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# Coisotropic Ekeland–Hofer capacities

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For subsets in the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$ whose closures are intersecting with coisotropic subspace $\mathbb{R}^{n,k}$ we construct relative versions of the Ekeland–Hofer capacities of the subsets with respect to $\mathbb{R}^{n,k}$, establish representation formulas for such capacities of bounded convex domains intersecting with $\mathbb{R}^{n,k}$. We also prove a product formula and a fact that the value of this capacity on a hypersurface $\mathcal{S}$ of restricted contact type containing the origin is equal to the action of a generalized leafwise chord on $\mathcal{S}$.

*Keywords:* Symplectic invariant; Ekeland–Hofer capacities; coisotropic

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## 1. Introduction

### 1.1. Coisotropic capacity

Recently, Lisi and Rieser [**29**] introduced the notion of a coisotropic capacity (i.e. a symplectic capacity relative to a coisotropic submanifold of a symplectic manifold), and discussed their motivations and backgrounds. Let $(M, \omega)$ be a symplectic manifold and $N \subset M$ a coisotropic submanifold. (*In this paper all manifolds are assumed to be connected without special statements*!) An equivalence relation $\sim$ on $N$ was called a *coisotropic equivalence relation* if $x$ and $y$ are on the same leaf then $x \sim y$ (cf. [**29**, definition 1.4]). Special examples are the trivial relation defined by $x \sim y$ for every pair $x, y \in N$ and the so-called *leaf relation* defined by $x \sim y$ if and only if $x$ and $y$ are on the same leaf. For two tuples $(M_0, N_0, \omega_0, \sim_0)$ and $(M_1, N_1, \omega_1, \sim_1)$ as above, a relative symplectic embedding from $(M_0, N_0, \omega_0)$ to $(M_1, N_1, \omega_1)$ is a symplectic embedding $\psi : (M_0, \omega_0) \to (M_1, \omega_1)$ satisfying $\psi^{-1}(N_1) = N_0$ [**29**, definition 1.5]. Such an embedding $\psi$ is said to *respect the pair of coisotropic equivalence relations* $(\sim_0, \sim_1)$ if for every $x, y \in N_0$,

$$\psi(x) \sim_1 \psi(y) \quad \Longrightarrow \quad x \sim_0 y.$$

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The standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$ has coisotropic linear subspaces

$$\mathbb{R}^{n,k} = \{x \in \mathbb{R}^{2n} \,|\, x = (q_1, \ldots, q_n, p_1, \ldots, p_k, 0, \ldots, 0)\}$$

for $k = 0, \ldots, n$, where we understand $\mathbb{R}^{n,0} = \{x \in \mathbb{R}^{2n} \,|\, x = (q_1, \ldots, q_n, 0, \ldots, 0)\}$. Denote by $\sim$ the leaf relation on $\mathbb{R}^{n,k}$, and by

$$V_0^{n,k} = \{x \in \mathbb{R}^{2n} \,|\, x = (0, \ldots, 0, q_{k+1}, \ldots, q_n, 0, \ldots, 0)\}, \tag{1.1}$$

$$V_1^{n,k} = \{x \in \mathbb{R}^{2n} \,|\, x = (q_1, \ldots, q_k, 0, \ldots, 0, p_1, \ldots, p_k, 0, \ldots, 0)\}. \tag{1.2}$$

Hereafter it is understood that $V_0^{n,0} = \{x \in \mathbb{R}^{2n} \,|\, x = (q_1, \ldots, q_n, 0, \ldots, 0)\} = \mathbb{R}^{n,0}$, $V_0^{n,n} = \{0\}$ and $V_1^{n,0} = \{0\}$, $V_1^{n,n} = \mathbb{R}^{2n}$. Then $L_0^n := V_0^{n,0}$ is a Lagrangian subspace, and two points $x, y \in \mathbb{R}^{n,k}$ satisfy $x \sim y$ if and only if their difference $x - y$ sits in $V_0^{n,k}$. Observe that $\mathbb{R}^{2n}$ has the orthogonal decomposition $\mathbb{R}^{2n} = J_{2n}V_0^{n,k} \oplus \mathbb{R}^{n,k} = J_{2n}\mathbb{R}^{n,k} \oplus V_0^{n,k}$ with respect to the standard inner product, where $J_{2n}$ denotes the standard complex structure on $\mathbb{R}^{2n}$ given by $(q_1, \ldots, q_n, p_1, \ldots, p_n) \mapsto (p_1, \ldots, p_n, -q_1, \ldots, -q_n)$.

For $a \in \mathbb{R}$ we write $\mathbf{a} := (0, \ldots, 0, a) \in \mathbb{R}^{2n}$. Denote by $B^{2n}(\mathbf{a}, r)$ and $B^{2n}(r)$ the open balls of radius $r$ centred at $\mathbf{a}$ and the origin in $\mathbb{R}^{2n}$ respectively, and by

$$W^{2n}(R) := \left\{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n} \,|\, x_n^2 + y_n^2 < R^2 \text{ or } y_n < 0\right\}, \tag{1.3}$$

$$W^{n,k}(R) := W^{2n}(R) \cap \mathbb{R}^{n,k} \quad \text{and} \quad B^{n,k}(r) := B^{2n}(r) \cap \mathbb{R}^{n,k}. \tag{1.4}$$

($W^{2n}(R)$ was written as $W(R)$ in [**29**, definition 1.1]).

According to [**29**, definition 1.7], a coisotropic capacity is a functor $c$, which assigns to every tuple $(M, N, \omega, \sim)$ as above a non-negative (possibly infinite) number $c(M, N, \omega, \sim)$, such that the following conditions hold:

(i) *Monotonicity*. If there exists a relative symplectic embedding $\psi$ from $(M_0, N_0, \omega_0, \sim_0)$ to $(M_1, N_1, \omega_1, \sim_1)$ respecting the coisotropic equivalence relations where $\dim M_0 = \dim M_1$, then $c(M_0, N_0, \omega_0, \sim_0) \leqslant c(M_1, N_1, \omega_1, \sim_1)$.

(ii) *Conformality*. $c(M, N, \alpha\omega, \sim) = |\alpha| c(M, N, \omega, \sim)$, $\forall\ \alpha \in \mathbb{R}\backslash\{0\}$.

(iii) *Non-triviality*. With the leaf relation $\sim$ it holds that for $k = 0, \ldots, n - 1$,

$$c(B^{2n}(1), B^{n,k}(1), \omega_0, \sim) = \frac{\pi}{2} = c(W^{2n}(1), W^{n,k}(1), \omega_0, \sim). \tag{1.5}$$

As remarked in [**29**, remark 1.9], any symplectic capacity cannot serve as a coisotropic capacity because of the non-triviality (iii).

From now on, we abbreviate $c(M, N, \omega, \sim)$ as $c(M, N, \omega)$ if $\sim$ is the leaf relation on $N$. In particular, for domains $D \subset \mathbb{R}^{2n}$ we also abbreviate $c(D, D \cap \mathbb{R}^{n,k}, \omega_0)$ as $c(D, D \cap \mathbb{R}^{n,k})$ for simplicity.

Given a $(n+k)$-dimensional coisotropic submanifold $N$ in a symplectic manifold $(M,\omega)$ of dimension $2n$ we defined in [26, definition 1.3]

$$w_G(N;M,\omega):=\sup\left\{\pi r^2\;\middle|\;\begin{array}{l}\exists\text{ a relative symplectic embedding}\\ (B^{2n}(r),B^{n,k}(r))\to(M,N)\text{ respecting}\\ \text{the leaf relations on }B^{n,k}(r)\text{ and }N\end{array}\right\}$$

the *relative Gromov width* of $(M,N,\omega)$. Here we always assume $k\in\{0,1\ldots,n-1\}$. (If $k=n$ then $w_G(N;M,\omega)$ is equal to the Gromov width $w_G(N,\omega|_N)$ of $(N,\omega|_N)$.)

When $k=0$, $N$ is a Lagrangian submanifold and this relative Gromov width was introduced by Barraud, Biran and Cornea [6–9]. It is easily seen that $w_G$ satisfies monotonicity, conformality and

$$w_G(B^{2n}(r)\cap\mathbb{R}^{n,k};B^{2n}(r),\omega_0)=\pi r^2,\quad\forall\ r>0.$$

In fact $w_G(N;M,\omega)/2$ is the smallest coisotropic capacity by the nonsqueezing theorem in [29]. Rizell [33] observed that the Lagrangian submanifolds of $\mathbb{C}^3$ constructed by Ekholm, Eliashberg, Murphy and Smith [15] have infinite relative Gromov width.

Similar to the construction of the Hofer–Zehnder capacity, Lisi and Rieser [29] constructed an analogue relative to a coisotropic submanifold, called the *coisotropic Hofer–Zehnder capacity*, and denoted by $c_{\mathrm{LR}}$ in this paper. By properties of this coisotropic capacity, they also studied symplectic embeddings relative to coisotropic constraints and got some corresponding dynamical results. The coisotropic capacity $c_{\mathrm{LR}}$ also played a key role in the proof of Humiliére–Leclercq–Seyfaddini's important rigidity result that symplectic homeomorphisms preserve coisotropic submanifolds and their characteristic foliations [21].

For the coisotropic capacity $c_{\mathrm{LR}}(D,D\cap\mathbb{R}^{n,k})$ of a bounded convex domain $D\subset\mathbb{R}^{2n}$, we [26] proved a representation formula, some interesting corollaries and corresponding versions of a Brunn–Minkowski type inequality by Artstein–Avidan and Ostrover and a theorem by Evgeni Neduv.

### 1.2. A relative version of the Ekeland–Hofer capacity with respect to a coisotropic submanifold $\mathbb{R}^{n,k}$

Prompted by Gromov's work [17], Ekeland and Hofer [13, 14] constructed a sequence of symplectic invariants for subsets in the standard symplectic space $(\mathbb{R}^{2n},\omega_0)$, the so-called Ekeland and Hofer symplectic capacities. (In this paper, the Ekeland and Hofer symplectic capacity always means the first Ekeland and Hofer symplectic capacity without special statements.) We introduced the generalized Ekeland–Hofer and the symmetric Ekeland–Hofer symplectic capacities and developed corresponding results [24, 25]. The aim of this paper is to construct a coisotropic analogue of the Ekeland–Hofer capacity for subsets in $(\mathbb{R}^{2n},\omega_0)$ relative to a coisotropic submanifold $\mathbb{R}^{n,k}$, the coisotropic Ekeland–Hofer capacity.

Fix an integer $0\leqslant k\leqslant n$. For each subset $B\subset\mathbb{R}^{2n}$ whose closure $\overline{B}$ has nonempty intersection with $\mathbb{R}^{n,k}$, we define a number $c^{n,k}(B)$, called *coisotropic Ekeland–Hofer capacity* of $B$ (though it does not satisfy the stronger monotonicity

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as in (i) above (1.5)), which is equal to the Ekeland–Hofer capacity of $B$ if $k = n$. The coisotropic capacity $c^{n,k}$ satisfies $c^{n,k}(B) = c^{n,k}(\overline{B})$ and the following:

PROPOSITION 1.1. *Let $\lambda > 0$ and $B \subset A \subset \mathbb{R}^{2n}$ satisfy $\overline{B} \cap \mathbb{R}^{n,k} \neq \emptyset$. Then*

(i) (Monotonicity) $c^{n,k}(B) \leqslant c^{n,k}(A)$.

(ii) (Conformality) $c^{n,k}(\lambda B) = \lambda^2 c^{n,k}(B)$.

(iii) (Exterior regularity) $c^{n,k}(B) = \inf\{c^{n,k}(U_\epsilon(B)) \,|\, \epsilon > 0\}$ *and so* $c^{n,k}(B) = c^{n,k}(\overline{B})$, *where* $U_\epsilon(B)$ *is the* $\epsilon$*-neighbourhood of* $B$.

(iv) (Translation invariance) $c^{n,k}(B + w) = c^{n,k}(B)$ *for all* $w \in \mathbb{R}^{n,k}$, *where* $B + w = \{z + w \,|\, z \in B\}$.

The group $\mathrm{Sp}(2n) = \mathrm{Sp}(2n, \mathbb{R})$ of symplectic matrices in $\mathbb{R}^{2n}$ is a connected Lie group. Kun Shi shows in Appendix A that its subgroup

$$\mathrm{Sp}(2n, k) := \{A \in \mathrm{Sp}(2n) \,|\, Az = z \;\forall\; z \in \mathbb{R}^{n,k}\} \tag{1.6}$$

is also connected.

THEOREM 1.2 (Symplectic invariance). *Let $B \subset \mathbb{R}^{2n}$ satisfy $\overline{B} \cap \mathbb{R}^{n,k} \neq \emptyset$. Suppose that $\phi \in \mathrm{Symp}(\mathbb{R}^{2n}, \omega_0)$ satisfies for some $w_0 \in \mathbb{R}^{n,k}$,*

$$\phi(w) = w - w_0 \quad \forall\; w \in \mathbb{R}^{n,k} \quad \textit{and} \quad \mathrm{d}\phi(w_0) \in \mathrm{Sp}(2n, k).$$

*Then $c^{n,k}(\phi(B)) = c^{n,k}(B)$.*

COROLLARY 1.3. *For a subset $A \subset \mathbb{R}^{2n}$ satisfying $\overline{A} \cap \mathbb{R}^{n,k} \neq \emptyset$, suppose that there exists a star-shaped open neighbourhood $U$ of $\overline{A}$ with respect to some point $w_0 \in \mathbb{R}^{n,k}$ and a symplectic embedding $\varphi$ from $U$ to $\mathbb{R}^{2n}$ such that*

$$\varphi(w) = w - w_0 \quad \forall w \in \mathbb{R}^{n,k} \cap U \quad \textit{and} \quad \mathrm{d}\varphi(w_0) \in \mathrm{Sp}(2n, k). \tag{1.7}$$

*Then $c^{n,k}(\varphi(A)) = c^{n,k}(A)$. In particular, for a subset $A \subset \mathbb{R}^{2n}$ satisfying $\overline{A} \cap \mathbb{R}^{n,k} \neq \emptyset$, if it is star-shaped with respect to some point $w_0 \in \mathbb{R}^{n,k}$ and there exists a symplectic embedding $\varphi$ from some open neighbourhood $U$ of $\overline{A}$ to $\mathbb{R}^{2n}$ such that (1.7) holds, then $c^{n,k}(\varphi(A)) = c^{n,k}(A)$.*

There exists a natural class of symplectic mappings satisfying the conditions in corollary 1.3. For $\epsilon > 0$ small, let $\mathbb{R}^{n,k}_\epsilon = \{(q_1, \dots, q_n, p_1, \dots, p_n) \,|\, p_{k+1}^2 + \cdots + p_n^2 < \epsilon^2\}$, that is, the tubular open neighbourhood of $\mathbb{R}^{n,k}$ of radius $\epsilon$. Let $U$ be as in corollary 1.3, and let $H : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ be any smooth Hamiltonian that vanishes in $[0, 1] \times (\mathbb{R}^{n,k}_\epsilon \cap U)$. Suppose that $X_H$ can determine a 1-parameter family of symplectic mappings $\phi^t_H$ for $t \in [0, 1]$ as usual (e.g. this can be satisfied if $H$ has compact support). Then $\varphi := (\psi_{w_0} \circ \phi^1_H)|_U$ satisfies the conditions in corollary 1.3, where $\psi_{w_0} \in \mathrm{Symp}(\mathbb{R}^{2n}, \omega_0)$ is the translation defined by $\psi_{w_0}(w) = w - w_0$ for $w \in \mathbb{R}^{2n}$.

For a bounded convex domain $D$ in $(\mathbb{R}^{2n}, \omega_0)$ with boundary $\mathcal{S}$, recall that a nonconstant absolutely continuous curve $z : [0, T] \to \mathbb{R}^{2n}$ (for some $T > 0$) is said

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to be a *generalized characteristic* on $\mathcal{S}$ if $z([0,T]) \subset \mathcal{S}$ and $\dot{z}(t) \in J_{2n} N_{\mathcal{S}}(z(t))$ a.e., where $N_{\mathcal{S}}(x) = \{y \in \mathbb{R}^{2n} \mid \langle u - x, y\rangle \leqslant 0 \forall\, u \in D\}$ is the normal cone to $D$ at $x \in \mathcal{S}$ [**24**, definition 1.1]. When $D \cap \mathbb{R}^{n,k} \neq \emptyset$, such a generalized characteristic $z : [0,T] \to \mathcal{S}$ is called a *generalized leafwise chord* (abbreviated GLC) on $\mathcal{S}$ for $\mathbb{R}^{n,k}$ if $z(0), z(T) \in \mathbb{R}^{n,k}$ and $z(0) - z(T) \in V_0^{n,k}$. (Generalized characteristics and generalized leafwise chords on $\mathcal{S}$ become characteristics and leafwise chords on $\mathcal{S}$ respectively if $\mathcal{S}$ is of class $C^1$.) The action of a GLC $z : [0,T] \to \mathcal{S}$ is defined by

$$A(z) = \frac{1}{2}\int_0^T \langle -J_{2n}\dot{z}, z\rangle\,\mathrm{d}t,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclid norm on $\mathbb{R}^{2n}$. As generalizations of representation formulas for the Ekeland–Hofer capacities of bounded convex domains we have:

Theorem 1.4. *Let $D \subset \mathbb{R}^{2n}$ be a bounded convex domain with $C^{1,1}$ boundary $S = \partial D$. If $D \cap \mathbb{R}^{n,k} \neq \emptyset$ (and so $\partial D$ contains at least two points of $\mathbb{R}^{n,k}$), then there exists a leafwise chord $x^*$ on $\partial D$ for $\mathbb{R}^{n,k}$ such that*

$$\begin{aligned} A(x^*) &= \min\{A(x) > 0 \mid x \textit{ is a leafwise chord on } \partial D \textit{ for } \mathbb{R}^{n,k}\} \\ &= c^{n,k}(D) \qquad (1.8) \\ &= c^{n,k}(\partial D). \qquad (1.9) \end{aligned}$$

*Moreover, if $D \subset \mathbb{R}^{2n}$ is only a bounded convex domain such that $D \cap \mathbb{R}^{n,k} \neq \emptyset$, then the above conclusions are still true after all words 'leafwise chord' are replaced by 'generalized leafwise chord'.*

This theorem may be false if the domain is not convex. Consider the following domain

$$A^{2n}(r_1, r_2) := \{z \in \mathbb{R}^{2n} \mid r_1^2 < |z| < r_2^2\} = B^{2n}(r_2) \setminus \overline{B^{2n}(r_1)},$$

where $0 < r_1 < r_2 < \infty$. Then by monotonicity of coisotropic Ekeland–Hofer capacity and theorem 1.4,

$$c^{n,k}(\partial B^{2n-1}(r_2)) \leqslant c^{n,k}(\overline{A^{2n}(r_1, r_2)}) \leqslant c^{n,k}(\overline{B^{2n}(r_2)}) = c^{n,k}(\partial B^{2n-1}(r_2))$$

and

$$c^{n,k}(A^{2n}(r_1, r_2)) = c^{n,k}(\overline{A^{2n}(r_1, r_2)}) = \begin{cases} \dfrac{\pi}{2} r_2^2, & k < n, \\ \pi r_2^2, & k = n. \end{cases}$$

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However, since $\partial A^{2n}(r_1, r_2) = \partial B^{2n-1}(r_1) \cup \partial B^{2n-1}(r_2)$,

$$\begin{aligned}
&\min\{A(x) > 0 \,|\, x \text{ is a leafwise chord on } \partial A^{2n}(r_1, r_2) \text{ for } \mathbb{R}^{n,k}\} \\
&\quad = \min\{\min\{A(x) > 0 \,|\, x \text{ is a leafwise chord on } \partial B^{2n-1}(r_1) \text{ for } \mathbb{R}^{n,k}\}, \\
&\qquad \min\{A(x) > 0 \,|\, x \text{ is a leafwise chord on } \partial B^{2n-1}(r_2) \text{ for } \mathbb{R}^{n,k}\}\} \\
&\quad = \min\{c^{n,k}(\partial B^{2n-1}(r_1)), c^{n,k}(\partial B^{2n-1}(r_2))\} \\
&\quad = \begin{cases} \dfrac{\pi}{2} r_1^2, & k < n, \\ \pi r_1^2, & k = n. \end{cases}
\end{aligned}$$

Hence theorem 1.4 is false for $A^{2n}(r_1, r_2)$.

