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SINGULARITIES OF HOROSPHERICAL HYPERSURFACES OF CURVES IN HYPERBOLIC 4-SPACE

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Abstract

We consider the contact between curves and hyperhorospheres in hyperbolic 4-space as an application of the theory of singularities of functions. We define the osculating hyperhorosphere and the horospherical hypersurface of the curve whose singular points correspond to the locus of polar vectors of osculating hyperhorospheres of the curve. One of the main results is a generic classification of singularities of horospherical hypersurfaces of curves.

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1. Introduction

In [2], we constructed some basic tools and applied singularity theory to local differential geometry on hypersurfaces in hyperbolic space. These tools work very well for hypersurfaces. The next step is to consider the case of submanifolds with higher codimensions. In this paper, we stick to hyperbolic space curves, the simplest case with higher codimensions. We study the contact between hyperbolic space curves and hyperhorospheres as an application of singularity theory of smooth functions. One of the basic tools that we gave in [2] is the notion of the horospherical height function on a hypersurface. We define the horospherical height function of a hyperbolic space curve. By using the techniques of singularity theory on such a function, we define osculating horospheres along a hyperbolic space curve (see Section 3). We also define the horospherical hypersurface of a hyperbolic space curve as the discriminant set of the horospherical height function on the curve. Compared with the case of curves in Euclidean space, the situation is rather different because the horospherical hypersurface is defined in the lightcone. It might be considered as a

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kind of dual hypersurface of the curve. The main result in this paper is Theorem 2.1, which gives a generic classification of singularities of horospherical hypersurfaces of hyperbolic space curves. Moreover, we study the geometric meanings of singularities of horospherical hypersurfaces of hyperbolic space curves and introduce a new invariant $\sigma(s)$. We show that $\sigma(s) = 0$ for all s if and only if the curve is located on a hyperhorosphere, under a certain generic assumption (see Section 3).

All maps considered here are of class C^∞ unless otherwise stated.

2. Basic notions and results

Let $\mathbb{R}^{n+1}$ be an $(n + 1)$-dimensional vector space with typical element $\mathbf{x}$ or $(x_0, x_1, \dots, x_n)$, where each $x_i \in \mathbb{R}$. The *pseudo-scalar product* of $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^{n+1}$ is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i.$$

The space $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$ is called *Minkowski* $(n + 1)$*-space* and is denoted by $\mathbb{R}^{n+1}_1$.

We say that a vector $\mathbf{x}$ in $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ is *spacelike*, *lightlike* or *timelike* when $\langle \mathbf{x}, \mathbf{x} \rangle$ is positive, zero, or negative, respectively. The norm of the vector $\mathbf{x} \in \mathbb{R}^{n+1}$ is defined by $\|\mathbf{x}\| = |\langle \mathbf{x}, \mathbf{x} \rangle|^{1/2}$. Given a vector $\mathbf{n} \in \mathbb{R}^{n+1}_1$ and a real number c, the hyperplane with pseudo-normal $\mathbf{n}$ is given by

$$HP(\mathbf{n}, c) = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 : \langle \mathbf{x}, \mathbf{n} \rangle = c\}.$$

We say that $HP(\mathbf{n}, c)$ is a *spacelike*, *timelike* or *lightlike hyperplane* when $\mathbf{n}$ is timelike, spacelike or lightlike, respectively. The *hyperbolic n-space* $H^n_+(-1)$ is defined by

$$H^n_+(-1) = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 : \langle \mathbf{x}, \mathbf{x} \rangle = -1, x_0 > 0\}.$$

Given vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in \mathbb{R}^{n+1}_1$, we may define a new vector $\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_n$ as follows:

$$\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_n = \begin{vmatrix} -\mathbf{e}_0 & \mathbf{e}_1 & \cdots & \mathbf{e}_n \\ a_0^1 & a_1^1 & \cdots & a_n^1 \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{vmatrix},$$

where $\{\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the canonical basis of $\mathbb{R}^{n+1}_1$ and $\mathbf{a}_i = (a_0^i, a_1^i, \dots, a_n^i)$ when $i = 1, \dots, n$. It is easy to check that

$$\langle \mathbf{a}, \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_n \rangle = \det \begin{pmatrix} \mathbf{a} \\ \dots \\ \mathbf{a}_n \end{pmatrix},$$

so $\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_n$ is pseudo-orthogonal to all $\mathbf{a}_i$ (where $i = 1, \dots, n$).

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We also define the set

$$LC_a = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 : \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = 0\},$$

which is called the *closed lightcone* with vertex $\mathbf{a}$. We denote

$$LC^*_+ = \{\mathbf{x} = (x_0, \dots, x_n) \in LC_0 : x_0 > 0\}$$

and call it the *future lightcone* at the origin.

