INITIAL VALUE PROBLEMS IN WATER WAVE THEORY

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1. Introduction

The work described in this paper grew out of an attempt to generalize some results obtained in an earlier paper [4] on the water entry problem of a thin wedge or cone into an incompressible fluid. The object of the generalization was to include the effect of gravity terms. In most papers on hydrodynamic impact it is considered permissible to neglect this effect since gravity terms might be expected to play a minor role in the initial stages of the motion. However, it seems desirable to investigate the effect of including gravity terms in order both to examine the later stages of the motion and to estimate to what extent their neglect is justified in the early stages. It will be seen that it is possible to develop a fairly complete solution for the normal entry of a thin symmetric body, both for two-dimensional and axially symmetric cases, on the basis of a linearized theory. The restriction to a linearized theory means that the whole field of analysis associated with the theory of surface waves of small amplitude becomes available. Most of the problems considered in this paper are initial value problems in which the whole fluid is at rest at $t = 0$.

An excellent survey of different types of intial value problem is given in [7]. Pioneer contributions were the subjects of classic memoirs by Cauchy and Poisson. In their work the agency applied at $t = 0$ to disturb the equilibrium is applied along the free surface $y = 0$. The main object of this paper is to discuss the effect of a general wavemaking agency $U(y, t)$ acting along the plane $x = 0$ (or, suitably interpreted, along the axis $r = 0$ in the case of axial symmetry). The difficulty in treating initial value problems in the theory of surface waves is in some respect due to the fact that t is not an active variable of the governing partial differential equation for the velocity potential ϕ (this being simply Laplace's equation) but appears in the boundary condition to be satisfied on the free surface and possibly elsewhere. These difficulties are largely surmounted by the use of integral transforms. It is reasonable to assume that motions starting from rest and caused by some localised disturbance will possess Fourier transforms in x (or Hankel transforms in r for axially symmetric flow) since the displacement at any fixed time will tend to zero for sufficiently large x (or r). In § 2 we develop this theory as it applies to the wavemaker $U(y, t)$ and obtain in particular equation (8). This equation is essentially equivalent to that obtained by Kennard [3] but it is derived here in a different manner and in a somewhat more general form suitable for applications other than those considered in [3]. In § 3 the results are applied to the two-dimensional water entry problem and the work of the earlier paper [4] is obtained as a special case by letting $g \to 0$. § 4 treats the case of axial symmetry which is almost identical with the two-dimensional theory except that a Hankel transform now replaces a Fourier cosine transform and the usual care has to be taken in interpreting boundary conditions on the axis of symmetry. Apart from obtaining formulae for the water entry problem analogous to those of § 3 we consider an example of a water exit problem in which a slim conical projectile rises vertically with constant speed from deep water. In § 5 we consider the case of a harmonic oscillation of the wavemaker. By regarding the motion as starting from rest at $t = 0$ we obtain results which in special cases reduce to those obtained by Stoker [6] and Miles [5]. Finally in § 6 certain results are obtained for the case of water of finite depth h. As $h \to 0$ motions are obtained which are identical with those which would have been obtained by using initially the linearized equation of shallow water theory, that is to say, the ordinary wave equation.

2. The two-dimensional wavemaker

In this section a relationship will be established between the behaviour of the free surface and the horizontal velocity of a vertical wavemaking agency. The mean free surface is taken as $y = 0$ and the y-axis points vertically downwards. The water is assumed to be of infinite depth and the velocity on $x = 0$ is given as $U(y, t)$. We consider only the motion of the water to the right of the wavemaker ($x > 0$). Then if ϕ is the velocity potential, the boundary conditions are:

(1) $$\frac{\partial^2 \phi}{\partial t^2} - g \frac{\partial \phi}{\partial y} = 0 \qquad (y = 0),$$

(2) $$\frac{\partial \phi}{\partial x} = U(y, t) \qquad (x = 0),$$

while ϕ satisfies

(3) $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

in the region $x > 0$, $y > 0$. If the equation of the free surface is $y = \eta(x, t)$, then

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(4) $$\eta(x, t) = \frac{1}{g}\left(\frac{\partial\phi}{\partial t}\right)_{y=0}.$$

Let the Fourier cosine transform of ϕ be defined by

$$\bar{\Phi}(\lambda, y, t) = \int_0^\infty \phi(x, y, t)\cos\lambda x \,\mathrm{d}x.$$