Theorem 1.4 and [**26**, theorem 1.5] show that $c^{n,k}(D) = c_{\mathrm{LR}}(D, D \cap \mathbb{R}^{n,k})$ for a bounded convex domain $D \subset \mathbb{R}^{2n}$ as in theorem 1.4. It follows from (3.18) and interior regularity of $c_{\mathrm{LR}}$ that

$$c^{n,k}(D) = c_{\mathrm{LR}}(D, D \cap \mathbb{R}^{n,k}) \tag{1.10}$$

for any convex domain $D \subset \mathbb{R}^{2n}$ such that $D \cap \mathbb{R}^{n,k} \neq \emptyset$. Hence theorems 1.6, 1.12 and corollaries 1.7–1.10 in [**26**] are still true if $c_{\mathrm{LR}}$ is replaced by suitable $c^{n,k}$.

Moser [**32**] first studied Hamiltonian leafwise chords for understanding perturbations of Hamiltonian dynamical systems, his framework has been extended in many directions, which promotes the research of symplectic topology, see [**3**, **4**, **12**, **16**, **18**, **19**, **27**, **29**, **38**, **39**] etc. Given two autonomous $C^{1,1}$ Hamiltonians $H, G : \mathbb{R}^{2n} \to \mathbb{R}$ and a regular energy surface $G^{-1}(c')$, one may ask the following natural mechanics problem: Is there a point on $G^{-1}(c')$ from which two particles start, move respectively along Hamiltonian trajectories of $X_H$ and $X_G$ and after some finite time return to an intersection point of these two trajectories?

Suppose for some $c \in \mathbb{R}$ that $D_0 := \{z \in \mathbb{R}^{2n} \,|\, H(z) < c\}$ is a bounded convex domain whose intersection with $G^{-1}(c')$ is a nonempty relative open subset in a $(2n-1)$-dimensional coisotropic subspace $V$ in $\mathbb{R}^{2n}$. Then theorem 1.4 implies an affirmative answer to the problem. In fact, since there exists a linear symplectic transformation $\Psi : (\mathbb{R}^{2n}, \omega_0) \to (\mathbb{R}^{2n}, \omega_0)$ such that $\Psi(V) = \mathbb{R}^{n,n-1}$, we can replace $H$ and $G$ by $H \circ \Psi^{-1}$ and $G \circ \Psi^{-1}$, respectively, and therefore reduce the question to the case $V = \mathbb{R}^{n,n-1}$. The desired conclusion follows from theorem 1.4.

Theorem 1.4 is also closely related to the famous Arnold's chord conjecture in [**5**, § 8]. Many cases for this problem have been proved to be true, see [**1**, **2**, **10**, **11**, **22**, **23**, **30**, **31**, **33**–**35**, **40**] etc. When $k = 0$, the intersection $S \cap \mathbb{R}^{n,0}$ is a closed $C^{1,1}$ Legendrian submanifold of dimension $n-1$ in the contact manifold $S$ with the standard contact form, which is diffeomorphic to the sphere $S^{n-1}$, and theorem 1.4 affirms the conjecture in this case though for the smooth $S$ it was proved by Mohnke [**31**] with a different method. Clearly, our result also gives the action of this chord.

As the Ekeland–Hofer capacity, $c^{n,k}$ satisfies the following product formulas, which play key roles for computations of $c_{\mathrm{LR}}$ and the proof of [**26**, theorem 1.12].

Theorem 1.5. *For convex domains* $D_i \subset \mathbb{R}^{2n_i}$ *containing the origin,* $i = 1, \ldots, m \geqslant 2$*, and integers* $0 \leqslant l_0 \leqslant n := n_1 + \cdots + n_m$*,* $l_j = \max\{l_{j-1} - n_j, 0\}$*,* $j = 1, \ldots, m-1$*, it holds that*

$$c^{n,l_0}(D_1 \times \cdots \times D_m) = \min_i c^{n_i, \min\{n_i, l_{i-1}\}}(D_i). \tag{1.11}$$

*Moreover, if all these domains* $D_i$ *are also bounded then*

$$c^{n,l_0}(\partial D_1 \times \cdots \times \partial D_m) = \min_i c^{n_i, \min\{n_i, l_{i-1}\}}(D_i). \tag{1.12}$$

Hereafter $\mathbb{R}^{2n_1} \times \mathbb{R}^{2n_2} \times \cdots \times \mathbb{R}^{2n_m}$ is identified with $\mathbb{R}^{2(n_1+\cdots+n_m)}$ via

$$\begin{aligned} \mathbb{R}^{2n_1} \times \mathbb{R}^{2n_2} \times \cdots \times \mathbb{R}^{2n_m} &\ni ((q^{(1)}, p^{(1)}), \ldots, (q^{(m)}, p^{(m)})) \\ &\mapsto (q^{(1)}, \ldots, q^{(m)}, p^{(1)}, \ldots, p^{(m)}) \in \mathbb{R}^{2n}. \end{aligned}$$

If $l_0 = n$ then $l_i = \sum_{j>i} n_j$ and thus $\min\{n_i, l_{i-1}\} = n_i$ for $i = 1, \ldots, m$. It follows that theorem 1.5 becomes theorem in [**37**, § 6.6]. We pointed out in [**26**, remark 1.11] that theorems 1.4, 1.5 and [**26**, theorem 1.5] can be combined together to improve some results therein.

Corollary 1.6. *Let* $S^1(r_i)$ *be boundaries of discs* $B^2(0, r_i) \subset \mathbb{R}^2$*,* $i = 1, \ldots, n \geqslant 2$*, and integers* $0 \leqslant l_0 \leqslant n$*,* $l_j = \max\{l_{j-1} - 1, 0\}$*,* $j = 1, \ldots, n-1$*. Then*

$$c^{n,l_0}(S^1(r_1) \times \cdots \times S^1(r_n)) = \min_i c^{1, \min\{1, l_{i-1}\}}(B^2(0, r_i)).$$

*Here* $c^{1,1}(B^2(0, r_i)) = \pi r_i^2$ *and* $c^{1,0}(B^2(0, r_i)) = \pi r_i^2/2$*. Precisely,*

$$\begin{aligned} &c^{n,0}(S^1(r_1) \times \cdots \times S^1(r_n)) = \min\{\pi r_1^2/2, \ldots, \pi r_n^2/2\}, \\ &c^{n,k}(S^1(r_1) \times \cdots \times S^1(r_n)) = \min\{\min_{i \leqslant k} \pi r_i^2, \min_{i>k} \pi r_i^2/2\}, \quad 0 < k < n, \\ &c^{n,n}(S^1(r_1) \times \cdots \times S^1(r_n)) = \min\{\pi r_1^2, \ldots, \pi r_n^2\}. \end{aligned}$$

Note that corollary 1.6 becomes [**37**, corollary 6.6] for $l_0 = n$.
Define $U^2(1) = \{(q_n, p_n) \in \mathbb{R}^2 \,|\, q_n^2 + p_n^2 < 1 \text{ or } -1 < q_n < 1 \text{and} p_n < 0\}$ and

$$U^{2n}(1) = \mathbb{R}^{2n-2} \times U^2(1) \quad \text{and} \quad U^{n,k}(1) = U^{2n}(1) \cap \mathbb{R}^{n,k}. \tag{1.13}$$

By (1.10) and [**26**, corollary 1.9] we obtain for $k = 0, 1, \ldots, n-1$,

$$c^{n,k}(U^{2n}(1)) = c_{\mathrm{LR}}(U^{2n}(1), U^{2n}(1) \cap \mathbb{R}^{n,k}) = \frac{\pi}{2}. \tag{1.14}$$

The proof of theorem 1.5 relies partially on the representation of coisotropic Ekeland–Hofer capacity of convex domains given by theorem 1.4. It is possible that theorem 1.5 is still true for some product of non-convex domains. For integers

$0 \leqslant l_0 \leqslant n := n_1 + n_2$ and $l_1 = \max\{l_0 - n_1, 0\}$, as the arguments below theorem 1.5 we can get

$$\begin{aligned} c^{n,l_0}(B^{2n_1}(r_2) \times B^{2n_2}(r_3)) &= c^{n,l_0}(\partial B^{2n_1-1}(r_2) \times \partial B^{2n_2-1}(r_3)) \\ &\leqslant c^{n,l_0}(\overline{A^{2n_1}(r_1, r_2)} \times \overline{B^{2n_2}(r_3)}) \\ &\leqslant c^{n,l_0}(\overline{B^{2n_1}(r_2)} \times \overline{B^{2n_2}(r_3)}) \end{aligned}$$

and therefore

$$\begin{aligned} c^{n,l_0}(A^{2n_1}(r_1, r_2) \times B^{2n_2}(r_3)) &= c^{n,l_0}(\overline{A^{2n_1}(r_1, r_2)} \times \overline{B^{2n_2}(r_3)}) \\ &= c^{n,l_0}(\overline{B^{2n_1}(r_2)} \times \overline{B^{2n_2}(r_3)}) \\ &= \min\{c^{n_1, \min\{n_1, l_0\}}(B^{2n_1}(r_2)), c^{n_2, \min\{n_2, l_1\}}(B^{2n_2}(r_3))\} \\ &= \begin{cases} \min\left\{\frac{\pi}{2}r_2^2, \frac{\pi}{2}r_3^2\right\}, & l_0 < n_1, \\ \min\{\pi r_2^2, c^{n_2, l_0 - n_1}(B^{2n_2}(r_3))\}, & l_0 \geqslant n_1. \end{cases} \end{aligned}$$

On the other hand

$$\begin{aligned} &\min\{c^{n_1, \min\{n_1, l_0\}}(A^{2n_1}(r_1, r_2)), c^{n_2, \min\{n_2, l_1\}}(B^{2n_2}(r_3))\} \\ &= \begin{cases} \min\left\{\frac{\pi}{2}r_2^2, \frac{\pi}{2}r_3^2\right\}, & l_0 < n_1, \\ \min\{\pi r_2^2, c^{n_2, l_0 - n_1}(B^{2n_2}(r_3))\}, & l_0 \geqslant n_1. \end{cases} \end{aligned}$$

Hence (1.11) is also true for the produce of $A^{2n_1}(r_1, r_2)$ and $B^{2n_2}(r_3)$.

Recall that a vector field $X$ defined on an open set $U \subset \mathbb{R}^{2n}$ is called a Liouville vector field if $L_X\omega_0 = \omega_0$. A hypersurface $\mathcal{S} \subset \mathbb{R}^{2n}$ is said to be of restricted contact type if there exists a Liouville vector field $X$ globally defined on $\mathbb{R}^{2n}$ which is transversal to $\mathcal{S}$. Corresponding to the representation of the Ekeland–Hofer capacity of a bounded domain in $\mathbb{R}^{2n}$ with boundary of restricted contact type we have:

Theorem 1.7. *Let $U \subset (\mathbb{R}^{2n}, \omega_0)$ be a bounded domain with $C^{2n+2}$ boundary $\mathcal{S}$ of restricted contact type. Suppose that $U$ contains the origin and that there exists a globally defined $C^{2n+2}$ Liouville vector field $X$ transversal to $\mathcal{S}$ whose flow $\phi^t$ maps $\mathbb{R}^{n,k}$ to $\mathbb{R}^{n,k}$ and preserves the leaf relation of $\mathbb{R}^{n,k}$. Then*

$$\Sigma_{\mathcal{S}} := \{A(x) > 0 \,|\, x \text{ is a leafwise chord on } \mathcal{S} \text{ for } \mathbb{R}^{n,k}\}. \tag{1.15}$$

*has empty interior and contains $c^{n,k}(U) = c^{n,k}(\mathcal{S})$.*

In order to show that $c^{n,k}$ is a coisotropic capacity (with the weaker monotonicity), we need to prove that $c^{n,k}$ satisfies the non-triviality as in (1.5). By theorem 1.4 we immediately obtain

$$c^{n,k}(B^{2n}(1)) = \frac{\pi}{2}, \quad k = 0, \ldots, n-1. \tag{1.16}$$

Proposition 1.1(i) and (1.14) also lead to $c^{n,k}(W^{2n}(1)) \geqslant c^{n,k}(U^{2n}(1)) = \pi/2$ directly. Using the extension monotonicity of $c_{\mathrm{LR}}$ in [**29**, lemma 2.4], Lisi and Rieser

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proved that

$$c_{\mathrm{LR}}\left(W^{2n}(1), W^{n,k}(1)\right) = c_{\mathrm{LR}}\left(U^{2n}(1), U^{n,k}(1)\right)$$

above [**29**, proposition 3.1]. However, our proposition 1.1 and theorem 1.2 cannot yield such strong extension monotonicity for $c^{n,k}$. Instead, we may use theorems 1.5 and 1.7 (though the latter does not hold for $c_{\mathrm{LR}}$ in general), to derive:

THEOREM 1.8. *For $k = 0, \dots, n-1$, it holds that*

$$c^{n,k}(W^{2n}(1)) = \frac{\pi}{2}.$$

By this theorem, corollary 1.6 and theorem 1.2 we deduce:

COROLLARY 1.9. *If $\min\{2\min_{i\leqslant k} r_i^2, \min_{i>k} r_i^2\} > 1$ for some $0 < k < n$, then there is no $\phi \in \mathrm{Symp}(\mathbb{R}^{2n}, \omega_0)$ which satisfies $\phi(w) = w - w_0\ \forall\ w \in \mathbb{R}^{n,k}$ and $\mathrm{d}\phi(w_0) \in \mathrm{Sp}(2n,k)_0$ for some $w_0 \in \mathbb{R}^{n,k}$, such that $\phi$ maps $S^1(r_1) \times \cdots \times S^1(r_n) = \{(x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{R}^{2n} \mid x_i^2 + y_i^2 = r_i^2,\ i = 1, \dots, n\}$ into $W^{2n}(1)$.*

Under the assumptions of corollary 1.9 it is easy to see that there always exists a $\phi \in \mathrm{Symp}(\mathbb{R}^{2n}, \omega_0)$ such that $\phi(S^1(r_1) \times \cdots \times S^1(r_n)) \subset W^{2n}(1)$.

Let $\tau_0 \in \mathcal{L}(\mathbb{R}^{2n})$ be the canonical involution on $\mathbb{R}^{2n}$ given by $\tau_0(x, y) = (x, -y)$. For a subset $B \subset \mathbb{R}^{2n}$ such that $\tau_0 B = B$ and $B \cap L_0^n \neq \emptyset$, let $c_{\mathrm{EH},\tau_0}(B)$ be the $\tau_0$-symmetrical Ekeland–Hofer capacity constructed in [**25**]. We shall prove in § 8:

THEOREM 1.10. *The $\tau_0$-symmetrical Ekeland–Hofer capacity $c_{\mathrm{EH},\tau_0}(B)$ of each subset $B \subset \mathbb{R}^{2n}$ satisfying $\tau_0 B = B$ and $B \cap L_0^n \neq \emptyset$ is greater than or equal to $c^{n,0}(B)$.*

*Structure of the paper*. In § 2 we provide necessary variational preparations on the basis of [**26**, **29**]. In § 3 we give the definition of the coisotropic Ekeland–Hofer capacity and proofs of proposition 1.1, theorem 1.2 and corollary 1.3. In § 4 we prove theorem 1.4. In § 5 we prove a product formula, theorem 1.5. In § 6 we prove theorem 1.7 about the representation of the coisotropic capacity $c^{n,k}$ of a bounded domain in $\mathbb{R}^{2n}$ with boundary of restricted contact type. In § 7 we prove theorem 1.8.

