0,0,4,0,0,0,0,0,0,0,0)	840
149	(5,4,0,1,0,1,5,4,1,0,0,1,0,1,1,0)	3360
150	(0,0,0,6,0,0,0,6,0,6,0,0,0,6,0,0)	140
151	(0,0,2,2,0,0,2,2,10,2,0,0,2,2,0,0)	240
152	(0,0,0,2,0,0,8,2,8,2,0,0,0,2,0,0)	2520
153	(0,12,4,2,0,0,0,2,0,2,0,0,0,2,0,0)	13440
154	(1,0,0,5,0,5,5,0,1,0,0,1,0,5,1,0)	1680
155	(1,12,0,1,0,1,1,0,1,0,0,1,0,1,1,4)	1680
156	(5,0,0,1,0,1,1,0,1,0,0,1,4,1,5,4)	20160
157	(0,0,0,2,0,0,0,2,0,2,16,0,0,2,0,0)	1680
158	(0,0,0,0,2,0,2,0,2,2,0,12,2,0,0,2)	4480
159	(1,0,4,7,0,1,1,2,1,2,0,1,0,3,1,0)	40320
160	(1,2,0,5,2,1,1,0,1,2,0,5,2,1,1,0)	5040

The following table gives the coefficients of the 9 Niemeier codes. The number in the first column refers to the 160 monomials as in Table I. The second line is the greatest common divisor of the coefficients of the 9 Niemeier codes. The last 9 columns give the coefficients divided by the greatest common divisor. For example the coefficient of the 127-th monomial $(1,1,1,1,1,1,1,1,9,1,1,1,1,1,1,1)$ (i.e. $F^8_{(1,0,0,0)} \prod F_a$) with respect to the code C_7 (this is the so-called Golay code) is given by $967\,680 \cdot 253$.

Table II

MON	GCD	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
1	96	3720	1715	882	297	34650	60	0	8526	6174
2	9216	12300	7301	4440	2064	73920	600	0	23030	18522
3	72	27660	24143	18996	13335	23100	6960	0	33236	35868
4	24	1560	1057	498	171	6930	0	0	3430	4158
5	120	12	1	0	0	792	0	0	56	0
6	768	186000	133525	100800	70836	1478400	39120	0	354760	321048
7	27648	232000	172725	117600	71028	739200	31760	0	378280	386904

https://doi.org/10.1017/S0027763000022133 Published online by Cambridge University Press

MON	GCD	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
8	2304	78375	78400	88515	99864	51975	115005	132825	75215	64827
9	96	2652	2856	3047	3225	1155	3390	3542	2205	2205
10	2160	8220	11515	13296	17010	27720	21856	28336	9800	16464
11	192	1920	1561	576	180	0	0	0	4592	7056
12	768	1080	1169	1008	810	0	480	0	1148	1764
13	27648	51200	55125	60000	62292	0	63000	61985	33320	24696
14	9216	410100	395675	401760	412938	1108800	430140	451605	473830	438354
15	2304	6000	5355	5136	4140	0	2520	0	3920	0
16	1152	339150	188650	182790	184356	2321550	238290	345345	558110	228438
17	288	120000	134995	117504	88848	0	47040	0	43120	16464
18	4608	7800	7350	16920	22608	0	26700	26565	4655	1029
19	9216	206400	184975	177600	180732	739200	192960	212520	288120	271656
20	768	6600	7805	9760	11898	0	14520	17710	3500	1764
21	384	88320	79135	70848	52668	0	28800	0	69776	49392
22	2304	27600	39445	54288	69480	0	86520	106260	3920	0
23	24	10	6	3	1	55	0	0	21	21
24	2304	58800	54145	41232	28908	0	15000	0	74480	98784
25	25920	44940	50225	54048	56022	9240	56672	56672	25480	16464
26	384	156000	132055	118368	87444	0	48240	0	121520	49392
27	1152	687750	513275	373230	240642	2321550	117990	0	1079470	1043406
28	1	1	1	1	1	1	1	1	1	1
29	4032	1650	280	702	486	4950	1230	3795	1550	1638
30	192	14400	10045	7680	4212	0	1440	0	14000	7056
31	48	9600	5929	2304	576	0	0	0	24304	35280
32	1536	159720	160475	160416	160314	147840	159960	159390	158564	160524
33	3456	27840	28175	22752	17352	0	9840	0	33712	49392
34	96	6000	2450	2187	1143	51975	450	0	10465	3969
35	4608	540	721	504	324	0	120	0	938	2646
36	576	51000	51695	43164	32805	69300	18180	0	51940	61740
37	192	36150	20825	23625	27846	155925	44640	79695	49735	27783
38	96	2400	3479	1728	792	0	0	0	6328	17640
39	4032	6600	8155	5796	4599	9900	2700	0	11900	22932
40	336	0	1	0	0	0	0	0	4	12
41	288	24000	29470	35145	41229	17325	47310	53130	16555	17787
42	768	61200	45325	45600	53892	0	74400	106260	72520	24696
43	221184	160800	144550	133800	125232	369600	119180	115115	199920	189336
44	384	3600	2695	1728	1836	0	1680	0	7000	3528
45	1152	5610	3283	2190	1047	11550	330	0	8526	6174
46	384	3120	500	288	0	63360	0	0	8533	2079
47	48	2460	2107	1710	1317	11550	780	0	4018	4410
48	2304	973200	1037575	1074960	1110312	0	1140600	1168860	827120	889056
49	1152	472800	496615	570816	641592	0	733440	850080	309680	148176
50	768	10320	12299	10224	8622	0	5400	0	13720	24696
51	48	52500	33565	21492	10575	207900	3240	0	83300	61740
52	768	9300	6475	4536	2340	0	720	0	8855	2205
53	4608	3000	2205	2792	3420	0	5220	8855	1960	0
54	384	3504000	2750125	1992960	1289196	8870400	623520	0	5225360	5383728
55	72	825	245	117	27	17325	5	0	2485	1421
56	3696	0	17	0	0	0	0	0	72	216
57	12	585	476	363	246	1155	125	0	791	791
58	384	16320	7831	2880	684	253440	0	0	50960	49392
59	13824	1327200	1275225	1205760	1137588	1478400	1065920	991760	1446480	1498224
60	3	213	221	229	237	165	245	253	197	197
61	144	4580	3675	2988	2075	7700	1080	0	5292	4116
62	6	105	68	39	18	495	5	0	203	203
63	24	180	97	36	9	1980	0	0	532	588
64	3456	36590	36505	37722	39309	53130	41510	44275	37338	34986
65	144	327900	354025	377820	397017	69300	412680	425040	257740	251076

