AN INTEGRAL INVOLVING A PRODUCT OF TWO MODIFIED BESSEL FUNCTIONS OF THE SECOND KIND

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The formula to be proved is

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}K_m(\lambda)K_n(z/\lambda)\,d\lambda$$

$$=\sum_{n,-n}\frac{\Gamma(\tfrac{1}{2})\,\Gamma(k+m+n)\,\Gamma(k-m+n)}{\Gamma(k+n+\tfrac{1}{2})2^{k+1}}\,\Gamma(n)z^{-n}$$

$$\times F\begin{pmatrix}\tfrac{3}{4}-\tfrac{1}{2}k-\tfrac{1}{2}n,\ \tfrac{1}{4}-\tfrac{1}{2}k-\tfrac{1}{2}n\ ;\ \tfrac{1}{4}z^2\\ 1-n,\ 1-\tfrac{1}{2}k-\tfrac{1}{2}m-\tfrac{1}{2}n,\ 1-\tfrac{1}{2}k+\tfrac{1}{2}m-\tfrac{1}{2}n,\ \tfrac{1}{2}-\tfrac{1}{2}k-\tfrac{1}{2}m-\tfrac{1}{2}n,\ \tfrac{1}{2}-\tfrac{1}{2}k+\tfrac{1}{2}m-\tfrac{1}{2}n\end{pmatrix}$$

$$+\sum_{m,-m}\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m+\tfrac{1}{2}n)\,\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m-\tfrac{1}{2}n)\ \Gamma(-m)2^{-m-3}\left(\frac{z}{2}\right)^{m+k}$$

$$\times F\begin{pmatrix}\tfrac{3}{4}+\tfrac{1}{2}m,\ \tfrac{1}{4}+\tfrac{1}{2}m\\ 1+\tfrac{1}{2}k+\tfrac{1}{2}m-\tfrac{1}{2}n,\ 1+\tfrac{1}{2}k+\tfrac{1}{2}m+\tfrac{1}{2}n,\ \tfrac{1}{2},\ \tfrac{1}{2}+m,\ 1+m\ ;\ \tfrac{1}{4}z^2\end{pmatrix}$$

$$-\sum_{m,-m}\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m-\tfrac{1}{2}n-\tfrac{1}{2})\ \Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m+\tfrac{1}{2}n-\tfrac{1}{2})\ \Gamma(-m)2^{-m-3}\left(\frac{z}{2}\right)^{m+k+1}$$

$$\times F\begin{pmatrix}\tfrac{5}{4}+\tfrac{1}{2}m,\ \tfrac{3}{4}+\tfrac{1}{2}m\\ \tfrac{3}{2}+\tfrac{1}{2}k+\tfrac{1}{2}m-\tfrac{1}{2}n,\ \tfrac{3}{2}+\tfrac{1}{2}k+\tfrac{1}{2}m+\tfrac{1}{2}n,\ \tfrac{3}{2},\ 1+m,\ \tfrac{3}{2}+m\ ;\ \tfrac{1}{4}z^2\end{pmatrix},\ \dots\dots\dots(1)$$

where $R(z)>0$.

We start with the formula

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}\,E(\gamma,\,\delta::\lambda)\,E(p\,;\ \alpha_r:q\,;\ \rho_s:z/\lambda^m)\,d\lambda$$

$$=(2\pi)^{\frac{1}{2}-\frac{1}{2}m}m^{k-\frac{1}{2}}\ \Gamma(\gamma)\ \Gamma(\delta)\ E(p+2m\,;\ \alpha_r:q+m\,;\ \rho_s:z/m^m),\ \dots\dots\dots\dots\dots\dots(2)$$

where m is a positive integer, $R(k+\gamma)>0$, $R(k+\delta)>0$,

$$\alpha_{p+\nu+1}=(\gamma+k+\nu)/m,\ \alpha_{p+m+\nu+1}=(\delta+k+\nu)/m,\ \rho_{q+\nu+1}=(\gamma+\delta+k+\nu)/m,\ \nu=0,1,2,\dots,m-1.$$