## 2. Variational preparations

We follow [**26**, **29**] to present necessary variational materials. Fix an integer $0 \leqslant k < n$. Consider the Hilbert space defined in [**29**, definition 3.6]

$$L^2_{n,k} = \left\{ x \in L^2([0,1], \mathbb{R}^{2n}) \,\middle|\, x \overset{L^2}{=} \sum_{m\in\mathbb{Z}} \mathrm{e}^{m\pi t J_{2n}} a_m + \sum_{m\in\mathbb{Z}} \mathrm{e}^{2m\pi t J_{2n}} b_m \right.$$
$$\left. a_m \in V_0^{n,k}, \quad b_m \in V_1^{n,k}, \sum_{m\in\mathbb{Z}} (|a_m|^2 + |b_m|^2) < \infty \right\} \tag{2.1}$$

with $L^2$-inner product. We proved in [**26**, proposition 2.3] that the Hilbert space $L^2_{n,k}$ is exactly $L^2([0,1], \mathbb{R}^{2n})$. (If $k = n$ this is clear as usual because $V_0^{n,n} = \{0\}$

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and $V_1^{n,n} = \mathbb{R}^{2n}$.) For any real $s \geqslant 0$ we follow [29, definition 3.6] to define

$$H^s_{n,k} = \left\{ x \in L^2([0,1], \mathbb{R}^{2n}) \,\middle|\, x \overset{L^2}{=} \sum_{m\in\mathbb{Z}} \mathrm{e}^{m\pi t J_{2n}} a_m + \sum_{m\in\mathbb{Z}} \mathrm{e}^{2m\pi t J_{2n}} b_m \right.$$
$$\left. a_m \in V_0^{n,k}, \quad b_m \in V_1^{n,k}, \quad \sum_{m\in\mathbb{Z}} |m|^{2s}(|a_m|^2 + |b_m|^2) < \infty \right\}. \tag{2.2}$$

LEMMA 2.1 [29, lemmas 3.8, 3.9]. *For each $s \geqslant 0$, $H^s_{n,k}$ is a Hilbert space with the inner product*

$$\langle \phi, \psi \rangle_{s,n,k} = \langle a_0, a_0' \rangle + \langle b_0, b_0' \rangle + \pi \sum_{m\neq 0} (|m|^{2s} \langle a_m, a_m' \rangle + |2m|^{2s} \langle b_m, b_m' \rangle).$$

*Furthermore, if $s > t$, then the inclusion $\jmath : H^s_{n,k} \hookrightarrow H^t_{n,k}$ and its Hilbert adjoint $\jmath^* : H^t_{n,k} \to H^s_{n,k}$ are compact.*

Let $\|\cdot\|_{s,n,k}$ denote the norm induced by $\langle \cdot, \cdot \rangle_{s,n,k}$. For $r \in \mathbb{N}$ or $r = \infty$ let $C^r_{n,k}([0,1], \mathbb{R}^{2n})$ denote the space of $C^r$ maps $x : [0,1] \to \mathbb{R}^{2n}$ such that $x(i) \in \mathbb{R}^{n,k}$, $i = 0, 1$, and $x(1) \sim x(0)$, where $\sim$ is the leaf relation on $\mathbb{R}^{n,k}$. (*Note*: $H^s_{n,n}$ is exactly the space $H^s$ on p. 83 of [20]; $C^r_{n,n}([0,1], \mathbb{R}^{2n})$ is $C^r(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$.)

LEMMA 2.2 [29, lemma 3.10]. *If $x \in H^s_{n,k}$ for $s > 1/2 + r$ where $r$ is an integer, then $x \in C^r_{n,k}([0,1], \mathbb{R}^{2n})$.*

LEMMA 2.3 [29, lemma 3.11]. *$\jmath^*(L^2) \subset H^1_{n,k}$ and $\|\jmath^*(y)\|_{1,n,k} \leqslant \|y\|_{L^2}$.*

Let

$$E = H^{1/2}_{n,k} \quad \text{and} \quad \|\cdot\|_E := \|\cdot\|_{1/2,n,k}. \tag{2.3}$$

It has an orthogonal decomposition $E = E^- \oplus E^0 \oplus E^+$, where

$$E^- = \left\{ x \in H^{1/2}_{n,k} \,\middle|\, x \overset{L^2}{=} \sum_{m<0} \mathrm{e}^{m\pi t J_{2n}} a_m + \sum_{m<0} \mathrm{e}^{2m\pi t J_{2n}} b_m \right\},$$
$$E^0 = \{ x = x_0 \in \mathbb{R}^{n,k} \},$$
$$E^+ = \left\{ x \in H^{1/2}_{n,k} \,\middle|\, x \overset{L^2}{=} \sum_{m>0} \mathrm{e}^{m\pi t J_{2n}} a_m + \sum_{m>0} \mathrm{e}^{2m\pi t J_{2n}} b_m \right\}.$$

Let $P^+$, $P^0$ and $P^-$ be the orthogonal projections to $E^+$, $E^0$ and $E^-$ respectively. For $x \in E$ we write

$$x^+ = P^+ x, \quad x^0 = P^0 x \quad \text{and} \quad x^- = P^- x.$$

Define a functional $\mathfrak{a} : E \to \mathbb{R}$ by

$$\mathfrak{a}(x) = \tfrac{1}{2}(\|x^+\|_E^2 - \|x^-\|_E^2).$$

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Then there holds

$$\mathfrak{a}(x) = \frac{1}{2}\int_0^1 \langle -J_{2n}\dot{x}, x\rangle \mathrm{d}t, \quad \forall\, x \in C^1_{n,k}([0,1],\mathbb{R}^{2n}).$$

(See [29].) The functional $\mathfrak{a}$ is differentiable with gradient $\nabla\mathfrak{a}(x) = x^+ - x^-$.

From now on we assume that for some $L > 0$,

$$H \in C^1(\mathbb{R}^{2n},\mathbb{R}) \quad \text{and} \quad \|\nabla H(x) - \nabla H(y)\|_{\mathbb{R}^{2n}} \leqslant L\|x-y\|_{\mathbb{R}^{2n}}\ \forall\, x,y \in \mathbb{R}^{2n}. \tag{2.4}$$

Then there exist positive real numbers $C_i$, $i = 1,2,3,4$, such that

$$|\nabla H(z)| \leqslant C_1|z| + C_2, \quad |H(z)| \leqslant C_3|z|^2 + C_4$$

for all $z \in \mathbb{R}^{2n}$. Define functionals $\mathfrak{b}, \Phi_H : E \to \mathbb{R}$ by

$$\mathfrak{b}(x) = \int_0^1 H(x(t))\,\mathrm{d}t \quad \text{and} \quad \Phi_H = \mathfrak{a} - \mathfrak{b}. \tag{2.5}$$

LEMMA 2.4 [20, § 3.3, lemma 4]. *The functional $\mathfrak{b}$ is differentiable. Its gradient $\nabla\mathfrak{b}$ is compact and satisfies a global Lipschitz condition on $E$. In particular, $\mathfrak{b}$ is $C^{1,1}$.*

LEMMA 2.5 [26, lemma 2.8]. *$x \in E$ is a critical point of $\Phi_H$ if and only if $x \in C^1_{n,k}([0,1],\mathbb{R}^{2n})$ and solves*

$$\dot{x} = X_H(x) = J_{2n}\nabla H(x).$$

*Moreover, if $H$ is of class $C^l$ $(l \geqslant 2)$ then each critical point of $\Phi_H$ on $E$ is $C^l$.*

Since $\nabla\Phi_H(x) = x^+ - x^- - \nabla\mathfrak{b}(x)$ satisfies the global Lipschitz condition, it has a unique global flow $\mathbb{R}\times E \to E : (u,x) \mapsto \varphi_u(x)$.

LEMMA 2.6 [29, lemma 3.25]. *$\varphi_u(x)$ has the following form*

$$\varphi_u(x) = \mathrm{e}^{-u}x^- + x^0 + \mathrm{e}^u x^+ + K(u,x),$$

*where $K : \mathbb{R}\times E \to E$ is continuous and maps bounded sets into precompact sets.*

This may follow from the proof of lemma 7 in [20, § 3.3] directly.

## 3. The Ekeland–Hofer capacity relative to a coisotropic subspace

We closely follow Sikorav's approach [37] to the Ekeland–Hofer capacity in [13]. Fix an integer $0 \leqslant k \leqslant n$. Let $E = H^{1/2}_{n,k}$ be as in (2.3) and $S^+ = \{x \in E^+ \mid \|x\|_E = 1\}$.

DEFINITION 3.1. *A continuous map $\gamma : E \to E$ is called an admissible deformation if there exists a homotopy $(\gamma_u)_{0\leqslant u\leqslant 1}$ such that $\gamma_0 = \mathrm{id}$, $\gamma_1 = \gamma$ and satisfies*

(i) $\forall u \in [0,1]$, $\gamma_u(E \setminus (E^- \oplus E^0)) \subset E \setminus (E^- \oplus E^0)$, *i.e.* $\gamma_u(x)^+ \neq 0$ *for any* $x \in E$ *such that* $x^+ \neq 0$.

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(ii) $\gamma_u(x) = a(x,u)x^+ + b(x,u)x^0 + c(x,u)x^- + K(x,u)$, *where* $(a,b,c,K)$ *is a continuous map from* $E \times [0,1]$ *to* $(0,+\infty)^3 \times E$ *and maps any closed bounded sets to compact sets.*

Let $\Gamma_{n,k}$ be the set of all admissible deformations on $E$. It is not hard to verify that the composition $\gamma \circ \tilde{\gamma} \in \Gamma_{n,k}$ for any $\gamma, \tilde{\gamma} \in \Gamma_{n,k}$. (If $k = n$, $\Gamma_{n,k}$ is equal to $\Gamma$ in [**37**].) Corresponding to [**37**, § 3, proposition 1] or [**13**, § II, proposition 1] we can easily prove the following intersection property.

Proposition 3.2. *$\gamma(S^+) \cap (E^- \oplus E^0 \oplus \mathbb{R}_+ e) \neq \emptyset$ for any $e \in E^+ \setminus \{0\}$ and $\gamma \in \Gamma_{n,k}$.*

Definition 3.3. *For $H \in C^0(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$, the $\mathbb{R}^{n,k}$-coisotropic capacity of $H$ is defined by*

$$c^{n,k}(H) := \sup_{h \in \Gamma_{n,k}} \inf_{x \in h(S^+)} \Phi_H(x) \tag{3.1}$$

*where $\Phi_H$ is as in* (2.5).

By proposition 1 in [**37**, § 3.3], for any $H \in C^0(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$ there holds

$$c^{n,n}(H) \leqslant \sup_{z \in \mathbb{C}^n} \left(\pi |z_1|^2 - H(z)\right), \tag{3.2}$$

where $z_1 \in \mathbb{C}$ is the projection of $z \in \mathbb{C}^n \equiv \mathbb{C} \times \mathbb{C}^{n-1}$ to $\mathbb{C}$. Correspondingly, we have

Proposition 3.4. *For any $H \in C^0(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$ there holds*

$$c^{n,k}(H) \leqslant \sup_{z \in \mathbb{C}^n} \left(\frac{\pi}{2}|z|^2 - H(z)\right), \quad k = 0, 1, \ldots, n-1. \tag{3.3}$$

*Proof.* Let $e(t) = \mathrm{e}^{\pi J_{2n} t} X$, where $X \in V_0^{n,k}$ and $|X| = 1$. For any $x = y + \lambda e$, where $y \in E^- \oplus E^0$ and $\lambda > 0$, it holds that

$$\mathfrak{a}(x) \leqslant \frac{1}{2}\|\lambda e\|_E^2 = \frac{\pi}{2}\lambda^2$$

and

$$\int_0^1 \langle x(t), \mathrm{e}^{\pi J_{2n} t} X \rangle \,\mathrm{d}t = \int_0^1 \langle \lambda\, \mathrm{e}^{\pi J_{2n} t} X, \mathrm{e}^{\pi J_{2n} t} X \rangle \,\mathrm{d}t = \lambda.$$

It follows that

$$\mathfrak{a}(x) \leqslant \frac{\pi}{2} \left( \int_0^1 \langle x(t), \mathrm{e}^{\pi J_{2n} t} X \rangle \,\mathrm{d}t \right)^2 \leqslant \frac{\pi}{2} \int_0^1 |x(t)|^2 \,\mathrm{d}t.$$

This and proposition 3.2 lead to

$$\inf_{x \in \gamma(S^+)} \Phi_H(x) \leqslant \sup_{x \in E^- \oplus E^0 \oplus \mathbb{R}_+ e} \Phi_H(x) \leqslant \sup_{z \in \mathbb{R}^{2n}} \left\{ \frac{\pi}{2}|z|^2 - H(z) \right\} \quad \forall\, \gamma \in \Gamma_{n,k},$$

and hence (3.3) is proved. □

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A function $H \in C^0(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$ is called $\mathbb{R}^{n,k}$-**admissible** if it satisfies:

(H1) $\mathrm{Int}(H^{-1}(0)) \neq \emptyset$ and intersects with $\mathbb{R}^{n,k}$,

(H2) there exists $z_0 \in \mathbb{R}^{n,k}$, real numbers $a, b$ such that $H(z) = a|z|^2 + \langle z, z_0\rangle + b$ outside a compact subset of $\mathbb{R}^{2n}$, where $a > \pi$ for $k = n$, and $a > \pi/2$ for $0 \leqslant k < n$.

Moreover, a $\mathbb{R}^{n,n}$-admissible $H$ is said to be *nonresonant* if $a$ in (H2) does not belong to $\pi\mathbb{N}$; and a $\mathbb{R}^{n,k}$-admissible $H$ with $k < n$ is called *strong nonresonant* if $a$ in (H2) does not sit in $\mathbb{N}\pi/2$.

Clearly, for any $\mathbb{R}^{n,k}$-admissible $H \in C^2(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$, $\nabla H : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfies a global Lipschitz condition.

Note that $c^{n,k}(H) < +\infty$ if $H \in C^0(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$ satisfies

$$H(z) \geqslant a|z|^2 + C, \quad \forall\, z \in \mathbb{R}^{2n} \tag{3.4}$$

for some constant $C$, where $a = \pi$ for $k = n$, and $a = \pi/2$ for $0 \leqslant k < n$. In particular, we have $c^{n,k}(H) < +\infty$ for any $H \in C^0(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$ satisfying (H2). In fact, for $k = n$ this can be derived from (3.2) (cf. [**37**]). For $0 \leqslant k < n$, since there exist constants $a > \pi/2, b$ such that $H(z) \geqslant a|z|^2 + \langle z, z_0\rangle + b$ for all $z \in \mathbb{R}^{2n}$, using the inequality

$$|\langle z, z_0\rangle| \leqslant \varepsilon|z|^2 + \frac{1}{4\varepsilon}|z_0|^2$$

for any $0 < \varepsilon < a - \frac{\pi}{2}$, we deduce that

$$\frac{\pi}{2}|z|^2 - H(z) \leqslant \left(\varepsilon - \left(a - \frac{\pi}{2}\right)\right)|z|^2 + \frac{|z_0|^2}{4\varepsilon} - b \leqslant \frac{|z_0|^2}{4\varepsilon} - b < \infty.$$

Then proposition 3.4 leads to $c^{n,k}(H) < +\infty$.

It is easy proved that $c^{n,k}(H)$ satisfies:

Proposition 3.5. *Let* $H, K \in C^0(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$ *satisfy* (H1) *and* (H2)*. Then the following holds*:

(i) (Monotonicity) *If* $H \leqslant K$ *then* $c^{n,k}(H) \geqslant c^{n,k}(K)$*.*

(ii) (Continuity) $|c^{n,k}(H) - c^{n,k}(K)| \leqslant \sup_{z\in\mathbb{R}^{2n}} |H(z) - K(z)|$.

(iii) (Homogeneity) $c^{n,k}(\lambda^2 H(\cdot/\lambda)) = \lambda^2 c^{n,k}(H)$ *for* $\lambda \neq 0$*.*

By proposition 2 in [**37**, § 3.3] the following proposition holds for $k = n$.

Proposition 3.6. *Suppose that* $H \in C^0(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$ *satisfies*

$$H(z_0 + z) \leqslant C_1|z|^2 \quad \textit{and} \quad H(z_0 + z) \leqslant C_2|z|^3 \quad \forall\, z \in \mathbb{R}^{2n} \tag{3.5}$$

*for some* $z_0 \in \mathbb{R}^{n,k}$ *and for constants* $C_1 > 0$ *and* $C_2 > 0$*. Then* $c^{n,k}(H) > 0$*. In particular,* $c^{n,k}(H) > 0$ *for any* $\mathbb{R}^{n,k}$*-admissible* $H \in C^2(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$*.*

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*Proof.* We assume $k < n$. For a constant $\varepsilon > 0$ define $\gamma_\varepsilon \in \Gamma_{n,k}$ by $\gamma_\varepsilon(x) = z_0 + \varepsilon x \ \forall\, x \in E$. We claim that

$$\inf_{y \in \gamma_\varepsilon(S^+)} \Phi_H(y) > 0 \tag{3.6}$$

for sufficiently small $\varepsilon$. Since

$$\Phi_H(z_0 + x) = \frac{1}{2}\|x\|_E^2 - \int_0^1 H(z_0 + x)\,\mathrm{d}t \quad \forall\, x \in E^+, \tag{3.7}$$

it suffices to prove that

$$\lim_{\|x\|_E \to 0} \frac{\int_0^1 H(z_0 + x)\,\mathrm{d}t}{\|x\|_E^2} = 0. \tag{3.8}$$

Otherwise, suppose there exists a sequence $(x_j) \subset E$ and $d > 0$ satisfying

$$\|x_j\|_E \to 0 \quad \text{and} \quad \frac{\int_0^1 H(z_0 + x_j)\,\mathrm{d}t}{\|x_j\|_E^2} \geqslant d > 0 \quad \forall\, j. \tag{3.9}$$

Let $y_j = x_j/\|x_j\|_E$ and hence $\|y_j\|_E = 1$. Then lemma 2.1 implies that $(y_j)$ has a convergent subsequence in $L^2$. By a standard result in $L^p$ theory, we have $w \in L^2$ and a subsequence of $(y_j)$, still denoted by $(y_j)$, such that $y_j(t) \to y(t)$ a.e. on $(0,1)$ and that $|y_j(t)| \leqslant w(t)$ a.e. on $(0,1)$ for each $j$. It follows from (3.5) that

$$\frac{H(z_0 + x_j(t))}{\|x_j\|_E^2} \leqslant C_1 \frac{|x_j(t)|^2}{\|x_j\|_E^2} = C_1|y_j(t)|^2 \leqslant C_1 w(t)^2, \quad \text{a.e. on } (0,1),\ \forall\, j,$$
$$\frac{H(z_0 + x_j(t))}{\|x_j\|_E^2} \leqslant C_2 \frac{|x_j(t)|^3}{\|x_j\|_E^2} = C_2|x_j(t)| \cdot |y_j(t)|^2 \leqslant C_2|x_j(t)| w(t)^2, \quad \text{a.e. on } (0,1),\ \forall j.$$

The first claim in (3.9) implies that $(x_j)$ has a subsequence such that $x_{j_l}(t) \to 0$, a.e. in $(0,1)$. Hence the Lebesgue dominated convergence theorem leads to

$$\int_0^1 \frac{H(z_0 + x_{j_l}(t))}{\|x_{j_l}\|_E^2}\mathrm{d}t \to 0.$$

This contradicts the second claim in (3.9).