If $\mathbf{x} = (x_0, x_1, \dots, x_n)$ is a lightlike vector, then $x_0 \neq 0$ and thus

$$\tilde{\mathbf{x}} = \frac{1}{x_0}(x_0, x_1, \dots, x_n) \in \{\mathbf{y} : \langle \mathbf{y}, \mathbf{y} \rangle = 0, y_0 = 1\} = S^{n-1}_+.$$

The subset S^{n-1}_+ is known as the *lightcone* $(n-1)$-*sphere*. There are three kinds of hypersurfaces in $H^n_+(-1)$ that are given by intersections of $H^n_+(-1)$ and hyperplanes in $\mathbb{R}^{n+1}_1$. A hypersurface $H^n_+(-1) \cap HP(\mathbf{v}, c)$ is called a *hypersphere*, an *equidistant hyperplane* or a *hyperhorosphere* if $HP(\mathbf{v}, c)$ is spacelike, timelike or lightlike, respectively. In particular, we write a hyperhorosphere as

$$HS^{n-1}(\mathbf{v}, c) = H^n_+(-1) \cap HP(\mathbf{v}, c).$$

If we consider a lightlike vector $\mathbf{v}$ and write $\mathbf{v}_0 = (-1/c)\mathbf{v}$, then

$$HS^{n-1}(\mathbf{v}, c) = HS^{n-1}(\mathbf{v}_0, -1).$$

We call $\mathbf{v}_0$ the *polar vector* of $HS^{n-1}(\mathbf{v}_0, -1)$.

Given a regular curve $\gamma : I \to H^n_+(-1)$, parametrized by arc length, where I is an open interval or the unit circle in the Euclidean plane, we define a pseudo-orthonormal frame

$$\{\gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \dots, \mathbf{n}_{n-1}(s)\}$$

for $\mathbb{R}^{n+1}_1$ along γ that satisfies the following Frenet–Serret type formulae:

$$\begin{aligned}
\gamma'(s) &= \mathbf{n}_0(s),\\
\mathbf{n}'_0(s) &= k_1(s)\mathbf{n}_1(s) + \gamma(s),\\
\mathbf{n}'_1(s) &= -k_1(s)\mathbf{t}(s) + k_2(s)\mathbf{n}_2(s),\\
\cdots &= \cdots,\\
\mathbf{n}'_i(s) &= -k_i(s)\mathbf{n}_{i-1}(s) + k_{i+1}(s)\mathbf{n}_{i+1}(s),\\
\cdots &= \cdots,\\
\mathbf{n}'_{n-2}(s) &= -k_{n-2}(s)\mathbf{n}_{n-3}(s) + k_{n-1}(s)\mathbf{n}_{n-1}(s),\\
\mathbf{n}'_{n-1}(s) &= -k_{n-1}(s)\mathbf{n}_{n-2}(s),
\end{aligned}$$

where $\mathbf{n}_0(s) = \mathbf{t}(s)$, while $k_i(s) = \|\mathbf{n}'_{i-1}(s) + k_{i-1}\,\mathbf{n}_{i-2}(s)\|$ when $i = 1, 2, \dots, n-1$, and

$$k_{n-1}(s) = -\frac{1}{k_1^{n-1}(s)k_2^{n-2}(s)\cdots k_{n-2}^2(s)} \det\begin{pmatrix} \gamma(s)\\ \gamma'(s)\\ \vdots\\ \gamma^{(n)}(s)\end{pmatrix}.$$

Consider the horospherical height function on γ,

$$H : I \times LC^*_+ \to \mathbb{R}$$
$$(s, \mathbf{v}) \mapsto \langle \gamma(s), \mathbf{v} \rangle + 1 = h_v(s).$$

It is tedious but straightforward to show that

$$h_v(s_0) = h_v'(s_0) = \cdots = h_v^{(n-1)}(s_0) = 0$$

if and only if $\mathbf{v} = \mathbf{v}_0 \in LC^*_+$, where

$$\mathbf{v}_0 = \gamma(s_0) + \sum_{j=1}^{n-2} \sigma_j \mathbf{n}_j(s_0) \pm \Big(1 - \sum_{j=1}^{n-2} \sigma_j\Big)^{1/2} \mathbf{n}_{n-1}(s_0),$$

the σ_j (where $j = 1, \ldots, n-2$) being real-valued functions that depend on the functions k_j and their derivatives (where $j = 1, \ldots, n-3$) and on k_{n-2}. Moreover,

$$h_v(s_0) = h_v'(s_0) = \cdots = h_v^{(n)}(s_0) = 0$$

if and only if $\mathbf{v}$ is as above and $\sigma_n(s_0) = 0$, where σ_n is a real-valued function that depends on the functions k_j and their derivatives (where $j = 1, \ldots, n-2$) and on k_{n-1}. The function σ_n gives a measure of how much the curve γ is contained in a hyperhorosphere. We conjecture that the function σ_n is a Lorentzian invariant of γ, and we call it the *hyperhorospherical torsion* of γ.

In this paper, we treat the case where $n = 4$.

The *horospherical flattenings* of a curve γ immersed in $H^4_+(-1)$ are the zeros of the hyperhorospherical torsion of γ. We consider the contact between hyperbolic space curves and hyperhorospheres. This is a special subject in hyperbolic differential geometry.

Let $\gamma : I \to H^4_+(-1)$ be a unit-speed hyperbolic space curve. We now define a map $HS_\gamma : I \times J \times K \to LC^*_+$ by

$$HS_\gamma(s, \theta, \phi) = \gamma(s) + \cos\theta\, \mathbf{n}_1(s) + \sin\theta \cos\phi\, \mathbf{n}_2(s) + \sin\theta \sin\phi\, \mathbf{n}_3(s),$$

where I and J are open intervals or the unit circle in the Euclidean plane. We call HS_γ the *horospherical hypersurface* of γ. We also introduce a hyperbolic invariant

$$\sigma(s) = ((k_1 k_1'' k_2 - k_1^2 k_2^3 - 2{k_1'}^2 k_2 - k_1 k_1' k_2')^2 - (k_1 k_2 k_3)^2 (k_1^4 k_2^2 - k_1^2 k_2^2 - {k_1'}^2))(s).$$

The geometric meaning of these will be discussed in Section 3. For the definition of A_k, where $k = 2, 3, 4$, see [1]. Our main result is the following theorem.