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This gives, if $p=1$, $q=0$, $m=2$, $R(k+\gamma+2\alpha_1)>0$, $R(k+\delta+2\alpha_1)>0$,

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}\,E(\gamma,\,\delta::\lambda)\,E(\alpha_1::z\lambda^2)\,d\lambda$$

$$=z^{\alpha_1}\int_0^\infty e^{-\lambda}\lambda^{k+2\alpha_1-1}\,E(\gamma,\,\delta::\lambda)\,E\left(\alpha_1::\frac{1}{z\lambda^2}\right)d\lambda$$

$$=(2\pi)^{-\frac{1}{2}}2^{k-\frac{1}{2}}\ \Gamma(\gamma)\ \Gamma(\delta)(4z)^{\alpha_1}$$

$$\times E\left(\begin{matrix}\alpha_1,\ \alpha_1+\dfrac{\gamma+k}{2},\ \alpha_1+\dfrac{\gamma+k+1}{2},\ \alpha_1+\dfrac{\delta+k}{2},\ \alpha_1+\dfrac{\delta+k+1}{2}\\ \alpha_1+\dfrac{\gamma+\delta+k}{2},\ \alpha_1+\dfrac{\gamma+\delta+k+1}{2}\end{matrix}:\frac{1}{4z}\right)$$

$$=\pi^{-\frac{1}{2}}2^{k-1}\,\Gamma(\gamma)\,\Gamma(\delta)\times\Bigg[\frac{\Gamma\left(\frac{\gamma+k}{2}\right)\Gamma\left(\frac{\gamma+k+1}{2}\right)\Gamma\left(\frac{\delta+k}{2}\right)\Gamma\left(\frac{\delta+k+1}{2}\right)}{\Gamma\left(\frac{\gamma+\delta+k}{2}\right)\Gamma\left(\frac{\gamma+\delta+k+1}{2}\right)}$$

$$\times\Gamma(\alpha_1)\,F\left(\begin{matrix}\alpha_1,\,1-\frac{\gamma+\delta+k}{2},\,\frac{1-\gamma-\delta-k}{2}\\ 1-\frac{\gamma+k}{2},\,\frac{1-\gamma-k}{2},\,1-\frac{\delta+k}{2},\,\frac{1-\delta-k}{2}\end{matrix};\,-\frac{1}{4z}\right)$$

$$+\sum_{\gamma,\delta}\frac{\Gamma\left(\frac{-\gamma-k}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\delta-\gamma}{2}\right)\Gamma\left(\frac{\delta-\gamma+1}{2}\right)}{\Gamma\left(\frac{\delta}{2}\right)\Gamma\left(\frac{\delta+1}{2}\right)(4z)^{(\gamma+k)/2}}$$

$$\times\Gamma\left(\alpha_1+\frac{\gamma+k}{2}\right)F\left(\begin{matrix}\alpha_1+\frac{\gamma+k}{2},\,1-\frac{\delta}{2},\,\frac{1-\delta}{2}\\ 1+\frac{\gamma+k}{2},\,\frac{1}{2},\,1+\frac{\gamma-\delta}{2},\,\frac{1+\gamma-\delta}{2}\end{matrix};\,-\frac{1}{4z}\right)$$

$$+\sum_{\gamma,\delta}\frac{\Gamma\left(\frac{-\gamma-k-1}{2}\right)\Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{\delta-\gamma-1}{2}\right)\Gamma\left(\frac{\delta-\gamma}{2}\right)}{\Gamma\left(\frac{\delta-1}{2}\right)\Gamma\left(\frac{\delta}{2}\right)(4z)^{(\gamma+k+1)/2}}$$

$$\times\Gamma\left(\alpha_1+\frac{\gamma+k+1}{2}\right)F\left(\begin{matrix}\alpha_1+\frac{\gamma+k+1}{2},\,\frac{3-\delta}{2},\,1-\frac{\delta}{2}\\ \frac{3+\gamma+k}{2},\,\frac{3}{2},\,\frac{3+\gamma-\delta}{2},\,1+\frac{\gamma-\delta}{2}\end{matrix};\,-\frac{1}{4z}\right)\Bigg]$$

