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A NOTE ON SPACES WITH RANK 2-DIAGONAL

WEI-FENG XUAN✉ and WEI-XUE SHI

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Abstract

We prove that if a space X with a rank 2-diagonal either has the countable chain condition or is star countable then the cardinality of X is at most $\mathfrak{c}$.

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1. Introduction

Diagonal properties are useful in estimating the cardinality of a space. For example, in 1977, Ginsburg and Woods [5] proved that the cardinality of a space with countable extent and a G_δ-diagonal is at most $\mathfrak{c}$. However, the cardinality of a regular ccc-space (defined below) with a G_δ-diagonal need not have an upper bound [7, 8]. In 2005, Buzyakova [3] proved that the cardinality of a ccc-space with a regular G_δ-diagonal is at most $\mathfrak{c}$. Rank 2-diagonal is stronger than G_δ-diagonal. However, the relationship between rank 2-diagonal and regular G_δ-diagonal is still not clear. A natural question then arises.

Question 1.1. *Is the cardinality of a ccc-space with a rank* 2*-diagonal at most* $\mathfrak{c}$*?*

In this paper, we prove that if X is a ccc-space or a star countable space with a rank 2-diagonal, then the cardinality of X is at most $\mathfrak{c}$. This gives a positive answer to Question 1.1.

2. Notation and terminology

All spaces are assumed to be Hausdorff unless otherwise stated.

The cardinality of a set X is denoted by $|X|$, and $[X]^2$ will denote the set of two-element subsets of X. We write ω for the first infinite cardinal and $\mathfrak{c}$ for the cardinality of the continuum.

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A space X has a rank 2-diagonal if there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that for each $x \in X$, $\{x\} = \bigcap\{\mathrm{St}^2(x, \mathcal{U}_n) : n \in \omega\}$. A space X has a strong rank 1-diagonal if there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that for each $x \in X$, $\{x\} = \bigcap\{\overline{\mathrm{St}(x, \mathcal{U}_n)} : n \in \omega\}$. Clearly, rank 2-diagonal implies strong rank 1-diagonal. A space X has the countable chain condition (abbreviated as ccc) if any disjoint family of open sets in X is countable, that is, the Souslin number (or cellularity) of X is at most ω. A space X is star countable if whenever $\mathcal{U}$ is an open cover of X, there is a countable subset A of X such that $\mathrm{St}(A, \mathcal{U}) = X$, where $\mathrm{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$.

All notation and terminology not explained here is given in [4].

3. Results

We will use the following countable version of a set-theoretic theorem due to Erdős and Radó.

Lemma 3.1 [6, Theorem 2.3]. *Let X be a set with $|X| > \mathfrak{c}$ and suppose that $[X]^2 = \bigcup\{P_n : n \in \omega\}$. Then there exist $n_0 < \omega$ and a subset S of X with $|S| > \omega$ such that $[S]^2 \subseteq P_{n_0}$.*

Proposition 3.2. *Let X be a space with a rank 2-diagonal. If $|X| > \mathfrak{c}$, then there exists an uncountable closed discrete subset of X which has a disjoint open expansion.*

Proof. Since X has a rank 2-diagonal, there exists a sequence $\{\mathcal{U}_m : m \in \omega\}$ of open covers of X such that $\{x\} = \bigcap\{\mathrm{St}^2(x, \mathcal{U}_m) : m \in \omega\}$ for every $x \in X$. We may assume that $\mathrm{St}^2(x, \mathcal{U}_{m+1}) \subseteq \mathrm{St}^2(x, \mathcal{U}_m)$ for any $m \in \omega$. For $n \in \omega$ let

$$P_n = \{\{x, y\} \in [X]^2 : n = \min\{m \in \omega : \mathrm{St}(x, \mathcal{U}_m) \cap \mathrm{St}(y, \mathcal{U}_m) = \emptyset\}\}.$$

Thus, $[X]^2 = \bigcup\{P_n : n \in \omega\}$. Then by Lemma 3.1 there exists a subset S of X with $|S| > \omega$ and $[S]^2 \subseteq P_{n_0}$ for some $n_0 \in \omega$.

We now show that S is closed and discrete and it has a disjoint open expansion.

Fact 1. Clearly, $\{\mathrm{St}(x, \mathcal{U}_{n_0}) : x \in S\}$ is an uncountable pairwise disjoint family of nonempty open sets of X.

Fact 2. S is closed and discrete. If not, let $x \in X$ and suppose that x is an accumulation point of S. Since X is T_1, each neighbourhood of x meets infinitely many members of S. Therefore there exist distinct points y and z in $S \cap \mathrm{St}(x, \mathcal{U}_{n_0})$. Thus $y, z \in \mathrm{St}(x, \mathcal{U}_{n_0})$; by symmetry, $x \in \mathrm{St}(y, \mathcal{U}_{n_0})$ and $x \in \mathrm{St}(z, \mathcal{U}_{n_0})$, which is a contradiction. Thus S has no accumulation points in X; equivalently, S is a closed and discrete subset of X. This completes the proof. □

Corollary 3.3. *Let X be a ccc-space with a rank 2-diagonal. Then the cardinality of X is at most $\mathfrak{c}$.*

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With the aid of the following lemma, we can deduce a further corollary.

LEMMA 3.4 [1]. *Suppose that X has an uncountable closed discrete subspace S whose points can be separated by pairwise disjoint open sets. Then X is not star countable.*

COROLLARY 3.5. *Let X be a star countable space with a rank 2-diagonal. Then the cardinality of X is at most $\mathfrak{c}$.*

Note that 'rank 2-diagonal' cannot be weakened to 'strong rank 1-diagonal' in Corollary 3.3, as can be seen in the following example.

EXAMPLE 3.6. For any cardinal κ, there exists a Tychonoff ccc-space X with a strong rank 1-diagonal and $|X| > \kappa$.

PROOF. By [8, Corollary], for any cardinal κ, there exists a Tychonoff F_σ-discrete ccc-space X with $|X| > \kappa$. We now show that X has a strong rank 1-diagonal. Since X is a countable union of closed discrete subspaces, it is a σ-space. By [6, Theorem 4.6] that every regular σ-space has a strong rank 1-diagonal, we obtain the conclusion. This completes the proof. □

COROLLARY 3.7. *Let X be a star countable Moore space. Then the cardinality of X is at most $\mathfrak{c}$.*

PROOF. This follows since every Moore space has a rank 2-diagonal [2, Proposition 1.1]. □

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WEI-FENG XUAN, Department of Mathematics, Nanjing University,
Nanjing 210093, PR China
e-mail: xuanwf8@gmail.com

WEI-XUE SHI, Department of Mathematics, Nanjing University,
Nanjing 210093, PR China
e-mail: wxshi@nju.edu.cn

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