For any fixed $\mathbb{R}^{n,k}$-admissible $H \in C^2(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$, pick some $z_0 \in \mathbb{R}^{n,k} \cap \mathrm{Int}(H^{-1}(0))$. Since (H1) implies that $H$ vanishes near $z_0$, by (H2) and the Taylor expansion of $H$ at $z_0 \in \mathbb{R}^{2n}$, we have constants $C_1 > 0$ and $C_2 > 0$ such that $H$ satisfies (3.5). □

By (3.2) and propositions 3.4 and 3.6 we see that $c^{n,k}(H)$ is a finite positive number for each $\mathbb{R}^{n,k}$-admissible $H$. The following is a generalization of lemma 3 in [**37**, § 3.4].

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Lemma 3.7. *Let $H \in C^0(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$ satisfy* (3.4) *and* (3.5). *Then*

$$c^{n,k}(H) = \sup_{F \in \mathcal{F}_{n,k}} \inf_{x \in F} \Phi_H(x),$$

*where*

$$\mathcal{F}_{n,k} := \{\gamma(S^+) \,|\, \gamma \in \Gamma_{n,k} \text{ and } \inf(\Phi_H|_{\gamma(S^+)}) > 0\}. \tag{3.10}$$

*Moreover, if $H$ is also of class $C^2$ and has bounded derivatives of second order, then $\mathcal{F}_{n,k}$ is positive invariant under the flow $\varphi_u$ of $\nabla\Phi_H$ (which must exist as pointed out above in lemma* 2.6*).*

*Proof.* Since $c^{n,k}(H)$ is a finite positive number by proposition 3.6, the first claim follows from the arguments above proposition 3.5.

When $H$ has bounded derivatives of second order, (2.4) is satisfied naturally. Then $\nabla\Phi_H$ satisfies the global Lipschitz condition, and thus has a unique global flow $\mathbb{R} \times E \to E : (u, x) \mapsto \varphi_u(x)$ satisfying lemma 2.6, that is, $\varphi_u(x) = \mathrm{e}^{-u}x^- + x^0 + \mathrm{e}^u x^+ + \widetilde{K}(u, x)$, where $\widetilde{K} : \mathbb{R} \times E \to E$ is continuous and maps bounded sets into precompact sets. For a set $F = \gamma(S^+) \in \mathcal{F}_{n,k}$ with $\gamma \in \Gamma_{n,k}$, we have $\alpha := \inf(\Phi_H|_{\gamma(S^+)}) > 0$ by the definition of $\mathcal{F}_{n,k}$. Let $\rho : \mathbb{R} \to [0, 1]$ be a smooth function such that $\rho(s) = 0$ for $s \leqslant 0$ and $\rho(s) = 1$ for $s \geqslant \alpha$. Define a vector field $V$ on $E$ by

$$V(x) = x^+ - x^- - \rho(\Phi_H(x))\nabla\mathfrak{b}(x).$$

Clearly $V$ is locally Lipschitz and has linear growth. These imply that $V$ has a unique global flow, denoted by $\Upsilon_u$. Moreover, it is obvious that $\Upsilon_u$ has the same property as $\varphi_u$ described in lemma 2.6. For $x \in E^- \oplus E^0$, we have $\Phi_H(x) \leqslant 0$ and hence $V(x) = -x^-$, which implies that $\Upsilon_u(E^- \oplus E^0) = E^- \oplus E^0$ and $\Upsilon_u(E \setminus E^- \oplus E^0) = E \setminus E^- \oplus E^0$ since $\Upsilon_u$ is a homeomorphism for each $u \in \mathbb{R}$. Therefore, $\Upsilon_u \in \Gamma_{n,k}$ for all $u \in \mathbb{R}$.

Note that $V|_{\Phi_H^{-1}([\alpha,\infty])} = \nabla\Phi_H(x)$. For each $u \geqslant 0$ we have $\Upsilon_u(x) = \varphi_u(x)$ for any $x \in \Phi_H^{-1}([\alpha, \infty])$, and especially $\Upsilon_u(F) = \varphi_u(F)$, that is, $(\Upsilon_u \circ \gamma)(S^+) = \varphi_u(F)$. Since $\Gamma_{n,k}$ is closed for the composition operation and

$$\inf(\Phi_H|_{(\Upsilon_u\circ\gamma)(S^+)}) = \inf(\Phi_H|_{\varphi_u(F)}) \geqslant \inf(\Phi_H|_F) > 0,$$

we obtain $\varphi_u(F) \in \mathcal{F}_{n,k}$, that is, $\mathcal{F}_{n,k}$ is positively invariant under the flow $\varphi_u$ of $\nabla\Phi_H$. □

Clearly, a $\mathbb{R}^{n,k}$-admissible $H \in C^2(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$ satisfies the conditions of lemma 3.7.

Theorem 3.8. *If an $\mathbb{R}^{n,k}$-admissible $H \in C^2(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$ is nonresonant for $k = n$, and strong nonresonant for $k < n$, then $c^{n,k}(H)$ is a positive critical value of $\Phi_H$.*

The case of $k = n$ was proved in [**13**, § II, proposition 2] (see also [**37**, § 3.4, proposition 1]). It remains to prove the case $k < n$. By lemma 2.4, the functional $\Phi_H$ is $C^{1,1}$ and its gradient $\nabla\Phi_H$ satisfies a global Lipschitz condition on $E$. By a standard minimax argument, theorem 3.8 follows from lemma 3.7 and the following

Lemma 3.9. *If an $\mathbb{R}^{n,k}$-admissible $H \in C^1(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$ is strong nonresonant, then each sequence $(x_j) \subset E$ with $\nabla\Phi_H(x_j) \to 0$ has a convergent subsequence. In particular, $\Phi_H$ satisfies the (PS) condition.*

*Proof.* The functional $\mathfrak{b}$ is differentiable. Its gradient $\nabla\mathfrak{b}$ is compact and satisfies a global Lipschitz condition on $E$. Since $\nabla\Phi_H(x) = x^+ - x^- - \nabla\mathfrak{b}(x)$ for any $x \in E$, we have

$$x_j^+ - x_j^- - \nabla\mathfrak{b}(x_j) \to 0. \tag{3.11}$$

*Case 1. $(x_j)$ is bounded in $E$.* Then $(x_j^0)$ is a bounded sequence in the space $\mathbb{R}^{n,k}$ of finite dimension. Hence $(x_j^0)$ has a convergent subsequence. Moreover, since $\nabla\mathfrak{b}$ is compact, $(\nabla\mathfrak{b}(x_j))$ has a convergent subsequence, and so both $(x_j^+)$ and $(x_j^-)$ have convergent subsequences in $E$. It follows that $(x_j)$ has a convergent subsequence.

*Case 2. $(x_j)$ is unbounded in $E$.* Without loss of generality, we may assume $\lim_{j\to+\infty} \|x_j\|_E = +\infty$. For $z_0 \in \mathbb{R}^{n,k}$ defined as in (H2), let

$$y_j = \frac{x_j}{\|x_j\|_E} - \frac{1}{2a} z_0.$$

Then $|y_j^0| \leqslant \|y_j\|_E \leqslant 1 + |z_0/2a|$, and (3.11) implies

$$y_j^+ - y_j^- - \jmath^*\left(\frac{\nabla H(x_j)}{\|x_j\|_E}\right) \to 0. \tag{3.12}$$

Also by (H2) there exist constants $C_1$ and $C_2$ such that

$$\left\|\frac{\nabla H(x_j)}{\|x_j\|_E}\right\|_{L^2}^2 \leqslant \frac{8a^2\|x_j\|_{L^2}^2 + C_1}{\|x_k\|_E^2} \leqslant C_2$$

that is, $(\nabla H(x_j)/\|x_j\|_E)$ is bounded in $L^2$. Hence the sequence $\jmath^*(\nabla H(x_j)/\|x_j\|_E)$ is compact. (3.12) implies that $(y_j)$ has a convergent subsequence in $E$. Without loss of generality, we may assume that $y_j \to y$ in $E$. Since (H2) implies that $H(z) = Q(z) := a|z|^2 + \langle z, z_0\rangle + b$ for $|z|$ sufficiently large, there exists a constant $C > 0$ such that $|\nabla H(z) - \nabla Q(z)| \leqslant C$ for all $z \in \mathbb{R}^{2n}$. It follows that as $j \to \infty$,

$$\begin{aligned}\left\|\frac{\nabla H(x_j)}{\|x_j\|_E} - \nabla Q(y)\right\|_{L^2} &\leqslant \left\|\frac{\nabla H(x_j)}{\|x_j\|_E} - \nabla Q(y_j)\right\|_{L^2} + \|\nabla Q(y_j) - \nabla Q(y)\|_{L^2}\\ &\leqslant \left\|\frac{\nabla H(x_j) - \nabla Q(x_j)}{\|x_j\|_E}\right\|_{L^2} + \frac{|z_0|}{\|x_j\|_E} + 2a\|y_j - y\|_{L^2}\\ &\leqslant \frac{C}{\|x_j\|_E} + \frac{|z_0|}{\|x_j\|_E} + 2a\|y_j - y\|_{L^2} \to 0.\end{aligned}$$

This implies that $\jmath^*(\nabla H(x_k)/\|x_k\|_E)$ tends to $\jmath^*(\nabla Q(y))$ in $E$, and thus we arrive at

$$y^+ - y^- - \jmath^*(\nabla Q(y)) = 0 \quad \text{and} \quad \left\|y + \frac{z_0}{2a}\right\|_E = 1.$$

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Then $y$ is smooth and satisfies

$$\dot{y} = J_{2n}\nabla Q(y) \quad \text{and} \quad y(1) \sim y(0),\ \ y(0),\ y(1) \in \mathbb{R}^{n,k}.$$

Clearly $y(t)$ is given by

$$y(t) + \frac{1}{2a}z_0 = \mathrm{e}^{2aJ_{2n}t}\left(y(0) + \frac{1}{2a}z_0\right).$$

Since $\|y + (1/2a)z_0\|_E = 1$ implies that $y + (1/2a)z_0$ is nonconstant, using the boundary condition satisfied by $y$ and the assumption that $z_0 \in \mathbb{R}^{n,k}$, we deduce that $2a \in m\mathbb{N}\pi$. This gives rise to a contradiction because $H$ is strong non-resonant. □

Corresponding to [**37**, § 3.5, lemma] we have

Lemma 3.10. *Suppose that $H : \mathbb{R}^{2n} \to \mathbb{R}$ is of class $C^{2n+2}$ and that $\nabla H : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfies a global Lipschitz condition. Then the set of critical values of $\Phi_H$ has empty interior in $\mathbb{R}$.*

*Proof.* The method is similar to that of [**26**, lemma 3.5]. For clearness we give it in details. By lemma 2.4, $\Phi_H$ is $C^{1,1}$. Lemma 2.5 implies that all critical points of $\Phi_H$ sit in $C^{2n+2}_{n,k}([0,1],\mathbb{R}^{2n})$. Thus the restriction of $\Phi_H$ to $C^1_{n,k}([0,1],\mathbb{R}^{2n})$, denoted by $\hat{\Phi}_H$, and $\Phi_H$ have the same critical value sets. As in the proof of [**24**, claim 4.4] we can deduce that $\hat{\Phi}_H$ is of class $C^{2n+1}$.

Let $P_0$ and $P_1$ be the orthogonal projections of $\mathbb{R}^{2n}$ to the spaces $V_0^{n,k}$ and $V_1^{2k}$ in (1.1) and (1.2), respectively. Take a smooth $g : [0,1] \to [0,1]$ such that $g$ equals 1 (resp. 0) near 0 (resp. 1). Denote by $\phi^t$ the flow of $X_H$. Since $X_H$ is $C^{2n+1}$, we have a $C^{2n+1}$ map

$$\psi : [0,1] \times \mathbb{R}^{n,k} \to \mathbb{R}^{2n}, \quad (t,z) \mapsto g(t)\phi^t(z) + (1-g(t))\phi^{t-1}(P_0\phi^1(z) + P_1 z).$$

For any $z \in \mathbb{R}^{n,k}$, since $\psi(0,z) = \phi^0(z) = z$ and $\psi(1,z) = P_0\phi^1(z) + P_1 z$, we have

$$\psi(1,z), \quad \psi(0,z) \in \mathbb{R}^{n,k} \quad \text{and} \quad \psi(1,z) \sim \psi(0,z).$$

These and [**24**, corollary B.2] show that $\psi$ gives rise to a $C^{2n}$ map

$$\Omega : \mathbb{R}^{n,k} \to C^1_{n,k}([0,1],\mathbb{R}^{2n}), \quad z \mapsto \psi(\cdot, z).$$

Hence $\Phi_H \circ \Omega : \mathbb{R}^{n,k} \to \mathbb{R}$ is of class $C^{2n}$. By Sard's theorem we deduce that the critical value sets of $\Phi_H \circ \Omega$ is nowhere dense (since $\dim \mathbb{R}^{n,k} < 2n$).

Let $z \in \mathbb{R}^{n,k}$ be such that $\phi^1(z) \in \mathbb{R}^{n,k}$ and $\phi^1(z) \sim z$. Then $P_0\phi^1(z) - P_0 z = \phi^1(z) - z$ and therefore $P_0\phi^1(z) + P_1 z = \phi^1(z)$, which implies $\psi(t,z) = \phi^t(z) \forall\, t \in [0,1]$.

For a critical point $y$ of $\Phi_H$, that is, $y \in C^{2n+2}_{n,k}([0,1],\mathbb{R}^{2n})$ and solves $\dot{y} = J_{2n}\nabla H(y) = X_H(y)$, with $z_y := y(0) \in \mathbb{R}^{n,k}$ we have $y(t) = \phi^t(z_y)\ \ \forall\ t \in [0,1]$, which implies that $\phi^1(z_y) \in \mathbb{R}^{n,k}$, $\phi^1(z_y) \sim z_y$ and therefore $y = \psi(\cdot, z_y) = \Omega(z_y)$. Hence $z_y$ is a critical point of $\Phi_H \circ \Omega$ and $\Phi_H \circ \Omega(z_y) = \Phi_H(y)$. Thus the critical value set of $\Phi_H$ is contained in that of $\Phi_H \circ \Omega$. The desired claim is obtained. □

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Having this lemma we can prove the following proposition, which corresponds to proposition 3 in [**13**, § II].

PROPOSITION 3.11. *Let* $H \in C^{2n+2}(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$ *be* $\mathbb{R}^{n,k}$*-admissible with* $k < n$ *and strong nonresonant. Suppose that* $[0,1] \ni s \mapsto \psi_s$ *is a smooth homotopy of the identity in* $\mathrm{Symp}(\mathbb{R}^{2n}, \omega_0)$ *satisfying*

$$\psi_s(\mathbb{R}^{n,k}) = \mathbb{R}^{n,k}, \quad \psi_s(w + V_0^{n,k}) = \psi_s(w) + V_0^{n,k} \quad \forall\, w \in \mathbb{R}^{n,k} \tag{3.13}$$

*and*

$$\psi_s(z) = z + w_s \quad \forall\, z \in \mathbb{R}^{2n} \setminus B^{2n}(0, R),$$

*where* $R > 0$ *and* $[0,1] \ni s \mapsto w_s$ *is a smooth path in* $\mathbb{R}^{n,k}$*. Then* $s \mapsto c^{n,k}(H \circ \psi_s)$ *is constant. Moreover, the same conclusion holds true if all* $\psi_s$ *are replaced by translations* $\mathbb{R}^{2n} \ni z \mapsto z + w_s$*, where* $[0,1] \ni s \mapsto w_s$ *is a smooth path in* $\mathbb{R}^{n,k}$*. In particular,* $c^{n,k}(H(\cdot + w)) = c^{n,k}(H)$ *for any* $w \in \mathbb{R}^{n,k}$*.*

*Proof.* By the assumptions each $H \circ \psi_s$ is also $\mathbb{R}^{n,k}$-admissible and strong nonresonant. Hence $c(H \circ \psi_s)$ is a positive critical value for each $s$. Let $x \in E$ be a critical point of $\Phi_{H\circ\psi_s}$ with critical value $c(H \circ \psi_s)$. Then $x \in C^{2n+2}_{n,k}([0,1], \mathbb{R}^{2n})$ and solves $\dot{x} = J_{2n}\nabla(H \circ \psi_s)(x) = X_{H\circ\psi_s}(x)$. Let $y_s = \psi_s \circ x$. Then $y_s \in C^{2n+2}_{n,k}([0,1], \mathbb{R}^{2n})$ and satisfies

$$\dot{y}_s(t) = (\mathrm{d}\psi_s(x(t))\dot{x}(t) = (\mathrm{d}\psi_s(x(t))X_{H\circ\psi_s}(x(t)) = X_H(\psi_s(x(t)) = J_{2n}\nabla H(y_s(t))$$

since $\mathrm{d}\psi_s(z)X_H(z) = X_H(\psi_s(z))$ for any $z \in \mathbb{R}^{2n}$ by [**20**, p. 9]. Therefore $y_s$ is a critical point of $\Phi_H$ on $E$. We claim that

$$\Phi_H(y_s) = \Phi_{H\circ\psi_s}(x). \tag{3.14}$$

Clearly, it suffices to prove the following equality:

$$A(y_s) = \frac{1}{2}\int_0^1 \langle -J_{2n}\dot{y}_s, y_s\rangle\, \mathrm{d}t = \frac{1}{2}\int_0^1 \langle -J_{2n}\dot{x}, x\rangle\, \mathrm{d}t = A(x). \tag{3.15}$$

Extend $x$ into a piecewise $C^{2n+2}$-smooth loop $x^* : [0,2] \to \mathbb{R}^{2n}$ by setting $x^*(t) = (2-t)x(1) + (t-1)x(0)$ for any $1 \leqslant t \leqslant 2$. We get a piecewise $C^{2n+2}$-smooth loop extending of $y_s$, $y_s^* = \psi_s(x^*)$. Clearly, we can extend $x^*$ into a piecewise $C^{2n+2}$-smooth $u : D^2 \to \mathbb{R}^{2n}$, where $D^2$ is a closed disc bounded by $\partial D^2 \equiv [0,2]/\{0,2\}$. Then $\psi_s \circ u : D^2 \to \mathbb{R}^{2n}$ is piecewise $C^{2n+2}$-smooth and $\psi_s \circ u|_{\partial D^2} = y_s^*$. Stokes theorem yields

$$\frac{1}{2}\int_0^2 \langle -J_{2n}\dot{x}^*, x^*\rangle\, \mathrm{d}t = \int_{D^2} u^*\omega_0,$$
$$\frac{1}{2}\int_0^2 \langle -J_{2n}\dot{y}_s^*, y_s^*\rangle\, \mathrm{d}t = \int_{D^2} (\psi_s \circ u)^*\omega_0 = \int_{D^2} u^*\omega_0.$$

Moreover, for any $t \in [1,2]$ we have $\dot{x}^*(t) = x(0) - x(1) \in V_0^{n,k}$ and $x^*(t) \in \mathbb{R}^{n,k}$, and therefore $\langle -J_{2n}\dot{x}^*(t), x^*(t)\rangle = 0$ because $\mathbb{R}^{2n}$ has the orthogonal decomposition $\mathbb{R}^{2n} = J_{2n}V_0^{n,k} \oplus \mathbb{R}^{n,k}$. Then (3.15) follows from these.