Thus, on generalising, if $q+3\geqq p\geqq q+1$, $R(k+\gamma+2\alpha_r)>0$, $R(k+\delta+2\alpha_r)>0$, $r=1,2,\ldots,p$,

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(\gamma,\delta::\lambda)\,E(p;\;\alpha_r:q;\;\rho_s:z\lambda^2)\,d\lambda=\pi^{-\frac{1}{2}}2^{k-1}\,\Gamma(\gamma)\,\Gamma(\delta)$$

$$\times\Bigg[\frac{\Gamma(\gamma+k)\,\Gamma(\delta+k)\,\Gamma(\frac{1}{2})}{\Gamma(\gamma+\delta+k)2^{k-1}}\;\frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_p)}{\Gamma(\rho_1)\ldots\Gamma(\rho_q)}$$

$$\times F\left(\begin{matrix}\alpha_1,\ldots,\alpha_p,\,1-\frac{\gamma+\delta+k}{2},\,\frac{1-\gamma-\delta-k}{2}\\ \rho_1,\ldots,\rho_q,\,1-\frac{\gamma+k}{2},\,\frac{1-\gamma-k}{2},\,1-\frac{\delta+k}{2},\,\frac{1-\delta-k}{2}\end{matrix};\,-\frac{1}{4z}\right)$$

$$+\sum_{\gamma,\delta}\frac{\Gamma\left(\frac{-\gamma-k}{2}\right)\Gamma(\frac{1}{2})\,\Gamma(\delta-\gamma)}{\Gamma(\delta)2^{-\gamma}(4z)^{(\gamma+k)/2}}\;\frac{\Gamma\left(\alpha_1+\frac{\gamma+k}{2}\right)\ldots\Gamma\left(\alpha_p+\frac{\gamma+k}{2}\right)}{\Gamma\left(\rho_1+\frac{\gamma+k}{2}\right)\ldots\Gamma\left(\rho_q+\frac{\gamma+k}{2}\right)}$$

$$\times F\left(\begin{matrix}\alpha_1+\frac{\gamma+k}{2},\ldots,\alpha_p+\frac{\gamma+k}{2},\,1-\frac{\delta}{2},\,\frac{1-\delta}{2}\\ \rho_1+\frac{\gamma+k}{2},\ldots,\rho_q+\frac{\gamma+k}{2},\,1+\frac{\gamma+k}{2},\,\frac{1}{2},\,1+\frac{\gamma-\delta}{2},\,\frac{1+\gamma-\delta}{2}\end{matrix};\,-\frac{1}{4z}\right)$$

$$+\sum_{\gamma,\delta}\frac{\Gamma\left(\frac{-\gamma-k-1}{2}\right)\Gamma(-\frac{1}{2})\Gamma(\delta-\gamma-1)}{\Gamma(\delta-1)2^{-\gamma}(4z)^{(\gamma+k+1)/2}}\frac{\Gamma\left(\alpha_1+\frac{\gamma+k+1}{2}\right)\dots\Gamma\left(\alpha_p+\frac{\gamma+k+1}{2}\right)}{\Gamma\left(\rho_1+\frac{\gamma+k+1}{2}\right)\dots\Gamma\left(\rho_q+\frac{\gamma+k+1}{2}\right)}$$

$$\left.\times F\begin{pmatrix}\alpha_1+\frac{\gamma+k+1}{2},\dots,\alpha_p+\frac{\gamma+k+1}{2},\frac{3-\delta}{2},1-\frac{\delta}{2} \\ \rho_1+\frac{\gamma+k+1}{2},\dots,\rho_q+\frac{\gamma+k+1}{2},\frac{3+\gamma+k}{2},\frac{3}{2},\frac{3+\gamma-\delta}{2},1+\frac{\gamma-\delta}{2};\ -\frac{1}{4z}\end{pmatrix}\right]$$

The result also holds for other values of p and q provided that the integral and the series are convergent.