MON	GCD	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
66	288	49500	40425	36990	27378	103950	15060	0	47530	26754
67	528	130	102	75	49	319	24	0	189	189
68	96	4160	2309	1152	368	28160	0	0	9044	8316
69	96	125040	96775	71082	46209	330330	22560	0	183358	179046
70	432	235500	197225	157680	108774	138600	55680	0	252840	214032
71	144	6600	3955	1818	648	34650	120	0	15750	17934
72	48	13980	12691	7632	4374	138600	1440	0	33320	49392
73	13824	7200	10780	13176	17226	0	21780	26565	6860	12348
74	1536	9240	7379	5616	3786	21120	1920	0	12740	12348
75	1152	3600	6125	5760	6048	0	4560	0	5320	10584
76	48	900	455	126	27	20790	0	0	3738	4410
77	4608	183000	110250	58320	24552	1108800	6240	0	404495	420861
78	192	9840	8827	7392	5364	0	2880	0	9436	8820
79	41472	301800	249900	224430	208632	1120350	203080	203665	457170	446586
80	24	4260	2107	828	225	69300	0	0	11900	8820
81	12	2835	1568	981	450	17325	135	0	5201	3633
82	1536	15840	18137	20184	22302	0	24420	26565	11564	12348
83	4	689	680	671	662	743	653	644	707	707
84	4608	4200	6335	9576	14130	0	19860	26565	1960	0
85	48	311100	229565	131328	62676	415800	17280	0	554680	691488
86	336	0	17	0	0	0	0	0	112	336
87	576	10950	5705	8298	6984	34650	4590	0	11410	882
88	720	945	0	379	108	10395	361	1771	735	343
89	48	30375	12250	9405	4212	155925	1515	0	54145	29841
90	24	345	175	81	27	3465	5	0	889	833
91	384	32160	23653	14688	7308	0	2160	0	46256	49392
92	2160	1620	1421	596	219	4620	0	0	4116	6860
93	48	59520	33019	20736	7200	0	0	0	74480	35280
94	4608	140000	99225	74880	47732	492800	23040	0	199920	148176
95	480	1440	2107	960	432	0	0	0	1624	3528
96	1344	0	0	0	0	0	0	0	1	3
97	48	11820	10045	8886	6687	20790	3780	0	11914	7938
98	48	1410	245	153	0	17325	0	0	3703	2079
99	24	780	371	228	75	4620	0	0	1316	588
100	384	307200	196735	108288	48636	1774080	12960	0	634256	642096
101	480	2220	917	675	225	10395	0	0	3157	441
102	72	620	105	64	0	9240	0	0	1400	0
103	576	80	63	32	12	0	0	0	140	196
104	768	276000	238385	188352	130446	0	66960	0	307720	321048
105	72	32700	26635	17424	9450	138600	2880	0	43400	35280
106	24	5265	5831	6473	7191	3465	7985	8855	4361	4361
107	43008	83600	77175	71280	65204	123200	59160	53130	95340	93492
108	576	1745850	1669675	1592955	1520154	2269575	1449660	1381380	1915165	1927317
109	192	3840	3115	2112	900	0	0	0	4592	7056
110	288	40	34	27	19	55	10	0	49	49
111	1728	16500	25725	39408	53076	46200	64720	70840	17640	32928
112	288	14400	11025	6912	3456	0	960	0	18032	16464
113	48	102300	84035	79488	64728	415800	40320	0	131320	98784
114	144	7500	4655	3660	3357	69300	2640	0	14700	4116
115	192	68250	67375	52245	40302	155925	23490	0	110495	163611
116	96	1380	938	597	297	3465	90	0	2135	1911
117	768	8160	6629	3168	1170	0	0	0	16072	24696
118	384	14400	13685	7776	4104	0	1200	0	27440	49392
119	3456	4150	1960	882	279	53130	50	0	11466	10290
120	4608	41400	47775	52200	59184	0	68100	79695	37240	49392
121	384	66240	27685	16704	5100	591360	0	0	126224	49392
122	576	18600	16415	16524	23301	69300	35580	53130	28420	12348
123	96	27360	29743	32826	37809	34650	44520	53130	28126	30870