Other barred quantities are defined similarly. Then from (2) and (3)

$$\frac{\mathrm{d}^2\bar{\Phi}}{\mathrm{d}y^2} - \lambda^2\bar{\Phi} = U(y, t).$$

This equation can be solved in terms of the value of $\bar{\Phi}$ at $y=0$ when the condition of boundedness at $y = \infty$ is used. The solution is

(5) $$\bar{\Phi}(\lambda, y, t) = \bar{\Phi}(\lambda, 0, t)\mathrm{e}^{-\lambda y} + \int_0^\infty U(\alpha, t)G(\alpha, y, \lambda)\mathrm{d}\alpha,$$

where $G(y, y_0, \lambda)$ is the Green's function which satisfies

$$\frac{\mathrm{d}^2G}{\mathrm{d}y^2} - \lambda^2 G = \delta(y-y_0)$$

and which vanishes at $y = 0$ and $y = \infty$. Explicitly it is given by

(6) $$\begin{aligned} G(y, y_0, \lambda) &= -\mathrm{e}^{-\lambda y_0}\sinh\lambda y/\lambda \qquad (y < y_0),\\ G(y, y_0, \lambda) &= -\mathrm{e}^{-\lambda y}\sinh\lambda y_0/\lambda \qquad (y > y_0).\end{aligned}$$

From (6) we see that, when $y = 0$, $G(\alpha, y, \lambda) = 0$ and $\mathrm{d}/\mathrm{d}y\{G(\alpha, y, \lambda)\} = -\mathrm{e}^{-\lambda\alpha}$. (We have to note the change in the position of y from second to first variable in G between equations (5) and (6).) Hence from (5)

(7) $$\left(\frac{\mathrm{d}\bar{\Phi}}{\mathrm{d}y}\right)_{y=0} = -\lambda\bar{\Phi}(\lambda, 0, t) - \int_0^\infty U(\alpha, t)\mathrm{e}^{-\lambda\alpha}\mathrm{d}\alpha.$$

Now the transform of equation (1) is obtained simply be replacing ϕ by $\bar{\Phi}$. If we denote $\bar{\Phi}(\lambda, 0, t)$ by $\gamma(\lambda, t)$, then from (5) and (7)

(8) $$\ddot{\gamma} + \lambda g\gamma = -g\tilde{U}(\lambda, t),$$

where the dots denote differentiation with respect to t and

$$\tilde{U}(\lambda, t) = \int_0^\infty U(\alpha, t)\mathrm{e}^{-\lambda\alpha}\mathrm{d}\alpha,$$

which is the Laplace transform of $U(y, t)$ with respect to y.

Equation (8) is the fundamental result of this paper in that all applications considered come from solving this equation or an allied one for various functions $\tilde{U}(\lambda, t)$. When we solve for $\gamma(\lambda, t)$ we can find $\bar{\eta}$ since $\bar{\eta} = g^{-1}\dot{\gamma}$ from (4) and η can then be found by the inversion formula. In particular, if

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the liquid is at rest in equilibrium at $t = 0$ when the wavemaker begins to act we have $\gamma = \dot{\gamma} = 0$ when $t = 0$ and then

$$\gamma(\lambda, t) = -g(\lambda g)^{-\frac{1}{2}} \int_0^t \bar{U}(\lambda, \tau) \sin\{(\lambda g)^{\frac{1}{2}}(t-\tau)\} \mathrm{d}\tau$$

from which we find

$$\bar{\eta}(\lambda, t) = -\int_0^t \bar{U}(\lambda, \tau) \cos\{(\lambda g)^{\frac{1}{2}}(t-\tau)\} \mathrm{d}\tau. \tag{9}$$

3. Application to the water entry problem

If a thin symmetric body is suddenly plunged with constant speed U along the y-axis into a liquid at rest then we can determine the function $U(y, t)$ as defined in the previous section. Suppose at the instant of entry ($t = 0$) the equation of the half of the body in $x > 0$ is $x = f(y)$ where $f(y)$ is small in some sense and is defined for $y < 0$. After time t the equation of the moving surface in the liquid is

$$F(x, y, t) \equiv x - f(y - Ut) = 0.$$

Since the total derivative of F with respect to t vanishes we get

$$\frac{\partial \phi}{\partial x} = -Uf'(y-Ut) \qquad (x = 0,\ 0 < y < Ut),$$

$$\frac{\partial \phi}{\partial x} = 0 \qquad (x = 0,\ y > Ut),$$

the first equation following from the neglect of all second order quantities in f and ϕ and the second equation from symmetry. These equations define $U(y, t)$ and so

$$\bar{U}(\lambda, t) = -U \int_0^{Ut} f'(y-Ut) \mathrm{e}^{-\lambda y} \mathrm{d}y. \tag{10}$$