THEOREM 2.1. *Let* $\mathrm{Emb}(I, H^4_+(-1))$ *be the space of proper embeddings* $\gamma : I \to H^4_+(-1)$, *equipped with the Whitney* C^∞*-topology. Then there exists an open dense subset* $\mathcal{O}$ *of* $\mathrm{Emb}(I, H^4_+(-1))$ *such that for all* $\gamma \in \mathcal{O}$, *the horospherical hypersurface* HS_γ *of* γ *at each singular point is locally diffeomorphic to a map germ of cusp type* A_2, *swallow tail type* A_3 *or butterfly type* A_4.

3. Horospherical height functions and invariants of hyperbolic space curves

In this section we introduce a family of functions on a curve that is useful for the study of invariants of hyperbolic space curves. Given a hyperbolic space curve $\gamma : I \to H^4_+(-1)$, we define the function $H : I \times LC^*_+ \to \mathbb{R}$ by $H(s, \mathbf{v}) = \langle \gamma(s), \mathbf{v}\rangle + 1$. We call H the *horospherical height function* on γ. We write $h_v(s) = H_v(s) = H(s, \mathbf{v})$ for all fixed vectors $\mathbf{v} \in LC^*_+$.

Proposition 3.1. *Let $\gamma : I \to H^4_+(-1)$ be a unit-speed hyperbolic space curve such that $k_1k_2(k_1^4k_2^2 - k_1^2k_2^2 - {k_1'}^2)(s) \neq 0$ for all s in I. Then the following hold.*

First, $h_{v_0}(s_0) = 0$ if and only if there exist real numbers λ, μ_i (where $i = 1, 2, 3$) such that $\lambda^2 + \sum_{i=1}^3 \mu_i^2 = 1$ and

$$\mathbf{v}_0 = \gamma(s_0) + \lambda\mathbf{t}(s_0) + \sum_{i=1}^{3} \mu_i\mathbf{n}_i(s_0).$$

Second, $h_{v_0}(s_0) = h'_{v_0}(s_0) = 0$ if and only if there exist $\theta_0, \phi_0 \in (0, 2\pi]$ such that

$$\mathbf{v}_0 = \gamma(s_0) + \cos\theta_0\, \mathbf{n}_1(s_0) + \sin\theta_0 \cos\phi_0\, \mathbf{n}_2(s_0) + \sin\theta_0 \sin\phi_0\, \mathbf{n}_3(s_0). \tag{3.1}$$

Third, $h_{v_0}(s_0) = h'_{v_0}(s_0) = h''_{v_0}(s_0) = 0$ if and only if (3.1) holds and

$$\sigma_1(s_0) = \cos\theta_0 = \frac{1}{k_1(s_0)}.$$

Fourth, $h_{v_0}(s_0) = h'_{v_0}(s_0) = h''_{v_0}(s_0) = h^{(3)}_{v_0}(s_0) = 0$ if and only if (3.1) holds, and

$$\begin{aligned} \sigma_1(s_0) &= \cos\theta_0 = \frac{1}{k_1(s_0)}, \\ \sigma_2(s_0) &= \sin\theta_0\cos\phi_0 = -\frac{{k_1}'(s_0)}{(k_1^2k_2)(s_0)}, \\ \sigma_3(s_0) &= \sin\theta_0\sin\phi_0 = \pm\left(1 - \frac{1}{k_1^2(s_0)} - \frac{{k_1}'(s_0)}{(k_1k_2)^2(s_0)}\right)^{1/2}. \end{aligned} \tag{3.2}$$

Fifth, $h_{v_0}(s_0) = h'_{v_0}(s_0) = \cdots = h^{(4)}_{v_0}(s_0) = 0$ if and only if both (3.1) and (3.2) hold and $\sigma(s_0) = 0$, where

$$\sigma(s) = (k_1k_1''k_2 - k_1^2k_2^3 - 2{k_1'}^2k_2 - k_1k_1'k_2')^2(s) - (k_1k_2k_3)^2(k_1^4k_2^2 - k_1^2k_2^2 - {k_1'}^2)(s). \tag{3.3}$$

Sixth, $h_{v_0}(s_0) = h'_{v_0}(s_0) = \cdots = h^{(5)}_{v_0}(s_0) = 0$ if and only if both (3.1) and (3.2) hold and $\sigma(s_0) = \sigma'(s_0) = 0$.

Proof. This follows by direct calculation. □

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The function

$$\sigma(s) = ((k_1 k_1'' k_2 - k_1^2 k_2^3 - 2{k_1'}^2 k_2 - k_1 k_1' k_2')^2 - (k_1 k_2 k_3)^2 (k_1^4 k_2^2 - k_1^2 k_2^2 - {k_1'}^2))(s)$$

on γ has a special geometric meaning, which we now try to understand. Let $\mathbf{v}$ be a lightlike vector and $\mathbf{w}$ be a spacelike vector. A surface $HS^3(\mathbf{v}, -1) \cap HP(\mathbf{w}, 0)$ is called a *horosphere*.