Since $s \mapsto c^{n,k}(H \circ \psi_s)$ is continuous by proposition 3.5, and a critical point $x$ of $\Phi_{H\circ\psi_s}$ with critical value $c(H \circ \psi_s)$ yields a critical point $y_s$ of $\Phi_H$ on $E$ satisfying (3.14), we deduce that each $c(H \circ \psi_s)$ is also a critical value of $\Phi_H$. Lemma 3.10 shows that $s \mapsto c^{n,k}(H \circ \psi_s)$ must be constant.

Finally, let $\psi_s(z) = z + w_s$. It is clear that $H \circ \psi_s$ is $\mathbb{R}^{n,k}$-admissible and strong nonresonant. Thus $c(H \circ \psi_s)$ is a positive critical value. If $x \in E$ is a critical point of $\Phi_{H\circ\psi_s}$ with critical value $c(H \circ \psi_s)$, then $y_s := \psi_s \circ x$ is a critical point of $\Phi_H$ on $E$ and (3.14) holds. Hence $s \mapsto c^{n,k}(H(\cdot + w_s))$ is constant. □

Let $\mathcal{F}_{n,k}(\mathbb{R}^{2n}) = \{H \in C^0(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0}) \mid H \text{ satisfies (H2)}\}$. For each bounded subset $B \subset \mathbb{R}^{2n}$ such that $\overline{B} \cap \mathbb{R}^{n,k} \neq \emptyset$, we define

$$\mathcal{F}_{n,k}(\mathbb{R}^{2n}, B) = \{H \in \mathcal{F}_{n,k}(\mathbb{R}^{2n}) \mid H \text{ vanishes near } \overline{B}\}. \tag{3.16}$$

Definition 3.12. *For each bounded subset $B \subset \mathbb{R}^{2n}$ such that $\overline{B} \cap \mathbb{R}^{n,k} \neq \emptyset$,*

$$c^{n,k}(B) := \inf\{c^{n,k}(H) \mid H \in \mathcal{F}_{n,k}(\mathbb{R}^{2n}, B)\} \in [0, +\infty) \tag{3.17}$$

*is called the coisotropic Ekeland–Hofer capacity of $B$ (relative to $\mathbb{R}^{n,k}$). For any unbounded subset $B \subset \mathbb{R}^{2n}$ such that $\overline{B} \cap \mathbb{R}^{n,k} \neq \emptyset$, its coisotropic Ekeland–Hofer capacity is defined by*

$$c^{n,k}(B) = \sup\{c^{n,k}(A) \mid A \subset B, A \text{ is bounded and } \overline{A} \cap \mathbb{R}^{n,k} \neq \emptyset\}. \tag{3.18}$$

Remark 3.13. When $k = n$ in the above definition, $c^{n,n}(B)$ is the (first) Ekeland–Hofer capacity of $B$.

For each bounded $B \subset \mathbb{R}^{2n}$ such that $\overline{B} \cap \mathbb{R}^{n,k} \neq \emptyset$, we write

$$\mathcal{E}_{n,k}(\mathbb{R}^{2n}, B) = \{H \in \mathcal{F}_{n,k}(\mathbb{R}^{2n}, B) \mid H \text{ is strong nonresonant}\} \quad \text{if } k < n,$$
$$\mathcal{E}_{n,n}(\mathbb{R}^{2n}, B) = \{H \in \mathcal{F}_{n,k}(\mathbb{R}^{2n}, B) \mid H \text{ is nonresonant}\}.$$

Clearly, each $H \in \mathcal{E}_{n,k}(\mathbb{R}^{2n}, B)$ satisfies (H1), and $\mathcal{E}_{n,k}(\mathbb{R}^{2n}, B)$ is a *cofinal family* of $\mathcal{F}_{n,k}(\mathbb{R}^{2n}, B)$, that is, for any $H \in \mathcal{F}_{n,k}(\mathbb{R}^{2n}, B)$ there exists $G \in \mathcal{E}_{n,k}(\mathbb{R}^{2n}, B)$ such that $G \geqslant H$. Moreover, for each $l \in \mathbb{N} \cup \{\infty\}$ the smaller subset $\mathcal{E}_{n,k}(\mathbb{R}^{2n}, B) \cap C^l(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$ is also a cofinal family of $\mathcal{F}_{n,k}(\mathbb{R}^{2n}, B)$. By the definition, we immediately get:

Proposition 3.14.

(i) $c^{n,k}(B) = c^{n,k}(\overline{B})$.

(ii) $\mathcal{F}_{n,k}(\mathbb{R}^{2n}, B)$ *in* (3.17) *can be replaced by any cofinal subset of it.*

(iii) *Suppose that $\overline{B} \subset B^{2n}(R)$. For each $l \in \mathbb{N} \cup \{\infty\}$ let $\mathcal{E}^l_{n,k}(\mathbb{R}^{2n}, B)$ consist of $H \in \mathcal{F}_{n,k}(\mathbb{R}^{2n}, B) \cap C^l(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$ for which there exists $z_0 \in \mathbb{R}^{n,k}$, real numbers $a, b$ such that $H(z) = a|z|^2 + \langle z, z_0\rangle + b$ outside the closed ball $\overline{B^{2n}(R)}$, where $a > \pi$ and $a \notin \pi\mathbb{N}$ for $k = n$, and $a > \pi/2$ and $a \notin \pi\mathbb{N}/2$ for $0 \leqslant k < n$. Then each $\mathcal{E}^l_{n,k}(\mathbb{R}^{2n}, B)$ is a cofinal subset of $\mathcal{F}_{n,k}(\mathbb{R}^{2n}, B)$.*

*Proof.* We only prove (iii). By (ii) it suffices to prove that for each given $H \in \mathcal{E}_{n,k}(\mathbb{R}^{2n}, B)$ there exists $G \in \mathcal{E}^l_{n,k}(\mathbb{R}^{2n}, B)$ such that $G \geqslant H$. We may assume that $H(z) = a|z|^2 + \langle z, z_0\rangle + b$ outside a larger closed ball $\overline{B^{2n}(R_1)}$, where $a > \pi$ and $a \notin \pi\mathbb{N}$ for $k = n$, and $a > \pi/2$ and $a \notin \pi\mathbb{N}/2$ for $0 \leqslant k < n$. Let $U_\epsilon(B)$ be the $\epsilon$-neighbourhood of $B$. We can also assume that $H$ vanishes in $U_{2\epsilon}(B)$. Since $\overline{B^{2n}(R_1)}$ is compact, we may find numbers $a' > a$, $b'$ such that $a' \notin \pi\mathbb{N}$ for $k = n$, $a' \notin \pi\mathbb{N}/2$ for $0 \leqslant k < n$, and $a'|z|^2 + \langle z, z_0\rangle + b' \geqslant H(z)$ for all $z \in \mathbb{R}^{2n}$. Take a smooth function $f : \mathbb{R}^{2n} \to \mathbb{R}_{\geqslant 0}$ such that it equals to zero in $U_\epsilon(B)$ and 1 outside $U_{2\epsilon}(B)$. Define $G(z) := f(z)(a'|z|^2 + \langle z, z_0\rangle + b')$ for $z \in \mathbb{R}^{2n}$. Then $G \geqslant H$ and $G \in \mathcal{E}^\infty_{n,k}(\mathbb{R}^{2n}, B)$. □

Remark 3.15. Let $\mathcal{H}_{n,k}(\mathbb{R}^{2n}, B)$ consist of $H \in C^\infty(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$ which vanishes near $\overline{B}$ and for which there exists $z_0 \in \mathbb{R}^{n,k}$ and a real number $a$ such that $H(z) = a|z|^2$ outside a compact subset, where $a > \pi$ and $a \notin \pi\mathbb{N}$ for $k = n$, and $a > \pi/2$ and $a \notin \pi\mathbb{N}/2$ for $0 \leqslant k < n$. As in the proof of proposition 1.1 it is not hard to prove that $\mathcal{H}_{n,k}(\mathbb{R}^{2n}, B)$ is a cofinal subset of $\mathcal{F}_{n,k}(\mathbb{R}^{2n}, B)$. When $k = n$ this shows that Sikorav's approach [**37**] to the Ekeland–Hofer capacity in [**13**] defines the same capacity.

*Proof of proposition 1.1.* Proposition 3.5(i)–(iii) lead to the first three claims. Let us prove (iv). We may assume that $B$ is bounded. By (3.17) we have a sequence $(H_j) \subset \mathcal{F}_{n,k}(\mathbb{R}^{2n}, B)$ such that $c^{n,k}(H_j) \to c^{n,k}(B)$. Note that $H_j(\cdot - w) \in \mathcal{F}_{n,k}(\mathbb{R}^{2n}, B + w)$ for each $j$. Hence

$$c^{n,k}(B + w) \leqslant \inf_j c^{n,k}(H_j(\cdot - w)) = \inf_j c^{n,k}(H_j) = c^{n,k}(B)$$

by the final claim in proposition 3.11. The same reasoning leads to $c^{n,k}(B) = c^{n,k}(B + w + (-w)) \leqslant c^{n,k}(B + w)$ and so $c^{n,k}(B + w) = c^{n,k}(B)$. □

Proposition 3.16 (Relative monotonicity). *Let subsets $A, B \subset \mathbb{R}^{2n}$ satisfy $\overline{A} \cap \mathbb{R}^{n,k} \neq \emptyset$ and $\overline{B} \cap \mathbb{R}^{n,k} \neq \emptyset$. If there exists a smooth homotopy of the identity in $\mathrm{Symp}(\mathbb{R}^{2n}, \omega_0)$ as in proposition 3.11, $[0, 1] \ni s \mapsto \psi_s$, such that $\psi_1(A) \subset B$, then $c^{n,k}(A) = c^{n,k}(\psi_s(A))$ for all $s \in [0, 1]$, and in particular $c^{n,k}(A) \leqslant c^{n,k}(B)$ by proposition 1.1(i).*

*Proof.* Note that

$$\mathcal{E}_{n,k}(\mathbb{R}^{2n}, A) \cap C^\infty(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0}) \to \mathcal{E}_{n,k}(\mathbb{R}^{2n}, \psi_s(A)) \cap C^\infty(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0}), \quad H \mapsto H \circ \psi_s^{-1}$$

is a one-to-one correspondence. Then

$$\begin{aligned} c^{n,k}(\psi_s(A)) &= \inf\{c^{n,k}(G) \mid G \in \mathcal{E}_{n,k}(\mathbb{R}^{2n}, \psi_s(A)) \cap C^\infty(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})\} \\ &= \inf\{c^{n,k}(H \circ \psi_s^{-1}) \mid H \in \mathcal{E}_{n,k}(\mathbb{R}^{2n}, A) \cap C^\infty(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})\} \\ &= \inf\{c^{n,k}(H) \mid H \in \mathcal{E}_{n,k}(\mathbb{R}^{2n}, A) \cap C^\infty(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})\} = c^{n,k}(A). \end{aligned}$$

Here the third equality comes from proposition 3.11. □

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*Proof of theorem 1.2.* We may assume that $B$ is bounded, and complete the proof in two steps.

*Step 1. Prove $c^{n,k}(\Phi(B)) = c^{n,k}(B)$ for every $\Phi \in \mathrm{Sp}(2n, k)$.* Take a smooth path $[0, 1] \ni t \mapsto \Phi_t \in \mathrm{Sp}(2n, k)$ such that $\Phi_0 = I_{2n}$ and $\Phi_1 = \Phi$. We have a smooth function $[0, 1] \times \mathbb{R}^{2n} \ni (t, z) \mapsto G_t(z) \in \mathbb{R}^{2n}$ such that the path $\Phi_t$ is generated by $X_{G_t}$ and that $G_t(z) = 0 \ \forall\ z \in \mathbb{R}^{n,k}$ (see step 2 below). Since $\cup_{t\in[0,1]}\Phi_t(\overline{B})$ is compact, it can be contained in a ball $B^{2n}(0, R)$ for some $R > 0$. Take a smooth cut function $\rho : \mathbb{R}^{2n} \to [0, 1]$ such that $\rho = 1$ on $B^{2n}(0, 2R)$ and $\rho = 0$ outside $B^{2n}(0, 3R)$. Define a smooth function $\tilde{G} : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ by $\tilde{G}(t, z) = \rho(z)G_t(z)$ for $(t, z) \in [0, 1] \times \mathbb{R}^{2n}$. Denote by $\psi_t$ the Hamiltonian path generated by $\tilde{G}$ in $\mathrm{Ham}^c(\mathbb{R}^{2n}, \omega_0)$. Then $\psi_t(z) = \Phi_t(z)$ for all $(t, z) \in [0, 1] \times B^{2n}(0, R)$. Moreover each $\psi_t$ restricts to the identity on $\mathbb{R}^{n,k}$ because $\tilde{G}(t, z) = \rho(z)G_t(z) = 0$ for all $(t, z) \in [0, 1] \times \mathbb{R}^{n,k}$. Hence we obtain $c^{n,k}(\Phi(B)) = c^{n,k}(\Phi_1(B)) = c^{n,k}(B)$ by proposition 3.16.

*Step 2. Prove $c^{n,k}(\phi(B)) = c^{n,k}(B)$ in case $w_0 = 0$.* Let $\Phi = (\mathrm{d}\phi(0))^{-1}$. Since $c^{n,k}(\Phi \circ \phi(B)) = c^{n,k}(\phi(B))$ by step 1, and $\Phi \circ \phi(w) = w \ \forall\ w \in \mathbb{R}^{n,k}$, replacing $\Phi \circ \phi$ by $\phi$, we may assume $\mathrm{d}\phi(0) = \mathrm{id}_{\mathbb{R}^{2n}}$. Define a continuous path in $\mathrm{Symp}(\mathbb{R}^{2n}, \omega_0)$,

$$\varphi_t(z) = \begin{cases} z & \text{if } t \leqslant 0, \\ \dfrac{1}{t}\phi(tz) & \text{if } t > 0, \end{cases} \tag{3.19}$$

which is smooth except possibly at $t = 0$. As in [**36**, proposition A.1] we can smoothen it with a smooth function $\eta : \mathbb{R} \to \mathbb{R}$ defined by

$$\eta(t) = \begin{cases} 0 & \text{if } t \leqslant 0, \\ e^2\, e^{-2/t} & \text{if } t > 0, \end{cases} \tag{3.20}$$

where $e$ is the Euler number. Namely, defining $\phi_t(z) := \varphi_{\eta(t)}(z)$ for $z \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}$, we get a smooth path $\mathbb{R} \ni t \mapsto \phi_t \in \mathrm{Symp}(\mathbb{R}^{2n}, \omega_0)$ such that

$$\phi_0 = \mathrm{id}_{\mathbb{R}^{2n}}, \quad \phi_1 = \phi, \quad \phi_t(z) = z, \quad \forall\ z \in \mathbb{R}^{n,k},\ \ \forall\ t \in \mathbb{R}. \tag{3.21}$$

Define $X_t(z) = ((\mathrm{d}/\mathrm{d}t)\phi_t)(\phi_t^{-1}(z))$ and

$$H_t(z) = \int_0^z i_{X_t}\omega_0, \tag{3.22}$$

where the integral is along any piecewise smooth curve from 0 to $z$ in $\mathbb{R}^{2n}$. Then $\mathbb{R} \times \mathbb{R}^{2n} \ni (t, z) \mapsto H_t(z) \in \mathbb{R}$ is smooth and $X_t = X_{H_t}$. By the final condition in (3.21), for each $(t, z) \in \mathbb{R} \times \mathbb{R}^{n,k}$ we have $X_t(z) = 0$ and therefore $H_t(z) = 0$. As in step 1, we can assume that $\cup_{t\in[0,1]}\phi_t(\overline{B})$ is contained a ball $B^{2n}(0, R)$. Take a smooth cut function $\rho : \mathbb{R}^{2n} \to [0, 1]$ as above, and define a smooth function $\tilde{H} : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ by $\tilde{H}(t, z) = \rho(z)H_t(z)$ for $(t, z) \in [0, 1] \times \mathbb{R}^{2n}$. Then the Hamiltonian path $\psi_t$ generated by $\tilde{H}$ in $\mathrm{Ham}^c(\mathbb{R}^{2n}, \omega_0)$ satisfies

$$\psi_t(z) = \phi_t(z), \quad \forall\ (t, z) \in [0, 1] \times B^{2n}(0, R) \quad \text{and} \quad \psi_t(z) = z, \quad \forall\ (t, z) \in [0, 1] \times \mathbb{R}^{n,k}.$$

It follows from proposition 3.16 that $c^{n,k}(\phi(B)) = c^{n,k}(\psi_1(B)) = c^{n,k}(B)$ as above.