For the special case of a thin wedge of angle 2ε, $f(y) = -\varepsilon y$ and (10) gives

$$\bar{U}(\lambda, t) = \frac{\varepsilon U}{\lambda}(1-\mathrm{e}^{-\lambda Ut}).$$

From (9) we get

$$\bar{\eta}(\lambda, t) = -\frac{\varepsilon U}{\lambda} \int_0^t (1-\mathrm{e}^{-\lambda U\tau}) \cos\{(\lambda g)^{\frac{1}{2}}(t-\tau)\} \mathrm{d}\tau.$$

On evaluating this integral and using the inversion theorem we arrive finally at the equation of the surface in the form

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$$(11) \qquad \eta(x, t) = \frac{2\varepsilon U^2}{\pi}\int_0^\infty \frac{\cos \lambda x}{\lambda(\lambda U^2+g)} \{\cos(\lambda g)^{\frac{1}{2}}t - U(\lambda/g)^{\frac{1}{2}}\sin(\lambda g)^{\frac{1}{2}}t - e^{-\lambda Ut}\}d\lambda.$$

As $g \to 0$, (11) becomes

$$\eta(x, t) = \frac{2\varepsilon}{\pi}\int_0^\infty \frac{\cos \lambda x}{\lambda^2}(1-\lambda Ut - e^{-\lambda Ut})\,d\lambda,$$

and after some manipulation this can be reduced to

$$(12) \qquad \eta(x, t) = \frac{\varepsilon}{\pi}\left\{2Ut - 2x\tan^{-1}\frac{Ut}{x} - Ut\log\left(1+\frac{U^2t^2}{x^2}\right)\right\},$$

which is the result obtained in [4] when gravity terms were neglected initially. It is possible to proceed from (9) and treat the more general case. If $g = 0$ in (9), then for the case of the thin body of arbitrary shape

$$\bar{\eta} = -\int_0^t \bar{U}(\lambda, \tau)\,d\tau = U\int_0^t d\tau \int_0^{U\tau} f'(y-U\tau)e^{-\lambda y}dy.$$

Using the inversion formula and integrating with respect to λ first, we obtain

$$\eta = \frac{2U}{\pi}\int_0^t d\tau \int_0^{U\tau}\frac{f'(y-U\tau)y\,dy}{y^2+x^2},$$

or

$$(13) \qquad \eta = -\frac{2}{\pi}\int_0^{Ut}\frac{yf(y-Ut)\,dy}{y^2+x^2}$$

since $f(0) = 0$ if the body is sharp. (13) gives the equation for the displacement of the free surface due to the penetration of a thin sharp body of arbitrary shape when gravity terms are neglected. It is easy to recover (12) when $f(y) = -\varepsilon y$.

It does not seem easy to obtain from (11) an expression giving the next term after (12) in the displacement for small g or t. Not altogether surprisingly a formal series in powers of g does not lead to a valid expansion as the second term indicates that the water level rises further on account of the presence of gravity terms! It is possible to expand the integral in (11) asymptotically using the method of stationary phase but this is an asymptotic expansion valid for large values of gt^2/x whereas we wish to examine what happens at a fixed x when t is small. Such an approach is not therefore fruitful.

4. Extension to axially symmetric flows

In axially symmetric flow the velocity potential satisfies the equation

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial y^2} = 0,$$

where r is the distance from the axis of symmetry. If this equation is multiplied by $rJ_0(\lambda r)$ and then integrated from 0 to ∞, we obtain

$$\frac{d^2\bar{\phi}}{dy^2} - \lambda^2\bar{\phi} = V(y, t)$$

where $\bar{\phi}$ is now the Hankel transform of order zero of ϕ defined by

$$\bar{\phi}(\lambda, y, t) = \int_0^\infty rJ_0(\lambda r)\phi(r, y, t)\,dr$$

and

$$V(y, t) = \lim_{r\to 0} r\frac{\partial\phi}{\partial r}.$$

(It should not cause confusion if, throughout this section, the bar is used to denote a Hankel transform of order zero instead of a Fourier cosine transform as in previous sections.)