Here replace λ by 2λ, z by $1/z^2$, k by $k-n-\frac{1}{2}$, γ and δ by $\frac{1}{2}+m$ and $\frac{1}{2}-m$, take $p=0$ and $q=1$ with $\rho_1=n+1$, and apply the formulae

$$\cos m\pi\, E(\tfrac{1}{2}+m,\tfrac{1}{2}-m::2\lambda)=\sqrt{(2\pi\lambda)}e^{\lambda}K_m(\lambda), \quad\dots\dots(3)$$

$$E(:n+1:4\lambda^2/z^2)=(2\lambda/z)^n J_n(z/\lambda). \quad\dots\dots(4)$$

Then, if z is real and positive and $R(k\pm m)>-\frac{3}{2}$,

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}K_m(\lambda)J_n(z/\lambda)\,d\lambda$$

$$=\frac{\Gamma(k+m-n)\,\Gamma(k-m-n)\,\Gamma(\frac{1}{2})}{\Gamma(k-n+\frac{1}{2})\,\Gamma(n+1)2^k}$$

$$\times z^n F\begin{pmatrix}\frac{3}{4}-\frac{1}{2}k+\frac{1}{2}n,\frac{1}{4}-\frac{1}{2}k+\frac{1}{2}n \\ n+1,1-\frac{1}{2}k-\frac{1}{2}m+\frac{1}{2}n,\frac{1}{2}-\frac{1}{2}k-\frac{1}{2}m+\frac{1}{2}n,1-\frac{1}{2}k+\frac{1}{2}m+\frac{1}{2}n,\frac{1}{2}-\frac{1}{2}k+\frac{1}{2}m+\frac{1}{2}n;\ -\frac{z^2}{4}\end{pmatrix}$$

$$+\sum_{m,-m}\frac{\Gamma(-\frac{1}{2}k-\frac{1}{2}m+\frac{1}{2}n)\,\Gamma(-2m)\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-m)\,\Gamma(1+\frac{1}{2}k+\frac{1}{2}m+\frac{1}{2}n)2^{k+1}}$$

$$\times z^{m+k}F\begin{pmatrix}\frac{3}{4}+\frac{1}{2}m,\frac{1}{4}+\frac{1}{2}m \\ 1+\frac{1}{2}k+\frac{1}{2}m+\frac{1}{2}n,1+\frac{1}{2}k+\frac{1}{2}m-\frac{1}{2}n,\frac{1}{2},\frac{1}{2}+m,1+m;\ -\frac{z^2}{4}\end{pmatrix}$$

$$+\sum_{m,-m}\frac{\Gamma(-\frac{1}{2}k-\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2})\,\Gamma(-2m-1)\,\Gamma(-\frac{1}{2})}{\Gamma(-\frac{1}{2}-m)\,\Gamma(\frac{3}{2}+\frac{1}{2}k+\frac{1}{2}m+\frac{1}{2}n)2^{k+2}}$$

$$\times z^{m+k+1}F\begin{pmatrix}\frac{5}{4}+\frac{1}{2}m,\frac{3}{4}+\frac{1}{2}m \\ \frac{3}{2}+\frac{1}{2}k+\frac{1}{2}m+\frac{1}{2}n,\frac{3}{2}+\frac{1}{2}k+\frac{1}{2}m-\frac{1}{2}n,\frac{3}{2},1+m,\frac{3}{2}+m;\ -\frac{z^2}{4}\end{pmatrix}. \quad\dots\dots(5)$$

Now

$$G_n(z)=\frac{\pi}{2\sin n\pi}\{J_{-n}(z)-e^{-in\pi}J_n(z)\}, \quad\dots\dots(6)$$

so that

$$i^nG_n(z)=\frac{\pi}{2}\sum_{n,-n}\frac{i^nJ_{-n}(z)}{\sin n\pi};$$

and therefore, if $0\leqq \operatorname{amp} z\leqq\pi$, $R(k\pm m)>-\frac{3}{2}$,

$$i^n\int_0^\infty e^{-\lambda}\lambda^{k-1}K_m(\lambda)G_n(z/\lambda)\,d\lambda$$