Proposition 3.2. *Let $\gamma : I \to H_+^4(-1)$ be a unit-speed hyperbolic space curve such that $k_1 k_2(s) \neq 0$ and $(k_1^4 k_2^2 - k_1^2 k_2^2 - {k_1'}^2)(s) \geq 0$ for all $s \in I$. We consider the vector field along γ given by*

$$\mathbf{v} = \gamma(s) + \cos\theta\, \mathbf{n}_1(s) + \sin\theta \cos\phi\, \mathbf{n}_2(s) + \sin\theta \sin\phi\, \mathbf{n}_3(s),$$

where

$$\begin{aligned} \cos\theta &= \frac{1}{k_1(s)}, \\ \sin\theta \sin\phi &= \pm\left(1 - \frac{1}{k_1^2(s)} - \frac{(k_1')^2(s)}{(k_1^2 k_2)^2(s)}\right)^{1/2}, \\ \sin\theta \cos\phi &= -\frac{k_1'(s)}{(k_1^2 k_2)(s)}. \end{aligned}$$

First, suppose that $(k_1^4 k_2^2 - k_1^2 k_2^2 - {k_1'}^2)(s) = 0$ for all s. Then the following conditions are equivalent:

(a) *$\mathbf{v}(s)$ is a constant vector;*
(b) *$k_3(s) = 0$ and $\sigma(s) = 0$ for all s;*
(c) *γ is a part of a horosphere.*

Second, suppose that the set $\{s \in I : (k_1^4 k_2^2 - k_1^2 k_2^2 - {k_1'}^2)(s) = 0\}$ consists of isolated points. Then the following conditions are equivalent:

(d) *$\mathbf{v}(s)$ is a constant vector;*
(e) *$\sigma(s) = 0$ for all s, where σ is given by (3.3);*
(f) *γ is located on a hyperhorosphere.*

Proof. Suppose that $(k_1^4 k_2^2 - k_1^2 k_2^2 - {k_1'}^2)(s) = 0$ for all s. Then

$$\mathbf{v}(s) = \gamma(s) + \frac{1}{k_1(s)}\, \mathbf{n}_1(s) - \frac{k_1'(s)}{(k_1^2 k_2)(s)} \mathbf{n}_2(s),$$

so that

$$\mathbf{v}'(s) = \frac{(k_1 k_2 + k_1^3 k_1' k_2')(s)}{k_1^5 k_2^2(s)}\, \mathbf{n}_2(s) + \frac{(k_1'' + 2{k_1'}^2 k_2 - k_1' k_3)(s)}{k_1^2 k_2(s)}\, \mathbf{n}_3(s).$$

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Therefore $\mathbf{v}(s)$ is constant if and only if $k_1'k_3(s) = 0$ and

$$(k_1^2k_2^3 - k_1k_1''k_2 + 2{k_1'}^2k_2 + k_1k_1'k_2')(s) = 0$$

for all s. If $k_1'(s) = 0$, then $k_1(s) = 0$ or $k_2(s) = 0$ because

$$(k_1^2k_2^3 - k_1k_1''k_2 + 2{k_1'}^2k_2 + k_1k_1'k_2')(s) = 0;$$

this contradicts the assumption that $k_1k_2(s) \neq 0$. Thus $k_3(s) = 0$ and

$$(k_1^2k_2^3 - k_1k_1''k_2 + 2{k_1'}^2k_2 + k_1k_1'k_2')(s) = 0$$

for all s. This means that $k_3(s) = 0$ and $\sigma(s) = 0$. We consider the horosphere

$$HS^3(\mathbf{v}(s), -1) \cap \langle \gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \mathbf{n}_2(s)\rangle_{\mathbb{R}}$$

for all $s \in I$, where $\langle \gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \mathbf{n}_2(s)\rangle_{\mathbb{R}}$ is the space generated by $\gamma(s)$, $\mathbf{t}(s)$, $\mathbf{n}_1(s)$ and $\mathbf{n}_2(s)$. If $\mathbf{v}(s)$ is constant, then $k_3(s) = 0$. This means that $\mathbf{n}_3(s)$ is constant, so that the hyperplane $\langle \gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \mathbf{n}_2(s)\rangle_{\mathbb{R}}$ is also constant. In this case the hyperhorosphere $HS^3(\mathbf{v}(s), -1)$ is also constant. Thus the image of γ is a part of the horosphere given by $HS^3(\mathbf{v}(s), -1) \cap \langle \gamma(s), \mathbf{t}(s), \mathbf{n}_1(s), \mathbf{n}_2(s)\rangle_{\mathbb{R}}$. If γ is part of a horosphere, then it is a hyperbolic plane curve. Therefore $k_3(s) = 0$ for all s. This completes the proof of the first assertion.