If $\gamma(\lambda, t) = \bar{\phi}(\lambda, 0, t)$ as before, the bar now denoting the Hankel transform, the analysis is identical with that of § 2 and in particular $\gamma(\lambda, t)$ satisfies the equation

(14) $$\ddot{\gamma}+\lambda g\gamma = -g\tilde{V}(\lambda, t).$$

$\eta(\lambda, t)$ is given by $g^{-1}\dot{\gamma}$ and then $\eta(r, t)$ may be found from the Hankel inversion formula

(15) $$\eta(r, t) = \int_0^\infty \lambda J_0(\lambda r)\bar{\eta}(\lambda, t)\,d\lambda.$$

The essential difference between $V(y, t)$ and the function $U(y, t)$ of the two-dimensional theory is well understood in terms of slender body theory. $V(y, t)$ is clearly related to the local density of a source distribution along the axis of symmetry. If the source distribution is supposedly caused by a thin body of fixed shape whose equation at any given time t is $r = F(y, t)$ moving with speed V along the y-axis in the direction of increasing y, then

(16) $$V(y, t) = -VF\frac{\partial F}{\partial y}.$$

We might mention briefly the formula for $\eta(r, t)$ when a thin body whose shape at $t = 0$ is $r = f(y)\,(y < 0)$ is plunged with speed V into liquid at rest. From work similar to that in the previous section and with the aid of (16) we find

$$\bar{\eta}(\lambda, t) = V\int_0^t \cos\{(\lambda g)^{\frac{1}{2}}(t-\tau)\}d\tau\int_0^{V\tau} f(y-V\tau)f'(y-V\tau)e^{-\lambda y}dy.$$

If we neglect gravity terms we can set $g = 0$ and in the inversion formula perform the integration with respect to λ first to get

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$$\eta(r,t) = V\int_0^t \mathrm{d}\tau \int_0^{V\tau} \frac{f(y-V\tau)f'(y-V\tau)y\,\mathrm{d}y}{(y^2+r^2)^{\frac{3}{2}}}$$

by means of an elementary integral for Bessel functions. This further reduces to

$$\eta(r,t) = -\tfrac{1}{2}\int_0^{Vt} \frac{f^2(y-Vt)y\,\mathrm{d}y}{(y^2+r^2)^{\frac{3}{2}}}$$

if the body is sharp which implies $f(0) = 0$. For the special case of a conical projectile $f(y) = -\varepsilon y$ and

$$\eta(r,t) = \varepsilon^2\left\{r+Vt\sinh^{-1}\frac{Vt}{r} - (r^2+V^2t^2)^{\frac{1}{2}} - V^2t^2/2r\right\},$$

which is the result obtained by a somewhat different approach in [4].

As a further application of the general theory we now consider the problem of a missile launched vertically upwards from deep water with constant speed V. The shape of the missile will be taken as that of a cone of semi-angle ε and length L. It will also be assumed to have an infinite tail of circular cross-section of the same area as the base so as to simulate to some extent the presence of a wake. If the tip is assumed to reach the surface at $t = 0$, then at time $t < 0$

$$\begin{aligned} F(y,t) &= 0 && (0 < y < -Vt), \\ &= \varepsilon(y+Vt) && (-Vt < y < -Vt+L), \\ &= \varepsilon L && (y > -Vt+L). \end{aligned}$$

Hence

$$\bar{V}(\lambda,t) = \varepsilon^2 V\int_{-Vt}^{-Vt+L} (y+Vt)\mathrm{e}^{-\lambda y}dy$$

from (16), with V replaced by $-V$ because the missile is moving upwards. For this problem equation (14) becomes

$$\ddot{\gamma}+\lambda g\gamma = \frac{\varepsilon^2 Vge^{\lambda Vt}}{\lambda^2}(\mathrm{e}^{-\lambda L}-1+\lambda L\mathrm{e}^{-\lambda L}).$$

We must now solve this problem with the conditions $\gamma = \dot{\gamma} = 0$ at $t = -\infty$ instead of $t = 0$ as previously. The solution is

$$\gamma(\lambda,t) = \frac{\varepsilon^2 Vg}{\lambda^3(\lambda V^2+g)}(\mathrm{e}^{-\lambda L}-1+\lambda L\mathrm{e}^{-\lambda L})\mathrm{e}^{\lambda Vt},$$

from which $\bar{\eta}$ is easily obtained and then from (15)

$$\eta(r,t) = \varepsilon^2V^2\int_0^\infty \frac{\mathrm{e}^{\lambda Vt}J_0(\lambda r)(\mathrm{e}^{-\lambda L}-1+\lambda L\mathrm{e}^{-\lambda L})\mathrm{d}\lambda}{\lambda(\lambda V^2+g)}.$$