$$=\sum_{n,-n}\frac{\Gamma(\frac{1}{2})\Gamma(k+m+n)\,\Gamma(k-m+n)\,\Gamma(n)}{\Gamma(k+n+\frac{1}{2})\,2^{k+1}}$$

$$\times\left(\frac{i}{z}\right)^n F\begin{pmatrix}\frac{3}{4}-\frac{1}{2}k-\frac{1}{2}n,\frac{1}{4}-\frac{1}{2}k-\frac{1}{2}n \\ 1-n,1-\frac{1}{2}k-\frac{1}{2}m-\frac{1}{2}n,\frac{1}{2}-\frac{1}{2}k-\frac{1}{2}m-\frac{1}{2}n,1-\frac{1}{2}k+\frac{1}{2}m-\frac{1}{2}n,\frac{1}{2}-\frac{1}{2}k+\frac{1}{2}m-\frac{1}{2}n;\ -\frac{z^2}{4}\end{pmatrix}$$

$$+\sum_{m,-m}\frac{\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m-\tfrac{1}{2}n)\,\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m+\tfrac{1}{2}n)\,\Gamma(-m)}{\sin n\pi\,2^{2m+k+3}}$$
$$\times\{\sin(\tfrac{1}{2}n-\tfrac{1}{2}m-\tfrac{1}{2}k)\pi\,.\,e^{\frac{1}{2}n\pi i}+\sin(\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{1}{2}k)\pi\,.\,e^{-\frac{1}{2}n\pi i}\}$$
$$z^{m+k}F\left(\begin{matrix}\tfrac{3}{4}+\tfrac{1}{2}m,\ \tfrac{1}{4}+\tfrac{1}{2}m\\ 1+\tfrac{1}{2}k+\tfrac{1}{2}m-\tfrac{1}{2}n,\ 1+\tfrac{1}{2}k+\tfrac{1}{2}m+\tfrac{1}{2}n,\ \tfrac{1}{2},\ \tfrac{1}{2}+m,\ 1+m\end{matrix};\ -\frac{z^2}{4}\right)$$
$$-\sum_{m,-m}\frac{\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m-\tfrac{1}{2}n-\tfrac{1}{2})\,\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m+\tfrac{1}{2}n-\tfrac{1}{2})\,\Gamma(-m)}{\sin n\pi\,2^{2m+k+4}}$$
$$\times\{-\cos(\tfrac{1}{2}n-\tfrac{1}{2}m-\tfrac{1}{2}k)\pi\,.\,e^{\frac{1}{2}n\pi i}+\cos(\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{1}{2}k)\pi\,.\,e^{-\frac{1}{2}n\pi i}\}$$
$$\times z^{m+k+1}F\left(\begin{matrix}\tfrac{5}{4}+\tfrac{1}{2}m,\ \tfrac{3}{4}+\tfrac{1}{2}m\\ \tfrac{3}{2}+\tfrac{1}{2}k+\tfrac{1}{2}m+\tfrac{1}{2}n,\ \tfrac{3}{2}+\tfrac{1}{2}k+\tfrac{1}{2}m-\tfrac{1}{2}n,\ \tfrac{3}{2},\ 1+m,\ \tfrac{3}{2}+m\end{matrix};\ -\frac{z^2}{4}\right).\qquad\ldots\ldots\ldots(7)$$

On replacing z by iz and applying the formulae

$$K_n(z)=i^nG_n(iz)\qquad\ldots\ldots\ldots(8)$$
$$\sin(\tfrac{1}{2}n-\tfrac{1}{2}m-\tfrac{1}{2}k)\pi\,.\,e^{\frac{1}{2}n\pi i}+\sin(\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{1}{2}k)\pi\,.\,e^{-\frac{1}{2}n\pi i}=\sin n\pi\,.\,i^{-m-k},$$
$$-\cos(\tfrac{1}{2}n-\tfrac{1}{2}m-\tfrac{1}{2}k)\pi\,.\,e^{\frac{1}{2}n\pi i}+\cos(\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{1}{2}k)\pi\,.\,e^{-\frac{1}{2}n\pi i}=\sin n\pi\,.\,i^{-m-k-1},$$

formula (1) is obtained.

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