Now we consider the second assertion. If $(k_1^4k_2^2 - k_1^2k_2^2 - {k_1'}^2)(s) \neq 0$, then

$$\mathbf{v} = \gamma(s) + \frac{1}{k_1(s)}\mathbf{n}_1(s) - \frac{k_1'(s)}{(k_1^2k_2)(s)}\,\mathbf{n}_2(s) \pm \left(1 - \frac{1}{k_1^2(s)} - \frac{({k_1}')^2(s)}{(k_1^4k_2^2)(s)}\right)^{1/2}\mathbf{n}_3(s).$$

Hence

$$\mathbf{v}'(s) = -\frac{\sigma(s)}{(k_1^3k_2^2)(s)}\,\mathbf{n}_2(s) \mp \frac{k_1'(s)\sigma(s)}{(k_1^5k_2^3)(s)}\,\mathbf{n}_3(s).$$

Therefore, $\mathbf{v}'(s) = 0$ if and only if $\sigma(s) = 0$. Conditions (d) and (e) are equivalent for these s.

By assumption, the set of points s where $(k_1^4k_2^2 - k_1^2k_2^2 - {k_1'}^2)(s) \neq 0$ is an open dense subset of I. Therefore, conditions (d) and (e) are equivalent at all points of I.

We now consider the horospherical height function $H(s, \mathbf{v})$ on γ. If γ is located on a hyperhorosphere $HS^3(\mathbf{v}_0, c)$, then we may choose $c = -1$. This means that $H(s, \mathbf{v}_0) = 0$ for all s. By the fifth assertion of Proposition 3.1,

$$((k_1k_1''k_2 - k_1^2k_2^3 - 2{k_1'}^2k_2 - k_1k_1'k_2')^2 - (k_1k_2k_3)^2(k_1^4k_2^2 - k_1^2k_2^2 - {k_1'}^2))(s) = 0$$

for all s. This means that condition (f) implies condition (e). If $\mathbf{v}(s)$ is a constant vector $\mathbf{v}_0$, then γ is located on $HS^3(\mathbf{v}_0, -1)$. □

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Remark 3.3. Let $\gamma : I \to H^4_+(-1)$ be a unit-speed hyperbolic space curve such that $(k_1^4k_2^2 - k_1^2k_2^2 - {k_1'}^2)(s) = 0$ and $k_1(s) \geq 1$ for all s. We consider the vector field along γ given by $\mathbf{v} = \gamma(s) + \cos\theta\,\mathbf{n}_1(s) + \sin\theta\,\mathbf{n}_2(s)$, where $\cos\theta = 1/k_1(s)$.

First, suppose that $k_1(s) = 1$ for all s. Then γ is a part of a horocycle.

Second, suppose that the set $\{s \in I : k_1(s) = 1\}$ consists of isolated points. Then γ is located on a horosphere (see [3]).

Let $F : H^4_+(-1) \to \mathbb{R}$ be a submersion and $\gamma : I \to H^4_+(-1)$ be a regular curve. We say that γ and $F^{-1}(0)$ have *at least k-point contact* at t_0 if the function $g(t) = F \circ \gamma(t)$ satisfies

$$g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0.$$

If γ and $F^{-1}(0)$ have at least k-point contact at t_0 and $g^{(k)}(t_0) \neq 0$, then we say that γ and $F^{-1}(0)$ have *k-point contact* when $t = t_0$. If a hyperhorosphere $HS^3(\mathbf{v}_0, -1)$ and a hyperbolic space curve γ have at least four-point contact at t_0, we call $HS^3(\mathbf{v}_0, -1)$ the *osculating hyperhorosphere of* γ at $\gamma(t_0)$.

Proposition 3.4. *Let $\gamma : I \to H^4_+(-1)$ be a unit-speed hyperbolic space curve.*

First, there exists an osculating hyperhorosphere of γ at a point $\gamma(s_0)$ if and only if $(k_1^4k_2^2 - k_1^2k_2^2 - {k_1'}^2)(s_0) > 0$.

Second, suppose that $(k_1^4k_2^2 - k_1^2k_2^2 - {k_1'}^2)(s_0) > 0$. Then the osculating hyperhorosphere and γ have five-point contact at $s = s_0$ if and only if $\sigma(s_0) = 0$ and $\sigma'(s_0) \neq 0$.

Proof. Define the function $\mathfrak{H} : H^4_+(-1) \times LC^*_+ \to \mathbb{R}$ by

$$\mathfrak{H}(\mathbf{x}, \mathbf{v}) = \langle \mathbf{x}, \mathbf{v} \rangle + 1.$$

For all $\mathbf{v} \in LC^*_+$, $\mathfrak{h}_{v_0}(\mathbf{x}) = \mathfrak{H}(\mathbf{x}, \mathbf{v}_0)$ is a submersion and $\mathfrak{h}^{-1}_{v_0}(0)$ is a hyperhorosphere. Moreover, each hyperhorosphere may be realized as the zero level set of $\mathfrak{h}_{v_0}$ for some $\mathbf{v}_0 \in LC^*_+$. Now $\mathfrak{h}_{v_0} \circ \gamma(s) = h(s)$, where $h(s) = H(s, \mathbf{v}_0)$, for all γ. Therefore $h^{-1}_{v_0}(0)$ is an osculating hyperhorosphere of γ at $\gamma(s_0)$ if and only if

$$h(s_0) = h'(s_0) = h''(s_0) = h^{(3)}(s_0) = 0.$$

By Proposition 3.1, this condition is equivalent to the condition that

$$\mathbf{v}_0 = \gamma(s_0) + \sigma_1(s_0)\mathbf{n}_1(s_0) + \sigma_2(s_0)\mathbf{n}_2(s_0) + \sigma_3(s_0)\mathbf{n}_3(s_0),$$

where

$$\sigma_2(s_0) = \sin\theta_0 \cos\phi_0 = -\frac{{k_1}'(s_0)}{(k_1^2k_2)(s_0)},$$

$$\sigma_3(s_0) = \sin\theta_0 \sin\phi_0 = \pm\left(1 - \frac{1}{k_1^2(s_0)} - \frac{{k_1}'^2(s_0)}{(k_1^4k_2^2)(s_0)}\right)^{1/2}.$$