*Step 3. Prove $c^{n,k}(\phi(B)) = c^{n,k}(B)$ in case $w_0 \neq 0$.* Define $\varphi(w) = \phi(w + w_0)$ for $w \in \mathbb{R}^{2n}$. Then $\mathrm{d}\varphi(0) = \mathrm{d}\phi(w_0) \in \mathrm{Sp}(2n, k)$ and $\varphi(w) = \phi(w + w_0) = w \forall\ w \in \mathbb{R}^{n,k}$.

By step 2 we arrive at $c^{n,k}(\varphi(B-w_0))=c^{n,k}(B-w_0)$. The desired equality follows because $\phi(B)=\varphi(B-w_0)$ and $c^{n,k}(B-w_0)=c^{n,k}(B)$ by proposition 1.1. $\square$

*Proof of corollary 1.3.* As discussed above the proof is reduced to the case $w_0=0$. Moreover we can assume that both sets $A$ and $U$ are bounded and that $U$ is also star-shaped with respect to the origin $0\in\mathbb{R}^{2n}$.

Next the proof can be completed following [**36**, proposition A.1]. Now $[0,1]\ni t\mapsto\phi_t(\cdot):=\varphi_{\eta(t)}(\cdot)$ given by (3.19) and (3.20) is a smooth path of symplectic embeddings from $U$ to $\mathbb{R}^{2n}$ with properties

$$\phi_0=\mathrm{id}_U,\quad \phi_1=\varphi,\quad \phi_t(z)=z,\quad \forall\ z\in\mathbb{R}^{n,k}\cap U,\ \forall\ t\in\mathbb{R}. \tag{3.23}$$

Thus $X_t(z):=((\mathrm{d}/\mathrm{d}t)\phi_t)(\phi_t^{-1}(z))$ is a symplectic vector field defined on $\phi_t(U)$, and (3.22) (where the integral is along any piecewise smooth curve from 0 to $z$ in $\phi_t(U)$) defines a smooth function $H_t$ on $\phi_t(U)$ in the present case. Observe that $H:\cup_{t\in[0,1]}(\{t\}\times\phi_t(U))\to\mathbb{R}$ defined by $H(t,z)=H_t(z)$ is smooth and generates the path $\phi_t$. Since $K=\cup_{t\in[0,1]}\{t\}\times\phi_t(\overline{A})$ is a compact subset in $[0,1]\times\mathbb{R}^{2n}$ we can choose a bounded and relative open neighbourhood $W$ of $K$ in $[0,1]\times\mathbb{R}^{2n}$ such that $W\subset\cup_{t\in[0,1]}(\{t\}\times\phi_t(U))$. Take a smooth cut function $\chi:[0,1]\times\mathbb{R}^{2n}\to\mathbb{R}$ such that $\chi|_K=1$ and $\chi$ vanishes outside $W$. Define $\hat{H}:[0,1]\times\mathbb{R}^{2n}\to\mathbb{R}$ by $\hat{H}(t,z)=\chi(t,z)H(t,z)$. It generates a smooth homotopy $\psi_t$ ($t\in[0,1]$) of the identity in $\mathrm{Ham}^c(\mathbb{R}^{2n},\omega_0)$ such that $\psi_t(z)=\phi_t(z)$ for all $(t,z)\in[0,1]\times A$. Moreover, the final condition in (3.21) implies that $\mathbb{R}^{n,k}\cap U\subset\phi_t(U)$ and $X_t(z)=0$ for any $t\in[0,1]$ and $z\in\mathbb{R}^{n,k}\cap U$. Hence for any $(t,z)\in[0,1]\times\mathbb{R}^{n,k}$ we have $\hat{H}(t,z)=\chi(t,z)H(t,z)=0$ and so $\psi_t(z)=z$. Then proposition 3.16 leads to $c^{n,k}(A)=c^{n,k}(\psi_1(A))=c^{n,k}(\phi_1(A))=c^{n,k}(\varphi(A))$. $\square$

## 4. Proof of theorem 1.4

The case of $k=n$ was proved in [**13**, **14**, **37**]. We assume $k<n$ below. By proposition 1.1(iv), $c^{n,k}(D)=c^{n,k}(D+w)$ for any $w\in\mathbb{R}^{n,k}$. Moreover, for each $x\in C^1_{n,k}([0,1])$ there holds

$$A(x)=\frac{1}{2}\int_0^1\langle -J_{2n}\dot{x},x\rangle\,\mathrm{d}t=\frac{1}{2}\int_0^1\langle -J_{2n}\dot{x},x+w\rangle\,\mathrm{d}t=A(x+w),\quad \forall\ w\in\mathbb{R}^{n,k}.$$

Recalling that $D\cap\mathbb{R}^{n,k}\neq\emptyset$, we may assume that $D$ contains the origin 0 below.

Let $j_D$ be the Minkowski functional associated to $D$, $H:=j_D^2$ and $H^*$ be the Legendre transform of $H$. Then $\partial D=H^{-1}(1)$, and there exists a constant $R\geqslant 1$ such that

$$\frac{|z|^2}{R}\leqslant H(z)\leqslant R|z|^2\quad\text{and so}\quad \frac{|z|^2}{4R}\leqslant H^*(z)\leqslant\frac{R}{4}|z|^2 \tag{4.1}$$

for all $z\in\mathbb{R}^{2n}$. Moreover $H$ is $C^{1,1}$ with uniformly Lipschitz constant.

By [**26**, theorem 1.5]

$$\Sigma^{n,k}_{\partial D}:=\{A(x)>0\,|\,x\text{ is a leafwise chord on }\partial D\text{ for }\mathbb{R}^{n,k}\}$$

contains a minimum number $\varrho$, that is, there exists a leafwise chord $x^*$ on $\partial D$ for $\mathbb{R}^{n,k}$ such that $A(x^*)=\min\Sigma^{n,k}_{\partial D}=\varrho$. Actually, the arguments therein shows that

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there exists $w \in C^1_{n,k}([0,1])$ such that

$$A(w) = 1 \quad \text{and} \quad I(w) := \int_0^1 H^*(-J_{2n}\dot{w})\,\mathrm{d}t = A(x^*) = \varrho. \tag{4.2}$$

Let us prove (1.8) and (1.9) by the following two steps. As done in [**24**, **25**] (see also step 4 below), by approximating arguments we can assume that $\partial D$ is smooth and strictly convex. In this case $\Sigma^{n,k}_{\partial D}$ has no interior points in $\mathbb{R}$ because of [**26**, lemma 3.5], and we give a complete proof though the ideas which are similar to those of the proof of [**37**] (and [**25**, theorem 1.11] and [**24**, theorem 1.17]).

*Step 1. Prove that $c^{n,k}(D) \geqslant \varrho$.* By the monotonicity of $c^{n,k}$ it suffices to prove $c^{n,k}(\partial D) \geqslant \varrho$. For a given $\epsilon > 0$, consider a cofinal family of $\mathcal{F}_{n,k}(\mathbb{R}^{2n}, \partial D)$,

$$\mathcal{E}^{n,k}_{\epsilon}(\mathbb{R}^{2n}, \partial D) \tag{4.3}$$

consisting of $\overline{H} = f \circ H$, where $f \in C^\infty(\mathbb{R}, \mathbb{R}_{\geqslant 0})$ satisfies

$$\left.\begin{array}{l} f(s) = 0 \text{ for } s \text{ near} 1 \in \mathbb{R}, \\ f'(s) \leqslant 0\ \forall\ s \leqslant 1, \quad f'(s) \geqslant 0\ \forall\ s \geqslant 1, \\ f'(s) = \alpha \in \mathbb{R} \setminus \Sigma^{n,k}_{\partial D} \text{ if } f(s) \geqslant \epsilon \quad s > 1 \end{array}\right\} \tag{4.4}$$

and where $\alpha$ is required to satisfy for some constant $C > 0$

$$\alpha H(z) \geqslant \frac{\pi}{2}|z|^2 - C \quad \text{for } |z| \text{ sufficiently large} \tag{4.5}$$

because of (4.1) and $\mathrm{Int}(\Sigma^{n,k}_{\partial D}) = \emptyset$.

Then each $\overline{H} \in \mathcal{E}^{n,k}_{\epsilon}(\mathbb{R}^{2n}, \partial D)$ satisfies all conditions in lemma 3.7. Indeed, it belongs to $C^\infty(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$, restricts to zero near $\partial D$ and thus satisfies (H1). Note that $f(s) = \alpha s + \epsilon - \alpha s_0$ for $s \geqslant s_0$, where $s_0 = \inf\{s > 1 \mid f(s) \geqslant \epsilon\}$. (4.5) implies that $\overline{H}(z) \geqslant (\pi/2)|z|^2 - C'\ \ \forall\ \ z \in \mathbb{R}^{2n}$ for some constant $C' > 0$, and therefore $c^{n,k}(\overline{H}) < +\infty$ by the arguments above proposition 3.5. Moreover, it is clear that $\mathbb{R}^{n,k} \cap \mathrm{Int}(\overline{H}^{-1}(0)) \neq \emptyset$ and $|\overline{H}_{zz}(z)|$ is bounded on $\mathbb{R}^{2n}$. Then (3.5) is satisfied with any $z_0 \in \mathbb{R}^{n,k} \cap \mathrm{Int}(\overline{H}^{-1}(0))$ by the arguments at the end of proof of proposition 3.6. Hence $c^{n,k}(\overline{H}) > 0$.

By combining proofs of lemma 3.9 and [**26**, lemma 3.7] we can obtain the first claim of the following.

Lemma 4.1. *For every $\overline{H} \in \mathcal{E}^{n,k}_{\epsilon}(\mathbb{R}^{2n}, \partial D)$, $\Phi_{\overline{H}}$ satisfies the $(PS)$ condition and hence $c^{n,k}(\overline{H})$ is a positive critical value of $\Phi_{\overline{H}}$.*

Lemma 4.2. *For every $\overline{H} \in \mathcal{E}^{n,k}_{\epsilon}(\mathbb{R}^{2n}, \partial D)$, any positive critical value $c$ of $\Phi_{\overline{H}}$ is greater than $\min \Sigma^{n,k}_{\partial D} - \epsilon$. In particular, $c^{n,k}(\overline{H}) > \min \Sigma^{n,k}_{\partial D} - \epsilon$.*

*Proof.* For a critical point $x$ of $\Phi_{\overline{H}}$ with positive critical values there holds

$$-J_{2n}\dot{x}(t) = \nabla\overline{H}(x(t)) = f'(H(x(t)))\nabla H(x(t)), \quad x(1) \sim x(0), \quad x(1), x(0) \in \mathbb{R}^{n,k}$$

and $H(x(t)) \equiv c_0$ (a positive constant). Since

$$\begin{aligned} 0 < \Phi_{\overline{H}}(x) &= \frac{1}{2}\int_0^1 \langle J_{2n}x(t), \dot{x}(t)\rangle\,\mathrm{d}t - \int_0^1 \overline{H}(x(t))\,\mathrm{d}t \\ &= \frac{1}{2}\int_0^1 \langle x(t), f'(c_0)\nabla H(x(t))\rangle\,\mathrm{d}t - \int_0^1 f(s)\,\mathrm{d}t \\ &= f'(c_0)c_0 - f(c_0), \end{aligned}$$

we deduce $\beta := f'(c_0) > 0$, and so $c_0 > 1$. Define $y(t) = (1/\sqrt{c_0})x(t/\beta)$ for $0 \leqslant t \leqslant \beta$. Then

$$H(y(t)) = 1, \quad -J_{2n}\dot{y} = \nabla H(y(t)), \quad y(\beta) \sim y(0), \quad y(\beta), y(0) \in \mathbb{R}^{n,k}$$

and therefore $f'(c_0) = \beta = A(y) \in \Sigma_{\partial D}^{n,k}$. By the definition of $f$ this implies $f(c_0) < \epsilon$ and so

$$\Phi_{\overline{H}}(x) = f'(c_0)c_0 - f(c_0) > f'(c_0) - \epsilon \geqslant \min \Sigma_{\partial D}^{n,k} - \epsilon. \qquad \square$$

Since for any $\epsilon > 0$ and $G \in \mathcal{F}_{n,k}(\mathbb{R}^{2n}, \partial D)$, there exists $\overline{H} \in \mathcal{E}_{\epsilon}^{n,k}(\mathbb{R}^{2n}, \partial D)$ such that $\overline{H} \geqslant G$, we deduce that $c^{n,k}(G) \geqslant c^{n,k}(\overline{H}) \geqslant \min \Sigma_{\partial D}^{n,k} - \epsilon$. Hence $c^{n,k}(\partial D) \geqslant \min \Sigma_{\partial D}^{n,k} = \varrho$.

*Step 2. Prove that $c^{n,k}(D) \leqslant \varrho$.* Denote by $w^*$ the projections of $w$ in (4.2) onto $E^*$ (according to the decomposition $E = E^{1/2} = E^+ \oplus E^- \oplus E^0$), $* = 0, -, +$. Then $w^+ \neq 0$. (Otherwise, a contradiction occurs because $1 = A(w) = A(w^0 \oplus w^-) = -\frac{1}{2}\|w^-\|^2$.) Define $y := w/\sqrt{\varrho}$. Then $y \in C_{n,k}^1([0,1])$ satisfies $I(y) = 1$ and $A(y) = 1/\varrho$. It follows from the definition of $H^*$ that for any $\lambda \in \mathbb{R}$ and $x \in E$,

$$\lambda^2 = I(\lambda y) = \int_0^1 H^*(-\lambda J_{2n}\dot{y}(t))\,\mathrm{d}t \geqslant \int_0^1 \{\langle x(t), -\lambda J\dot{y}(t)\rangle - H(x(t))\}\,\mathrm{d}t$$

and so

$$\int_0^1 H(x(t))\,\mathrm{d}t \geqslant \int_0^1 \langle x(t), -\lambda J_{2n}\dot{y}(t)\rangle\,\mathrm{d}t - \lambda^2 = \lambda\int_0^1 \langle x(t), -J_{2n}\dot{y}(t)\rangle\,\mathrm{d}t - \lambda^2.$$

In particular, taking $\lambda = \frac{1}{2}\int_0^1 \langle x(t), -J_{2n}\dot{y}(t)\rangle \mathrm{d}t$ we arrive at

$$\int_0^1 H(x(t))\,\mathrm{d}t \geqslant \left(\frac{1}{2}\int_0^1 \langle x(t), -J_{2n}\dot{y}(t)\rangle\,\mathrm{d}t\right)^2, \quad \forall\, x \in E. \tag{4.6}$$

Since $y^+ = w^+/\sqrt{\varrho} \neq 0$ and $E^- \oplus E^0 + \mathbb{R}_+ y = E^- \oplus E^0 \oplus \mathbb{R}_+ y^+$, by proposition 3.2(ii),

$$\gamma(S^+) \cap (E^- \oplus E^0 + \mathbb{R}_+ y) \neq \emptyset, \quad \forall\, \gamma \in \Gamma_{n,k}.$$

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Fixing $\gamma \in \Gamma_{n,k}$ and $x \in \gamma(S^+) \cap (E^- \oplus E^0 + \mathbb{R}_+ y)$, write $x = x^{-0} + sy = x^{-0} + sy^{-0} + sy^+$ where $x^{-0} \in E^- \oplus E^0$, and consider the polynomial

$$P(t) = \mathfrak{a}(x + ty) = \mathfrak{a}(x) + t \int_0^1 \langle x, -J_{2n}\dot{y}\rangle \, \mathrm{d}t + \mathfrak{a}(y)t^2 = \mathfrak{a}(x^{-0} + (t+s)y).$$

Since $\mathfrak{a}|_{E^- \oplus E^0} \leqslant 0$ implies $P(-s) \leqslant 0$, and $\mathfrak{a}(y) = 1/\varrho > 0$ implies $P(t) \to +\infty$ as $|t| \to +\infty$, there exists $t_0 \in \mathbb{R}$ such that $P(t_0) = 0$. It follows that

$$\left(\int_0^1 \langle x, -J_{2n}\dot{y}\rangle \, \mathrm{d}t\right)^2 \geqslant 4\mathfrak{a}(y)\mathfrak{a}(x).$$

This and (4.6) lead to

$$\mathfrak{a}(x) \leqslant (\mathfrak{a}(y))^{-1} \left(\frac{1}{2}\int_0^1 \langle x, -J_{2n}\dot{y}\rangle \, \mathrm{d}t\right)^2 \leqslant \varrho \int_0^1 H(x(t)) \, \mathrm{d}t. \tag{4.7}$$

In order to prove that that $c^{n,k}(D) \leqslant \varrho$, it suffices to prove that for any $\varepsilon > 0$ there exists $\tilde{H} \in \mathfrak{F}_{n,k}(\mathbb{R}^{2n}, D)$ such that $c^{n,k}(\tilde{H}) < \varrho + \varepsilon$, which is reduced to prove: for any given $\gamma \in \Gamma_{n,k}$ there exists $x \in h(S^+)$ such that

$$\Phi_{\tilde{H}}(x) < \varrho + \varepsilon. \tag{4.8}$$

Now for $\tau > 0$ there exists $H_\tau \in \mathfrak{F}_{n,k}(\mathbb{R}^{2n}, D)$ such that

$$H_\tau \geqslant \tau \left(H - \left(1 + \frac{\varepsilon}{2\varrho}\right)\right). \tag{4.9}$$

For $\gamma \in \Gamma_{n,k}$ choose $x \in h(S^+)$ satisfying (4.7). We shall prove that for $\tau > 0$ large enough $\tilde{H} = H_\tau$ satisfies the requirements.