When the missile first breaks the surface, the surface elevation H is obtained by setting $r = t = 0$ in the above integral and changing the sign since η is negative for a positive height. With the change of variable $\theta = \lambda L$ this gives

$$H = \varepsilon^2 L \int_0^\infty \frac{1-(1+\theta)e^{-\theta}}{\theta(\theta+Fr^{-1})} d\theta,$$

where Fr is the Froude number V^2/gL based on the missile's length and velocity. This gives one simple formula in which the effect of varying the Froude number can be assessed. It would be possible to extend the analysis in order to describe the motion both while the missile is emerging and subsequently after the conical part has cleared the surface. For this it would be necessary to repeat the above work, suitably redefining $F(y, t)$, or to treat the whole problem as one composite one, using Heaviside functions.

Before leaving motions possessing axial symmetry we point out the connexion between this work and that of Finkelstein [1] on time-dependent Green's functions. Finkelstein obtained the velocity potential for a source switched on at $t = 0$ at some interior point of the fluid which may be taken on $r = 0$ without loss of generality. It would have been possible to have used this directly for the problems of this section and to have used a two-dimensional equivalent for the problems of other sections. However, it was found more convenient for our purposes to proceed as in § 2 and derive equation (8) or (14). To obtain Finkelstein's result from (14) we set $V(y, t) = \delta(y-y_0)$ and solve (14), which becomes

$$\ddot{\gamma}+\lambda g\gamma = -ge^{-\lambda y_0},$$

with the conditions $\gamma = \dot{\gamma} = 0$ at $t = 0$ (the instant chosen for the "switch-on" of the source). The solution is

$$\gamma(\lambda, t) = -\frac{e^{-\lambda y_0}}{\lambda}(1-\cos(\lambda g)^{\frac{1}{2}}t)$$

whence

$$\eta(r, t) = g^{-\frac{1}{2}} \int_0^\infty \lambda^{\frac{1}{2}} e^{-\lambda y_0} \sin(\lambda g)^{\frac{1}{2}}t\, J_0(\lambda r) d\lambda.$$

We can go somewhat further than this. We have up to now concentrated on obtaining the equation of the surface only. It is, however, a simple matter to write down the complete solution for ϕ. Indeed it is given by inverting (5) with $U(\alpha, t) = \delta(\alpha-y_0)$ and the bar interpreted as a Hankel transform. The equation for $\bar{\phi}$ is

$$\bar{\phi}(\lambda, y, t) = \gamma(\lambda, t)e^{-\lambda y}+G(y_0, y, \lambda) \qquad (t > 0).$$

For $y < y_0$,

$$G(y_0, y, \lambda) = -e^{-\lambda y_0} \sinh \lambda y/\lambda.$$

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The inversion then gives

$$\phi(r, y, t) = -\tfrac{1}{2}\{(y-y_0)^2+r^2\}^{-\frac{1}{2}}+\tfrac{1}{2}\{(y+y_0)^2+r^2\}^{-\frac{1}{2}} - \int_0^\infty e^{-\lambda(y+y_0)} \times (1-\cos(\lambda g)^{\frac{1}{2}}t)J_0(\lambda r)d\lambda.$$

This is equivalent to Finkelstein's result.

5. Harmonic oscillations

We return now to the problem of the two-dimensional wavemaker, assuming it to be of the special form $U(y)e^{i\omega t}$ and that it starts to operate at $t = 0$ when the water in $x > 0$ is at rest in equilibrium. As has been mentioned, this problem has been studied by Kennard [3] who recovered Havelock's "steady state" solution [2] by considering the behaviour for large values of t. It is introduced here to show how a special case is related to some recent work by Stoker [6] and Miles [5].