By Proposition 3.2,

$$1 - \frac{1}{k_1^2(s_0)} - \frac{({k_1}')^2(s_0)}{(k_1^4k_2^2)(s_0)} > 0,$$

that is, $(k_1^4k_2^2 - k_1^2k_2^2 - {k_1'}^2)(s_0) > 0$.

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The second assertion follows from the fourth and fifth assertions of Proposition 3.1. □

REMARK 3.5. The osculating hyperhorosphere of γ is located in de Sitter 4-space S^4_1 if and only if $(k_1^4k_2^2 - k_1^2k_2^2 - {k'_1}^2)(s_0) < 0$, where

$$S^4_1 = \{\mathbf{x} \in \mathbb{R}^5_1 : \langle \mathbf{x}, \mathbf{x} \rangle = 1\}.$$

However, we do not consider this case in this paper.

Theorem 2.1 asserts that the set of singular points of the horospherical hypersurface of γ is the locus of the polar vectors of osculating hyperhorospheres of γ. Moreover, the butterfly point of the horospherical hypersurface of γ corresponds to the point $\gamma(s_0)$ where the osculating hyperhorosphere and γ have 5-point contact.

On the other hand, we consider the hyperhorosphere

$$HS^3(\mathbf{v}(s_0), -1) \cap \langle \gamma(s_0), \mathbf{t}(s_0), \mathbf{n}_1(s_0), \mathbf{n}_2(s_0)\rangle_{\mathbb{R}}$$

at a point $s_0 \in I$ at which $(k_1^4k_2^2 - k_1^2k_2^2 - {k'_1}^2)(s_0) > 0$. We call it the *osculating hyperhorosphere* of γ at $\gamma(s_0)$. The first assertion of Proposition 3.1 suggests that the invariants $k_i(s_0)$, where $i = 1, 2, 3$ describe the contact between curves and hyperhorospheres. We do not, however, study this topic here.

4. Generating families and generic properties

Proposition 3.1 means that the discriminant set of the horospherical height function H is given by

$$\mathcal{D}_H = \Big\{\mathbf{v} : \mathbf{v} = \gamma(s) + \sum_{i=1}^{3} \sigma_i\, \mathbf{n}_i(s), \sigma_i \in \mathbb{R}, \sum_{i=1}^{3} \sigma_i^2 = 1, s \in I\Big\},$$

which is the image of the horospherical hypersurface along γ. Therefore a singular point of the horospherical hypersurface is a point

$$\mathbf{v}_0 = \mathbf{v} = \gamma(s) + \sum_{i=1}^{3} \mu_i\, \mathbf{n}_i(s_0),$$

at which $\sum_{i=1}^{3} \mu_i^2 = 1$. We now explain the reason why such a correspondence exists from the viewpoint of contact geometry. Given a point $\mathbf{v} = (v_0, v_1, \dots, v_4) \in LC^*_+$, we take the projective cotangent bundle

$$\pi : PT^*(LC^*_+) \to LC^*_+$$

with its canonical contact structure. We review the geometric properties of this space. Consider the tangent bundle $\tau : TPT^*(LC^*_+) \to PT^*(LC^*_+)$ and the differential map $d\pi : TPT^*(LC^*_+) \to TLC^*_+$ of π. For all $X \in TPT^*(LC^*_+)$, there exists an element $\alpha \in T^*(LC^*_+)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x(LC^*_+)$, the property $\alpha(V) = 0$

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does not depend on the choice of representative of the class $[\alpha]$. Thus we may define the canonical contact structure on $PT^*(LC^*_+)$ by

$$K = \{X \in TPT^*(LC^*_+) : \tau(X)(d\pi(X)) = 0\}.$$

Via the coordinates $(v_0, v_1, \ldots, v_4)$, there is a trivialization

$$PT^*(LC^*_+) \cong LC^*_+ \times P(\mathbb{R}^3)^*,$$

and $((v_0, v_1, \ldots, v_4), [\xi_0 : \xi_1 : \cdots : \xi_4])$, where $[\xi_0 : \xi_1 : \cdots : \xi_4]$ are the homogeneous coordinates of the dual projective space $P(\mathbb{R}^3)^*$, are known as *homogeneous coordinates*.

It is easy to show that $X \in K_{(x,[\xi])}$ if and only if $\sum_{i=1}^4 \mu_i\xi_i = 0$, where $d\pi(X) = \sum_{i=1}^4 \mu_i\partial/\partial v_i$. An immersion $i : L \to PT^*(LC^*_+)$ is said to be *Legendrian* if $\dim L = 4$ and $di_q(T_qL) \subset K_{i(q)}$ for all $q \in L$. The map $\pi \circ i$ is also called the *Legendrian map* and the set $W(i) = \text{image } \pi \circ i$ is called the *wave front* of i. Moreover, i (or its image) is called the *Legendrian lift* of $W(i)$.