- If $\int_0^1 H(x(t)) \, \mathrm{d}t \leqslant (1 + \frac{\varepsilon}{\varrho})$, then by $H_\tau \geqslant 0$ and (4.7), we have

$$\Phi_{H_\tau}(x) \leqslant \mathfrak{a}(x) \leqslant \varrho \int_0^1 H(x(t)) \, \mathrm{d}t \leqslant \varrho \left(1 + \frac{\varepsilon}{\varrho}\right) < \varrho + \varepsilon.$$

- If $\int_0^1 H(x(t)) \, \mathrm{d}t > (1 + \varepsilon/\varrho)$, then (4.9) implies

$$\begin{aligned} \int_0^1 H_\tau(x(t)) \, \mathrm{d}t &\geqslant \tau \left(\int_0^1 H(x(t)) \, \mathrm{d}t - \left(1 + \frac{\varepsilon}{2\varrho}\right)\right) \\ &\geqslant \tau \frac{\varepsilon}{2a} \left(1 + \frac{\varepsilon}{\varrho}\right)^{-1} \int_0^1 H(x(t)) \, \mathrm{d}t \end{aligned} \tag{4.10}$$

  because

$$\begin{aligned} \left(1 + \frac{\varepsilon}{2\varrho}\right) &= \left(1 + \frac{\varepsilon}{2\varrho}\right)\left(1 + \frac{\varepsilon}{\varrho}\right)^{-1}\left(1 + \frac{\varepsilon}{\varrho}\right) \\ &< \left(1 + \frac{\varepsilon}{2\varrho}\right)\left(1 + \frac{\varepsilon}{\varrho}\right)^{-1} \int_0^1 H(x(t)) \, \mathrm{d}t \end{aligned}$$

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and

$$1 - \left(1 + \frac{\varepsilon}{2\varrho}\right)\left(1 + \frac{\varepsilon}{\varrho}\right)^{-1} = \left(1 + \frac{\varepsilon}{\varrho}\right)^{-1}\left[\left(1 + \frac{\varepsilon}{\varrho}\right) - \left(1 + \frac{\varepsilon}{2\varrho}\right)\right]$$
$$= \frac{\varepsilon}{2\varrho}\left(1 + \frac{\varepsilon}{\varrho}\right)^{-1}.$$

Choose $\tau > 0$ so large that the right side of the last equality is more than $\varrho$. Then

$$\int_0^1 H_\tau(x(t))\,\mathrm{d}t \geqslant \varrho \int_0^1 H(x(t))\,\mathrm{d}t$$

by (4.10), and hence (4.7) leads to

$$\Phi_{H_\tau}(x) = \mathfrak{a}(x) - \int_0^1 H_\tau(x(t))\,\mathrm{d}t \leqslant \mathfrak{a}(x) - \varrho \int_0^1 H(x(t))\,\mathrm{d}t \leqslant 0.$$

In summary, in the above two cases we have $\Phi_{H_\tau}(x) < \varrho + \varepsilon$. (4.8) is proved. *Step 3. Prove the final claim.* By [**26**, theorem 1.5] we have

$$c_{\mathrm{LR}}(D, D \cap \mathbb{R}^{n,k}) = \min\{A(x) > 0 \,|\, x \text{ is a leafwise chord on } \partial D \text{ for } \mathbb{R}^{n,k}\}.$$

Using proposition 1.12 and corollary 2.41 in [**28**] we can choose two sequences of $C^\infty$ strictly convex domains with boundaries, $(D_j^+)$ and $(D_j^-)$, such that

(i) $D_1^- \subset D_2^- \subset \cdots \subset D$ and $\cup_{j=1}^\infty D_j^- = D$,

(ii) $D_1^+ \supseteq D_2^+ \supseteq \cdots \supseteq D$ and $\cap_{j=1}^\infty D_j^+ = D$,

(iii) for any small neighbourhood $O$ of $\partial D$ there exists an integer $N > 0$ such that $\partial D_k^+ \cup \partial D_k^- \subset O \;\forall\, k \geqslant N$.

Now step 1–step 2 and [**26**, theorem 1.5] give rise to $c_{\mathrm{LR}}(D_j^+, D \cap \mathbb{R}^{n,k}) = c^{n,k}(D_j^+)$ and $c_{\mathrm{LR}}(D_j^-, D \cap \mathbb{R}^{n,k}) = c^{n,k}(D_j^-)$ for each $j = 1, 2, \ldots$. We have also that the sequence $c_{\mathrm{LR}}(D_j^+, D \cap \mathbb{R}^{n,k})$ converges decreasingly to $c_{\mathrm{LR}}(D, D \cap \mathbb{R}^{n,k})$ as $j \to \infty$ and that the sequence $c_{\mathrm{LR}}(D_j^-, D \cap \mathbb{R}^{n,k})$ converges increasingly to $c_{\mathrm{LR}}(D, D \cap \mathbb{R}^{n,k})$ as $j \to \infty$. Moreover for each $j$ there holds $c^{n,k}(D_j^-) \leqslant c^{n,k}(D) \leqslant c^{n,k}(D_j^+)$ by the monotonicity of $c^{n,k}$. These lead to $c^{n,k}(D) = c_{\mathrm{LR}}(D, D \cap \mathbb{R}^{n,k})$.

## 5. Proof of theorem 1.5

Clearly, the proof of theorem 1.5 can be reduced to the case that $m = 2$ and all $D_i$ are also bounded. Moreover, by an approximation argument in step 3 of § 4 we only need to prove the following:

THEOREM 5.1. *For bounded strictly convex domains $D_i \subset \mathbb{R}^{2n_i}$ with $C^2$-smooth boundary and containing the origin, $i = 1, 2$, and any integer $0 \leqslant k \leqslant n := n_1 + n_2$*

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*it holds that*

$$\begin{aligned} c^{n,k}(\partial D_1 \times \partial D_2) &= c^{n,k}(D_1 \times D_2) \\ &= \min\{c^{n_1,\min\{n_1,k\}}(D_1), c^{n_2,\max\{k-n_1,0\}}(D_2)\}. \end{aligned}$$

We first prove two lemmas. For convenience we write $E = H^{1/2}_{n,k}$ as $E_{n,k}$, and $E^*$ as $E^*_{n,k}$, $* = +, -, 0$. As a generalization of lemma 2 in [**37**, § 6.6] we have:

Lemma 5.2. *Let $D \subset \mathbb{R}^{2n}$ be a bounded strictly convex domain with $C^2$-smooth boundary and containing 0. Then for any given integer $0 \leqslant k \leqslant n$, function $H \in \mathfrak{F}_{n,k}(\mathbb{R}^{2n}, \partial D)$ and any $\epsilon > 0$ there exists $\gamma \in \Gamma_{n,k}$ such that*

$$\Phi_H|_{\gamma(B^+_{n,k} \setminus \epsilon B^+_{n,k})} \geqslant c^{n,k}(D) - \epsilon \quad \textit{and} \quad \Phi_H|_{\gamma(B^+_{n,k})} \geqslant 0, \tag{5.1}$$

*where $B^+_{n,k}$ is the closed unit ball in $E^+_{n,k}$.*

*Proof.* The case $k = n$ was proved in lemma 2 of [**37**, § 6.6]. We assume $k < n$ below. Let $S^+_{n,k} = \partial B^+_{n,k}$ and $\mathcal{E}^{n,k}_{\epsilon/2}(\mathbb{R}^{2n}, \partial D)$ be as in (4.3). Replacing $H$ by a greater function we may assume $H \in \mathcal{E}^{n,k}_{\epsilon/2}(\mathbb{R}^{2n}, \partial D)$. Since $H = 0$ near $\partial D$, by the arguments at the end of proof of proposition 3.6, the condition (3.5) may be satisfied with any $z_0 \in \mathbb{R}^{n,k} \cap \mathrm{Int}(H^{-1}(0))$. Fix such a $z_0 \in \mathbb{R}^{n,k} \cap \mathrm{Int}(H^{-1}(0))$. It follows that there exists $\alpha > 0$ such that

$$\inf \Phi_H|_{(z_0 + \alpha S^+_{n,k})} > 0 \quad \text{and} \quad \Phi_H|_{(z_0 + \alpha B^+_{n,k})} \geqslant 0, \tag{5.2}$$

(see (3.6)–(3.8) in the proof of proposition 3.6). Define $\gamma_\varepsilon : E_{n,k} \to E_{n,k}$ by $\gamma_\varepsilon(z) = z_0 + \alpha z$. It is easily seen that $\gamma_\varepsilon \in \Gamma_{n,k}$. The first inequality in (5.2) shows that $\gamma_\varepsilon(S^+_{n,k})$ belongs to the set $\mathcal{F}_{n,k} = \{\gamma(S^+_{n,k}) \,|\, \gamma \in \Gamma_{n,k} \text{ and } \inf(\Phi_H|_{\gamma(S^+_{n,k})}) > 0\}$ in (3.10). Lemma 3.7 shows that

$$c^{n,k}(H) = \sup_{F \in \mathcal{F}_{n,k}} \inf_{x \in F} \Phi_H(x),$$

and $\mathcal{F}_{n,k}$ is positively invariant under the flow $\varphi_u$ of $\nabla\Phi_H$. Define $S_u = \varphi_u(z_0 + \alpha S^+_{n,k})$ and $\mathrm{d}(H) = \sup_{u \geqslant 0} \inf(\Phi_H|_{S_u})$. It follows from these and (5.2) that

$$0 < \inf \Phi_H|_{S_0} \leqslant \mathrm{d}(H) \leqslant \sup_{F \in \mathcal{F}_{n,k}} \inf_{x \in F} \Phi_H(x) = c^{n,k}(H) < \infty. \qquad \square$$

Since $\Phi_H$ satisfies the (PS) condition by lemma 4.1, $\mathrm{d}(H)$ is a positive critical value of $\Phi_H$, and $\mathrm{d}(H) \geqslant c^{n,k}(D) - \epsilon/2$ by lemma 4.2. Moreover, by the definition of

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$d(H)$ there exists $r > 0$ such that $\Phi_H|_{S_r} \geqslant \mathrm{d}(H) - \epsilon/2$ and thus

$$\Phi_H|_{S_r} \geqslant c^{n,k}(D) - \epsilon. \tag{5.3}$$

Because $\Phi_H$ is nondecreasing along the flow $\varphi_u$, we arrive at

$$\Phi_H|_{S_u} \geqslant \Phi_H|_{S_0} \geqslant \inf(\Phi_H|_{S_0}) > 0, \quad \forall\ u \geqslant 0. \tag{5.4}$$

Define $\gamma : E_{n,k} \to E_{n,k}$ by $\gamma(x^+ + x^0 + x^-) = \widetilde{\gamma}(x^+) + x^0 + x^-$, where

$$\widetilde{\gamma}(x) = z_0 + 2(\alpha/\epsilon)x \quad \text{if } x \in E^+_{n,k} \text{ and} \|x\|_{E_{n,k}} \leqslant \frac{1}{2}\epsilon,$$

$$\widetilde{\gamma}(x) = \varphi_{r(2\|x\|_{E_{n,k}}-\epsilon)/\epsilon}(z_0 + \alpha x/\|x\|_{E_{n,k}}) \quad \text{if } x \in E^+_{n,k} \text{ and } \frac{1}{2}\epsilon < \|x\|_{E_{n,k}} \leqslant \epsilon,$$

$$\widetilde{\gamma}(x) = \varphi_r(z_0 + \alpha\ x/\|x\|_{E_{n,k}}) \quad \text{if } x \in E^+_{n,k} \text{ and } \|x\|_{E_{n,k}} > \epsilon.$$

The first and second lines imply $\gamma((\epsilon/2)B^+_{n,k}) = (z_0 + \alpha B^+_{n,k})$ and $\gamma(B^+_{n,k} \setminus (\epsilon/2)B^+_{n,k}) = \bigcup_{0\leqslant u\leqslant r} S_u$, respectively, and so

$$\gamma(B^+_{n,k}) = (z_0 + \alpha B^+_{n,k}) \bigcup_{0\leqslant u\leqslant r} S_u;$$

the third line implies $\gamma(B^+_{n,k} \setminus \epsilon B^+_{n,k}) = S_r$. It follows from these, (5.2) and (5.3)–(5.4) that $\gamma$ satisfies (5.1).

Finally, we can also know that $\gamma \in \Gamma_{n,k}$ by considering the homotopy

$$\gamma_0(x) = 2(\alpha/\epsilon)x^+ + x^0 + x^-, \quad \gamma_u(x) = \frac{1}{u}(\gamma(ux) - z_0) + z_0, \quad 0 < u \leqslant 1.$$

LEMMA 5.3. *Let integers $n_1, n_2 \geqslant 1$, $0 \leqslant k \leqslant n := n_1 + n_2$. For a bounded strictly convex domain $D \subset \mathbb{R}^{2n_1}$ with $C^2$ smooth boundary $\mathcal{S}$ and containing* 0, *it holds that*

$$c^{n,k}(D \times \mathbb{R}^{2n_2}) = c^{n_1,\min\{n_1,k\}}(D). \tag{5.5}$$

*Moreover, if $\Omega \subset \mathbb{R}^{2n_2}$ is a bounded strictly convex domain with $C^2$ smooth boundary and containing* 0, *then*

$$c^{n,k}(\mathbb{R}^{2n_1} \times \Omega) = c^{n_2,\max\{k-n_1,0\}}(\Omega).$$

*Proof.* Let $H(z) = (j_D(z))^2$ for $z \in \mathbb{R}^{n_1}$ and define

$$E_R = \{(z, z') \in \mathbb{R}^{2n_1} \times \mathbb{R}^{2n_2} \mid H(z) + (|z'|/R)^2 < 1\}.$$

By the definition and the monotonicity of $c^{n,k}$ we have

$$c^{n,k}(D \times \mathbb{R}^{2n_2}) = \sup_R c^{n,k}(E_R).$$

Since the function $\mathbb{R}^{2n_1} \times \mathbb{R}^{2n_2} \ni (z, z') \mapsto G(z, z') := H(z) + (|z'|/R)^2 \in \mathbb{R}$ is convex and of class $C^{1,1}$, $E_R$ is convex and $\mathcal{S}_R = \partial E_R$ is of class $C^{1,1}$. By theorem 1.4

we arrive at

$$c^{n,k}(E_R) = \min \Sigma^{n,k}_{\mathcal{S}_R}.$$

Let $\lambda$ be a positive number and $u = (x, x') : [0, \lambda] \to \mathcal{S}_R$ satisfy

$$\dot{u} = X_G(u) \quad \text{and} \quad u(\lambda), u(0) \in \mathbb{R}^{n,k}, \quad u(\lambda) \sim u(0). \tag{5.6}$$

Namely, $u$ is a leafwise chord on $\mathcal{S}_R$ for $\mathbb{R}^{n,k}$ with action $\lambda$. Let $k_1 = \min\{n_1, k\}$ and $k_2 = \max\{k - n_1, 0\}$. Clearly, $k_1 + k_2 = k$, and (5.6) is equivalent to the following

$$\dot{x} = X_H(x) \quad \text{and} \quad x(\lambda), x(0) \in \mathbb{R}^{n_1,k_1}, \quad x(\lambda) \sim x(0), \tag{5.7}$$

$$\dot{x}' = 2J_{2n_2}x'/R^2 \quad \text{and} \quad x'(\lambda), x'(0) \in \mathbb{R}^{n_2,k_2}, \quad x'(\lambda) \sim x'(0) \tag{5.8}$$

because $\mathbb{R}^{n,k} \equiv (\mathbb{R}^{n_1,k_1} \times \{0\}^{2n_2}) + (\{0\}^{2n_1} \times \mathbb{R}^{n_2,k_2})$. Note that nonzero constant vectors cannot be solutions of (5.7) and (5.8) and that $H(z)$ and $(|z'|/R)^2$ take constant values along solutions of (5.7) and (5.8), respectively. There exist three possibilities for solutions of (5.7) and (5.8):

- $x \equiv 0$, $|x'| = R$ and so $2\lambda/R^2 \in \pi\mathbb{N}$ if $k < n_1 + n_2$, and $2\lambda/R^2 \in 2\pi\mathbb{N}$ if $k = n_1 + n_2$ by (5.8).
- $x' \equiv 0$, $H(x) \equiv 1$ and so $\lambda \in \Sigma^{n_1,\min\{n_1,k\}}_{\mathcal{S}}$ by (5.7).
- $H(x) \equiv \delta^2 \in (0, 1)$ and $|x'|^2 = R^2(1 - \delta^2)$, where $\delta > 0$. Then $y(t) := (1/\delta)x(t)$ and $y'(t) := x'(t/\delta)$ satisfy respectively the following two lines:

$$\dot{y} = X_H(y) \quad \text{and} \quad y(\lambda), y(0) \in \mathbb{R}^{n_1,k_1}, \quad y(\lambda) \sim y(0), \quad H(y) \equiv 1,$$
$$\dot{y}' = 2J_{2n_2}y'/R^2 \quad \text{and} \quad y'(\lambda), y'(0) \in \mathbb{R}^{n_2,k_2}, \quad y'(\lambda) \sim y'(0), \quad |y'| \equiv R.$$

Hence we have also $\lambda \in \Sigma^{n_1,\min\{n_1,k\}}_{\mathcal{S}}$ by the first line, and

$$\lambda \in \frac{R^2\pi}{2}\mathbb{N} \quad \text{if } k < n_1 + n_2, \quad \lambda \in \pi R^2\mathbb{N} \text{ if } k = n_1 + n_2$$

by the second line.