MON	GCD	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
124	2322432	93000	80850	71880	62292	184800	53140	44275	112770	102606
125	18	845	833	869	953	1925	1085	1265	1013	1013
126	24	15075	10045	10455	12177	51975	17355	26565	20335	7791
127	967680	0	70	24	72	0	148	253	80	168
128	120	6975	6097	4707	3321	10395	1755	0	9331	10731
129	24	9300	3745	1872	450	83160	0	0	18872	7056
130	384	566400	750925	941760	1095084	0	1209600	1275120	145040	49392
131	1728	470	210	126	45	2310	10	0	798	294
132	24	22020	14147	4896	1260	138600	0	0	68488	98784
133	144	175325	159250	156335	161348	544775	174545	194810	234955	223979
134	144	1060	245	288	100	15400	0	0	1960	0
135	2304	30480	42287	48672	60840	0	78720	106260	32536	74088
136	387072	6720	5082	5448	5928	21120	7100	8855	9072	5880
137	6	5	4	3	2	11	1	0	7	7
138	768	45600	61985	80352	103662	0	130320	159390	25480	24696
139	192	33600	39655	41472	37332	0	24480	0	14000	7056
140	48	35580	21805	11664	4050	138600	0	0	61544	49392
141	4608	61800	79625	93672	108414	0	121620	132825	37240	49392
142	336	0	51	0	0	0	0	0	112	336
143	192	282750	145775	86625	37386	2373525	10620	0	578935	398223
144	384	50400	45815	38592	28944	0	16320	0	50960	49392
145	72	1460	1029	596	285	4620	80	0	2660	2940
146	1152	64800	49735	35136	22104	177408	10368	0	100156	102900
147	4608	3300	3815	6336	9702	0	15900	26565	1330	2646
148	36	34385	34544	34727	34934	33935	35165	35420	34139	34139
149	576	86400	49735	23424	7368	591360	0	0	200312	205800
150	192	25922	24990	24057	23123	31493	22188	21252	27783	27783
151	12096	370	469	570	741	330	970	1265	386	546
152	144	8550	5329	2979	1332	48015	360	0	17437	17157
153	48	240	35	18	0	6930	0	0	798	294
154	1152	12192	14455	16992	18936	0	20400	21252	6076	4116
155	768	180	35	24	0	0	0	0	329	147
156	576	13440	10535	10944	8712	0	5280	0	12152	8232
157	24	30	5	3	0	495	0	0	77	21
158	216	140	63	16	2	3080	0	0	616	784
159	768	20400	23975	22896	18882	0	11160	0	13720	24696
160	2304	197520	139895	112896	77400	887040	41040	0	277144	172872

References

[BFW] R. Borcherds, E. Freitag, R. Weissauer, *A Siegel cusp form of degree* 12 *and weight* 12, J. reine angew. Math., **494** (1998), 141–153.

[Du] W. Duke, *On codes and Siegel modular forms*, Int. Math. Research Notoces (1993), 125–136.

[Gl] A. M. Gleason, *Weight polynomials of self-dual codes and the MacWilliams identities*, Actes Congrés Intern. des Mathématiciens (Nice 1970), Tome 3, Gauthier-Villars, Paris (1971), pp. 211–215.

[Ou] M. Oura, *The dimension formula for the ring of code polynomials in genus* 4, Osaka J. Math., **34** (1997), 53–72.

[PS] V. Pless and N. J. A. Sloane, *On the classification and enumeration of self-dual codes*, J. Comb. Th., **A18** (1975), 313–335.

[PY] C. Poor and D. Yuen, *Dimensions of spaces of Siegel modular forms of low weight in degree four*, Bull. Austral. Math. Soc., **54** (1996), 309–315.

[Ru1] B. Runge, *On Siegel modular forms, Part I*, J. reine angew. Math., **436** (1993), 57–85.

[Ru2] ———, *On Siegel modular forms, Part II*, Nagoya Math. J., **138** (1995), 179–197.

[Ru3] ———, *Codes and Siegel modular forms*, Disc. Math., **148** (1996), 175–204.

Eberhard Freitag
Mathematisches Institut
Im Neuenheimer Feld 288, D69120
Heidelberg
freitag@mathi.uni-heidelberg.de

Manabu Oura
Graduate School of Mathematics
Kyushu University
Japan
ohura@math.kyushu-u.ac.jp

https://doi.org/10.1017/S0027763000022133 Published online by Cambridge University Press