For additional definitions and basic results on generating families, we refer to [2] or [1]. By the previous arguments, the horospherical hypersurface HS_γ is the discriminant set of the horospherical height function H.

Proposition 4.1. *Let H be the horospherical height function on γ. Then H is a Morse family.*

Proof. Write $\gamma(s) = (x_0(s), x_1(s), \ldots, x_4(s))$ and $\mathbf{v} = (v_0, v_1, \ldots, v_4)$, where

$$\begin{aligned} v_0 &= (v_1^2 + v_2^2 + v_3^2 + v_4^2)^{1/2}, \\ x_0(s) &= (x_1^2(s) + x_2^2(s) + x_3^2(s) + x_4^2(s) + 1)^{1/2}. \end{aligned}$$

By definition, $H(s, \mathbf{v}) = -x_0(s)v_0 + \sum_{i=1}^4 x_i(s)v_i$. Thus, when $i = 1, \ldots, 4$,

$$\partial H/\partial v_i(s, \mathbf{v}) = -v_i x_0(s)/v_0 + x_i(s).$$

We now prove that the mapping

$$\Delta^* H = (H, \partial H/\partial s)$$

is nonsingular at $(u, \mathbf{v})$ in the singular set of the horospherical hypersurface. In fact, the Jacobian matrix of $\Delta^* H$ is given by

$$\begin{pmatrix} \langle\gamma', \mathbf{v}\rangle & -x_0v_1/v_0 + x_1 & -x_0v_2/v_0 + x_2 & \cdots & -x_0v_4/v_0 + x_4 \\ \langle\gamma'', \mathbf{v}\rangle & -x_0'v_1/v_0 + x_1' & -x_0'v_2/v_0 + x_2' & \cdots & -x_0'v_4/v_0 + x_4' \end{pmatrix}.$$

We will show that the rank of the matrix

$$A = \begin{pmatrix} x_0v_1/v_0 + x_1 & x_0v_2/v_0 + x_2 & \cdots & x_0v_4/v_0 + x_4 \\ x_0'v_1/v_0 + x_1' & x_0'v_2/v_0 + x_2' & \cdots & x_0'v_4/v_0 + x_4' \end{pmatrix}$$

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is two at $(u, \mathbf{v})$ in the singular set of the horospherical hypersurface. Since the vector $\mathbf{v} = \sum_{i=1}^{3} \sigma_i \mathbf{n}(s)$ is lightlike, we may assume that $\sigma_1 \neq 0$. We now write $\mathbf{a} = (x_0, x_0', n_{2_0}, n_{3_0})$ and $\mathbf{b}_i = (x_i, x_i', n_{2_i}, n_{3_i})$ where $i = 1, \ldots, 4$, and

$$\bar{A} = \begin{pmatrix} x_0v_1/v_0 + x_1 & x_0v_2/v_0 + x_2 & \cdots & x_0v_4/v_0 + x_4 \\ x_0'v_1/v_0 + x_1' & x_0'v_2/v_0 + x_2' & \cdots & x_0'v_4/v_0 + x_4' \\ n_{2_0}v_1/v_0 + n_{2_1} & n_{2_0}v_2/v_0 + n_{2_2} & \cdots & n_{2_0}v_4/v_0 + n_{2_4} \\ n_{3_0}v_1/v_0 + n_{3_1} & n_{3_0}v_2/v_0 + n_{3_2} & \cdots & n_{3_0}v_4/v_0 + n_{3_4} \end{pmatrix},$$

where $\mathbf{n}_i = (n_{i_0}, n_{i_1}, n_{i_2}, n_{i_3}, n_{i_4})$ when $i = 2, 3$. Then

$$\det \bar{A} = \frac{v_0}{v_0} \det \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{b}_4 \end{pmatrix} - \frac{v_1}{v_0} \det \begin{pmatrix} \mathbf{a} \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{b}_4 \end{pmatrix} - \cdots - \frac{v_4}{v_0} \det \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{a} \end{pmatrix}.$$

On the other hand,

$$(\gamma \wedge \gamma' \wedge \mathbf{n}_2 \wedge \mathbf{n}_3) = \left(-\det \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix}, -\det \begin{pmatrix} \mathbf{a} \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix}, -\det \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{a} \\ \mathbf{b}_3 \end{pmatrix} - \det \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{a} \end{pmatrix} \right).$$

Therefore

$$\det \bar{A} = \left\langle \frac{1}{v_0}(v_0, \ldots, v_4), \gamma \wedge \gamma' \wedge \mathbf{n}_2 \wedge \mathbf{n}_3 \right\rangle = \frac{1}{v_0} \langle \mathbf{v}, \gamma \wedge \mathbf{t} \wedge \mathbf{n}_2 \wedge \mathbf{n}_3 \rangle = \frac{1}{v_0\sigma_1} \neq 0$$

at $(u, \mathbf{v})$ in the singular set of the horospherical hypersurface. Since A is a submatrix of $\bar{A}$, which consists of the first and second rows of $\bar{A}$, the rank of the matrix A is two. This means that the Jacobi matrix of $\Delta^* H$ is nonsingular at $(u, \mathbf{v})$ in the singular set of the horospherical hypersurface. □

We observe that these consideration allow us to assert that the horospherical hypersurface HS_γ is a wave front and the horospherical height function H on γ gives a Minkowski canonical generating family for the Legendrian lift of HS_γ.