If $U(y, t) = U(y)e^{i\omega t}$ then $\ddot{\gamma}+\lambda g\gamma = -g\bar{U}(\lambda)e^{i\omega t}$ and if this is solved subject to $\gamma(0) = \dot{\gamma}(0) = 0$, then after some algebra

$$\bar{\eta}(\lambda, t) = \bar{U}(\lambda)\left(\frac{\partial}{\partial t}+i\omega\right)\frac{\cos\omega t-\cos(\lambda g)^{\frac{1}{2}}t}{\omega^2-\lambda g}. \tag{17}$$

If we take $\bar{U}(\lambda) = 1$, corresponding to a delta function for $U(y)$, that is to an impulsive force at the origin, then the subsequent expression for η is exactly that obtained by Stoker except for multiplication by a purely imaginary constant. Stoker's solution was obtained as the reult of the limit of a disturbance applied at $t = 0$ over an area of the free surface as this area shrank to zero. Thus fundamentally it is due to a singularity at the origin applied at $t = 0$. Miles discovered an error in Stoker's work arising from the fact that Stoker had assumed $\phi = \dot{\phi} = 0$ on $y = 0$ as initial conditions. Miles pointed out that the correct condition was $\phi = \eta = 0$ and that these were not equivalent since (in our present notation)

$$\frac{\partial\phi}{\partial t} - g\eta = -\frac{p}{\rho}$$

and p was non-zero at $t = 0$. It might not seem likely at first that this would be significant since p is zero everywhere except at $x = 0$. However, when the complex Fourier transform of the above is taken, we find that the transform of p is not zero because of the delta function singularity. Miles gave the correct formulation and his answer is $i\omega$ times Stoker's solution integrated from 0 to t. However, it should be noted that if Stoker's initial conditions had been a disturbance $\delta(x)\sin\omega t$, the problem would have been correctly formulated and this explains why the imaginary parts of Stoker's and Miles' solutions are in fact identical. From the remarks which follow (17) they are also equal to the real part of the solution obtained from (17), that is to say from an impulse from the wavemaker of the form $\delta(y)\cos\omega t$. An impulse $\delta(y)\sin\omega t$ would yield the real part of Stoker's solution — that part which is incorrect when regarded as the limit of an impulsive pressure on the surface. All these solutions have essentially the same character of an impulsive disturbance concentrated at the origin. Naturally, therefore, they have the same asymptotic behaviour for large t.

6. Case of finite depth

A theory analogous to that of § 2 can be developed for water of finite constant depth h. The Green's function is now

$$G(y, y_0, \lambda) = -\frac{\sinh\lambda y\cosh\lambda(h-y_0)}{\lambda\cosh\lambda h} \qquad (y < y_0),$$

$$= -\frac{\sinh\lambda y_0\cosh\lambda(h-y)}{\lambda\cosh\lambda h} \qquad (y > y_0),$$

and if Γ is the cosine (or Hankel) transform of ϕ at $y = 0$, then corresponding to (8)

$$\ddot{\Gamma}+\lambda g\tanh\lambda h\,\Gamma = -g\int_0^h \frac{U(y,t)\cosh\lambda(h-y)dy}{\cosh\lambda h}.$$

Let us consider the special case in two-dimensional motion when $U(y,t) = U$, a constant, and $\Gamma(0) = \dot{\Gamma}(0) = 0$. This corresponds to a wall $x = 0$ bounding the liquid at rest being pushed suddenly into it with speed U. This leads to the solution

$$\eta(x,t) = -\frac{2Uh^{\frac{1}{2}}}{\pi g^{\frac{1}{2}}}\int_0^\infty \lambda^{-\frac{3}{2}}\cos\lambda x(\tanh\lambda h)^{\frac{1}{2}}\sin\{(\lambda g\tanh\lambda h)^{\frac{1}{2}}t\}\,d\lambda.$$

This expression is somewhat clumsy but if we let $h\to 0$ the result is striking. The leading term is

$$\eta = -\frac{2Uh^{\frac{1}{2}}}{\pi g^{\frac{1}{2}}}\int_0^\infty \lambda^{-1}\cos\lambda x\sin\{(gh)^{\frac{1}{2}}t\lambda\}\,d\lambda$$

and this gives

$$\eta = -Uh/c \qquad (x < ct),$$

$$= 0 \qquad (x > ct),$$

where $c^2 = gh$. This is the result that would have been obtained from the equation of linearized shallow water theory, the ordinary one-dimensional wave equation. Disturbances subject to this equation travel at the finite

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speed c. In any exact solution of Laplace's equation a pulse travels with infinite speed and there is no sharply defined expanding wave. Nevertheless we see how the wave equation solution emerges as the leading term in the value of η derived from a solution of Laplace's equation as the depth $h \to 0$.

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