We now consider generic properties of curves in $H^4_+(-1)$. Our principal tool is a kind of transversality theorem. Denote by $\mathrm{Emb}(I, H^4_+(-1))$ the space of proper embeddings $\gamma : I \to H^4_+(-1)$ with the Whitney C^∞-topology. We also define the function $\mathcal{H} : H^4_+(-1) \times LC^*_+ \to \mathbb{R}$ by

$$\mathcal{H}(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle + 1.$$

We claim that $\mathcal{H}_u$ is a submersion for all $\mathbf{u} \in LC^*_+$, where $\mathcal{H}_u(\mathbf{v}) = \mathcal{H}(\mathbf{u}, \mathbf{v})$. Now $H = \mathcal{H} \circ (\gamma \times \mathrm{id}_{LC^*_+})$ for all $\gamma \in \mathrm{Emb}(I, H^4_+(-1))$, and the ℓ-jet extension

$$j^\ell_1 H : I \times LC^*_+ \to J^\ell(I, \mathbb{R})$$

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is defined by $j_1^\ell H(s,\mathbf{v}) = j^\ell h_v(s)$. We consider the trivialization

$$J^\ell(I,\mathbb{R}) \equiv I \times \mathbb{R} \times J^\ell(1,1).$$

For each submanifold Q of $J^\ell(1,1)$, we write $\widetilde{Q} = I \times \{0\} \times Q$. The following proposition is a corollary of Wassermann [5, Lemma 6] (see also Montaldi [4]).

Proposition 4.2. *Let Q be a submanifold of $J^\ell(1,1)$. Then the set*

$$T_Q = \{\gamma \in \mathrm{Emb}(I, H^4_+(-1)) : j_1^\ell H \text{ is transversal to } \widetilde{Q}\}$$

is a residual subset of $\mathrm{Emb}(I, H^4_+(-1))$. *If Q is a closed subset, then T_Q is open.*

Let $f : (\mathbb{R},0) \to (\mathbb{R},0)$ be a function germ with an A_k-singularity at 0. By the well-known classification of A_k-singularities, there exists a diffeomorphism germ $\phi : (\mathbb{R},0) \to (\mathbb{R},0)$ such that $f \circ \phi(s) = \pm s^{k+1}$. For all $z = j^\ell f(0) \in J^\ell(1,1)$, the orbit $L^\ell(z)$ is given by the action of the Lie group of ℓ-jets of diffeomorphism germs. If f has an A_k-singularity, then the codimension of the orbit is k. Now we give another characterization of versal unfoldings.

Proposition 4.3. *Let $F : (\mathbb{R} \times \mathbb{R}^r, 0) \to (\mathbb{R},0)$ be an r-parameter unfolding of the function germ $f : (\mathbb{R},0) \to (\mathbb{R},0)$, which has an A_k-singularity at* 0. *Then F is a versal unfolding if and only if $j_1^\ell F$ is transversal to the orbit $\widetilde{L}^\ell(j^\ell f(0))$ whenever $\ell \geq k+1$. Here, $j_1^\ell F : (\mathbb{R} \times \mathbb{R}^r, 0) \to J^\ell(\mathbb{R},\mathbb{R})$ is the ℓ-jet extension of F given by $j_1^\ell F(s,x) = j^\ell F_x(s)$.*

We prove Theorem 2.1 as a corollary of Proposition 4.2.

Proof of Theorem 2.1 For all $\ell \geq 5$, we consider the decomposition of the jet space $J^\ell(1,1)$ into $L^\ell(1)$ orbits. We now define the semialgebraic set Σ^ℓ to be the set of all jets $j^\ell f(0) \in J^\ell(1,1)$ of functions f with an A_k-singularity where $k \geq 4$. Then the codimension of Σ^ℓ is five. Therefore, the codimension of $\widetilde{\Sigma}_0 = I \times \{0\} \times \Sigma^\ell$ is six. The orbit $J^\ell(1,1) - \Sigma^\ell$ decomposes:

$$J^\ell(1,1) - \Sigma^\ell = L_0^\ell \cup L_1^\ell \cup \cdots \cup L_4^\ell,$$

where L_k^ℓ is the orbit through an A_k-singularity. Thus the codimension of $\widetilde{L}_k^\ell$ is $k+1$. We consider the ℓ-jet extension $j_1^\ell H$ of the horospherical height function H. By Proposition 4.2, there exists an open and dense subset O of $\mathrm{Emb}(I, H^4_+(-1))$ such that $j_1^\ell H$ is transversal to $\widetilde{L}_k^\ell$ (here $k = 0, 1, \ldots, 4$) and the orbit decomposition of $\widetilde{\Sigma}^\ell$. This means that $j_1^\ell H(I \times LC^*_+) \cap \widetilde{\Sigma}^\ell = \emptyset$ and H is a versal unfolding of h at each point (s_0, v_0). By Proposition 4.1 and the Legendrian singularity theory of Arnold [1], the discriminant set of H (that is, the horospherical hypersurface of γ) is locally diffeomorphic to a cuspidal edge, a swallow tail or a butterfly if the point is singular. □

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