In summary, we always have

$$\Sigma^{n,k}_{\mathcal{S}_R} \subset \Sigma^{n_1,\min\{n_1,k\}}_{\mathcal{S}} \bigcup \frac{R^2\pi}{2}\mathbb{N} \quad \text{if } k < n_1 + n_2, \tag{5.9}$$

$$\Sigma^{n,k}_{\mathcal{S}_R} \subset \Sigma^{n_1,\min\{n_1,k\}}_{\mathcal{S}} \bigcup R^2\pi\mathbb{N} \quad \text{if } k = n_1 + n_2. \tag{5.10}$$

A solution $x$ of (5.7) siting on $\mathcal{S}$ gives a solution $u = (x, 0)$ of (5.6) on $\mathcal{S}_R$. It follows that

$$\min \Sigma^{n,k}_{\mathcal{S}_R} = \min \Sigma^{n_1,\min\{n_1,k\}}_{\mathcal{S}}$$

for $R$ sufficiently large. (5.5) is proved.

The second claim can be proved in the similar way. □

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*Proof of theorem 5.1.* Since $D_1 \times D_2 \subset D_1 \times \mathbb{R}^{2n_2}$ and $D_1 \times D_2 \subset \mathbb{R}^{2n_1} \times D_2$, we get

$$c^{n,k}(D_1 \times D_2) \leqslant \min\{c^{n_1,\min\{n_1,k\}}(D_1), c^{n_2,\max\{k-n_1,0\}}(D_2)\}$$

by lemma 5.3. In order to prove the inverse direction inequality it suffices to prove

$$c^{n,k}(\partial D_1 \times \partial D_2) \geqslant \min\{c^{n_1,\min\{n_1,k\}}(D_1), c^{n_2,\max\{k-n_1,0\}}(D_2)\} \tag{5.11}$$

because $c^{n,k}(D_1 \times D_2) \geqslant c^{n,k}(\partial D_1 \times \partial D_2)$ by the monotonicity.

We assume $n_1 \leqslant k$. (The case $n_1 > k$ is similar!) Then (5.11) becomes

$$c^{n,k}(\partial D_1 \times \partial D_2) \geqslant \min\{c_{\mathrm{EH}}(D_1), c^{n_2,k-n_1}(D_2)\} \tag{5.12}$$

because $c^{n_1,n_1}(D_1) = c_{\mathrm{EH}}(D_1)$ by definition. Note that for each $H \in \mathcal{F}_{n,k}(\mathbb{R}^{2n}, \partial D_1 \times \partial D_2)$ we may choose $\widehat{H}_1 \in \mathcal{F}_{n_1,n_1}(\mathbb{R}^{2n_1}, \partial D_1)$ and $\widehat{H}_2 \in \mathcal{F}_{n_2,k-n_1}(\mathbb{R}^{2n_2}, \partial D_2)$ such that

$$\widehat{H}(z) := \widehat{H}_1(z_1) + \widehat{H}_2(z_2) \geqslant H(z), \quad \forall\ z.$$

Let $k_1 = n_1$ and $k_2 = n - k_1$. By lemma 5.2, for any

$$0 < \epsilon < \min\{c^{n_1,n_1}(D_1), c^{n_2,k-n_1}(D_2), 1/4\}$$

and each $i \in \{1,2\}$ there exists $\gamma_i \in \Gamma_{n_i,k_i}$ such that

$$\Phi_{\widehat{H}_i}|_{\gamma_i(B^+_{n_i,k_i} \setminus \epsilon B^+_{n_i,k_i})} \geqslant c^{n_i,k_i}(D_i) - \epsilon \quad \text{and} \quad \Phi_{\widehat{H}_i}|_{\gamma_i(B^+_{n_i,k_i})} \geqslant 0. \tag{5.13}$$

Put $\gamma = \gamma_1 \times \gamma_2$, which is in $\Gamma_{n,k}$. Since for any $x = (x_1, x_2) \in S^+_{n,k} \subset B^+_{n_1,k_1} \times B^+_{n_2,k_2}$ there exists some $j \in \{1,2\}$ such that

$$x_j \in B^+_{n_j,k_j} \setminus 4^{-1}B^+_{n_j,k_j} \subset B^+_{n_j,k_j} \setminus \epsilon B^+_{n_j,k_j},$$

it follows from this and (5.13) that

$$\Phi_{\widehat{H}}(\gamma(x)) = \Phi_{\widehat{H}_1}(\gamma_1(x_1)) + \Phi_{\widehat{H}_2}(\gamma_2(x_2)) \geqslant \min\{c^{n_1,n_1}(D_1), c^{n_2,k-n_1}(D_2)\} - \epsilon > 0$$

and hence

$$c^{n,k}(H) \geqslant c^{n,k}(\widehat{H}) = \sup_{h\in\Gamma_{n,k}} \inf_{y\in h(S^+_{n,k})} \Phi_{\widehat{H}}(y) \geqslant \min\{c^{n_1,n_1}(D_1), c^{n_2,k-n_1}(D_2)\} - \epsilon.$$

This leads to (5.12) because $c^{n_1,n_1}(D_1) = c_{\mathrm{EH}}(D_1)$. □

## 6. Proof of theorem 1.7

### 6.1. The interior of $\Sigma_{\mathcal{S}}$ is empty

Let $\lambda := \imath_X\omega_0$, and $\lambda_0 := \frac{1}{2}(qdp - pdq)$, where $(q,p)$ is the standard coordinate on $\mathbb{R}^{2n}$.

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CLAIM 6.1. For every leafwise chord on $\mathcal{S}$ for $\mathbb{R}^{n,k}$, $x:[0,T]\to\mathcal{S}$, there holds

$$A(x)=\int_x \lambda_0=\int_x \lambda. \tag{6.1}$$

*Proof.* Since $\mathcal{S}$ is of class $C^{2n+2}$, so is $x$. Define $y:[0,T]\to\mathbb{R}^{n,k}$ by $y(t)=tx(0)+(1-t)x(T)$. As below (3.15) we can take a piecewise $C^{2n+2}$-smooth map $u$ from a suitable closed disc $D^2$ to $\mathbb{R}^{2n}$ such that $u|\partial D^2$ is equal to the loop $x\cup(-y)$. Now it is easily checked that $\int_y \lambda_0=0$ and hence

$$\int_x \lambda_0=\int_{x\cup(-y)}\lambda_0=\int_{u(D^2)}\mathrm{d}\lambda_0=\int_{u(D^2)}\omega_0. \tag{6.2}$$

On the other hand, since the flow of $X$ maps $\mathbb{R}^{n,k}$ to $\mathbb{R}^{n,k}$, $X$ is tangent to $\mathbb{R}^{n,k}$ and therefore $\omega_0(X,\dot y)=0$, i.e. $y^*\lambda=0$. It follows that

$$\int_x \lambda=\int_{x\cup(-y)}\lambda=\int_{u(D^2)}\mathrm{d}\lambda=\int_{u(D^2)}\omega_0.$$

This and (6.2) lead to (6.1). □

Choosing $\varepsilon>0$ so small that $\mathbb{R}^{2n}\setminus\cup_{t\in(-\varepsilon,\varepsilon)}\phi^t(\mathcal{S})$ has two components, we obtain a very special parameterized family of $C^{2n+2}$ hypersurfaces modelled on $\mathcal{S}$, given by

$$\psi:(-\varepsilon,\varepsilon)\times\mathcal{S}\ni(s,z)\mapsto\psi(s,z)=\phi^s(z)\in\mathbb{R}^{2n}$$

which is $C^{2n+2}$ because both $\mathcal{S}$ and $X$ are $C^{2n+2}$. Define $U:=\cup_{t\in(-\varepsilon,\varepsilon)}\phi^t(\mathcal{S})$ and

$$K_\psi:U\to\mathbb{R},\quad w\mapsto\tau$$

if $w=\psi(\tau,z)\in U$ where $z\in\mathcal{S}$. This is $C^{2n+2}$. Denote by $X_{K_\psi}$ the Hamiltonian vector field of $K_\psi$ defined by $\omega_0(\cdot,X_{K_\psi})=dK_\psi$. Then it is not hard to prove

$$X_{K_\psi}(\psi(\tau,z))=\mathrm{e}^{-\tau}\,\mathrm{d}\phi^\tau(z)[X_{K_\psi}(z)]\quad\forall\ (\tau,z)\in(-\varepsilon,\varepsilon)\times\mathcal{S},$$

and for $w=\phi^\tau(z)=\psi(\tau,z)\in U$ there holds

$$\lambda_w(X_{K_\psi})=(\omega_0)_w(X(w),X_{K_\psi}(w))=\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}K_\psi(\phi^s(w))=1. \tag{6.3}$$

Let $\mathcal{S}_\tau:=\psi(\{\tau\}\times\mathcal{S})$. Since $\phi^t$ preserves the leaf of $\mathbb{R}^{n,k}$, $y:[0,T]\to\mathcal{S}_\tau$ satisfies

$$\dot y(t)=X_{K_\psi}(y(t)),\quad y(0),y(T)\in\mathbb{R}^{n,k}\quad\text{and}\quad y(T)\sim y(0)$$

if and only if $y(t)=\phi^\tau(x(\mathrm{e}^{-\tau}t))$, where $x:[0,\mathrm{e}^{-\tau}T]\to\mathcal{S}$ satisfies

$$\dot x(t)=X_{K_\psi}(x(t)),\quad x(0),x(\mathrm{e}^{-\tau}T)\in\mathbb{R}^{n,k}\quad\text{and}\quad x(\mathrm{e}^{-\tau}T)\sim x(0).$$

In addition, $y(t)=\phi^\tau(x(\mathrm{e}^{-\tau}t))$ implies $\int_y\lambda=\mathrm{e}^\tau\int_x\lambda$. By (6.1) and (6.3) we deduce

$$A(y)=\int_y\lambda_0=\int_y\lambda=\int_0^T\lambda(\dot y)\,\mathrm{d}t=\int_0^T\lambda_w(X_{K_\psi})\,\mathrm{d}t=T\quad\text{and}\quad A(x)=\mathrm{e}^{-\tau}T.$$

Fix $0<\delta<\varepsilon$. Let $\mathbf{A}_\delta$ and $\mathbf{B}_\delta$ denote the unbounded and bounded components of $\mathbb{R}^{2n}\setminus\cup_{t\in(-\delta,\delta)}\phi^t(\mathcal{S})$, respectively. Then $\psi(\{\tau\}\times\mathcal{S})\subset\mathbf{B}_\delta$ for $-\varepsilon<\tau<-\delta$. Let

$\mathcal{F}_{n,k}(\mathbb{R}^{2n})$ be given by (3.10). We call $H \in \mathcal{F}_{n,k}(\mathbb{R}^{2n})$ **adapted to** $\psi$ if

$$H(x) = \begin{cases} C_0 \geqslant 0 & \text{if } x \in \mathbf{B}_\delta, \\ f(\tau) & \text{if } x = \psi(\tau, y), \quad y \in \mathcal{S}, \;\; \tau \in [-\delta, \delta], \\ C_1 \geqslant 0 & \text{if } x \in \mathbf{A}_\delta \cap B^{2n}(0,R), \\ h(|x|^2) & \text{if } x \in \mathbf{A}_\delta \setminus B^{2n}(0,R), \end{cases} \tag{6.4}$$

where $f : (-1,1) \to \mathbb{R}$ and $h : [0, \infty) \to \mathbb{R}$ are smooth functions satisfying

$$f|_{(-1,-\delta]} = C_0, \quad f|_{[\delta,1)} = C_1, \tag{6.5}$$

$$sh'(s) - h(s) \leqslant 0 \quad \forall\ s. \tag{6.6}$$

Clearly, $H$ defined by (6.4) is $C^{2n+2}$ and its gradient $\nabla H : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfies a global Lipschitz condition.

Lemma 6.2.

(i) *If $x$ is a nonconstant critical point of $\Phi_H$ on $E$ such that $x(0) \in \psi(\{\tau\} \times \mathcal{S})$ for some $\tau \in (-\delta, \delta)$ satisfying $f'(\tau) > 0$, then*

$$\mathrm{e}^{-\tau} f'(\tau) \in \Sigma_{\mathcal{S}} \quad \textit{and} \quad \Phi_H(x) = f'(\tau) - f(\tau).$$

(ii) *If some $\tau \in (-\delta, \delta)$ satisfies $f'(\tau) > 0$ and $\mathrm{e}^{-\tau} f'(\tau) \in \Sigma_{\mathcal{S}}$, then there is a nonconstant critical point $x$ of $\Phi_H$ on $E$ such that $x(0) \in \psi(\{\tau\} \times \mathcal{S})$ and $\Phi_H(x) = f'(\tau) - f(\tau)$.*

*Proof.* (i) By lemma 2.5 $x$ is $C^{2n+2}$ and satisfies $\dot{x} = X_H(x) = f'(\tau) X_{K_\psi}(x)$, $x(j) \in \mathbb{R}^{n,k}$, $j = 0, 1$, and $x(1) \sim x(0)$. Moreover $x(0) \in \psi(\{\tau\} \times \mathcal{S})$ implies $H(x(1)) = H(x(0)) = f(\tau)$ and therefore $x(1) \in \psi(\{\tau\} \times \mathcal{S})$ by the construction of $H$ above. These show that $x$ is a leafwise chord on $\psi(\{\tau\} \times \mathcal{S})$ for $\mathbb{R}^{n,k}$. By the arguments below (6.3), $[0,1] \ni t \mapsto y(t) := \phi^{-\tau}(y(t))$ is a leafwise chord on $\mathcal{S}$ for $\mathbb{R}^{n,k}$. It follows from (6.3) and (6.1) that

$$\begin{aligned} f'(\tau) &= \int_0^1 f'(\tau) \lambda(X_{K_\psi})\,\mathrm{d}t = \int_0^1 \lambda(X_H)\,\mathrm{d}t = \int_{[0,1]} x^*\lambda = \int_{[0,1]} y^*(\phi^\tau)^*\lambda \\ &= \mathrm{e}^\tau \int_{[0,1]} y^*\lambda = \mathrm{e}^\tau A(y) \end{aligned}$$

These show that $\mathrm{e}^{-\tau} f'(\tau) = A(y) \in \Sigma_{\mathcal{S}}$. By (6.1) we have

$$\Phi_H(x) = A(x) - \int_0^1 H(x(t))\,\mathrm{d}t = \int_{[0,1]} x^*\lambda - \int_0^1 H(x(t))\,\mathrm{d}t = f'(\tau) - f(\tau).$$

(ii) By the assumption there exists $y : [0,1] \to \mathcal{S}$ satisfying

$$\dot{y}(t) = \mathrm{e}^{-\tau} f'(\tau) X_{K_\psi}(y(t)), \quad y(0), y(1) \in \mathbb{R}^{n,k} \quad \text{and} \quad y(1) \sim y(0).$$

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Hence $x(t) = \psi(\tau, y(t)) = \phi^\tau(y(t))$ satisfies

$$\begin{aligned}\dot{x}(t) &= \mathrm{d}\phi^\tau(y(t))[\dot{y}(t)] = \mathrm{e}^{-\tau} f'(\tau)\,\mathrm{d}\phi^\tau(y(t))[X_{K_\psi}(y)]\\ &= f'(\tau)X_{K_\psi}(\phi^\tau(y(t))) = f'(\tau)X_{K_\psi}(x(t)) = X_H(x(t)),\\ &x(0, x(1) \in \mathbb{R}^{n,k}, j = 0, 1, \quad x(1) \sim x(0) \in \phi^\tau(\mathcal{S}).\end{aligned}$$

By lemma 2.5, $x$ is a critical point of $\Phi_H$. Moreover $\Phi_H(x) = f'(\tau) - f(\tau)$ as in (i). □

Proposition 6.3. *Let $\mathcal{S}$ be as in theorem 1.7. Then the interior of $\Sigma_{\mathcal{S}}$ in $\mathbb{R}$ is empty.*

*Proof.* Otherwise, suppose that $T \in \Sigma_{\mathcal{S}}$ is an interior point of $\Sigma_{\mathcal{S}}$. Then for some small $0 < \epsilon_1 < \delta$ the open neighbourhood $O := \{\mathrm{e}^{-\tau}T \,|\, \tau \in (-\epsilon_1, \epsilon_1)\}$ of $T$ is contained in $\Sigma_{\mathcal{S}}$. Let us choose the function $f$ in (6.4) such that $f(u) = Tu + \overline{C} \geqslant 0\ \forall\, u \in [-\epsilon_1, \epsilon_1]$ (by shrinking $0 < \epsilon_1 < \delta$ if necessary). By lemma 6.2(ii) we deduce

$$\begin{aligned}(-\epsilon_1, \epsilon_1) \subset \big\{\tau \in (-\epsilon_1, \epsilon_1) \,|\, \mathrm{e}^{-\tau}T \in \Sigma_{\mathcal{S}}\big\} \subset \{\tau \in (-\epsilon_1, \epsilon_1) \,|\, T\\ -f(\tau) \text{ is a critical value of } \Phi_H\}\end{aligned}$$

It follows that the critical value set of $\Phi_H$ has nonempty interior. This is a contradiction by lemma 3.10. Hence $\Sigma_{\mathcal{S}}$ has empty interior. □

### 6.2. $c^{n,k}(U) = c^{n,k}(\mathcal{S})$ belongs to $\Sigma_{\mathcal{S}}$

This can be obtained by slightly modifying the proof of [**37**, theorem 7.5] (or [**25**, theorem 1.18] or [**24**, theorem 1.17]). For completeness we give it in detail. For $C > 0$ large enough and $\delta > 2\eta > 0$ small enough, define $H = H_{C,\eta} \in \mathcal{F}_{n,k}(\mathbb{R}^{2n})$ adapted to $\psi$ as follows:

$$H_{C,\eta}(x) = \begin{cases} C \geqslant 0 & \text{if } x \in \mathbf{B}_\delta,\\ f_{C,\eta}(\tau) & \text{if } x = \psi(\tau, y), \quad y \in \mathcal{S}, \ \ \tau \in [-\delta, \delta],\\ C & \text{if } x \in \mathbf{A}_\delta \cap B^{2n}(0, R),\\ h(|x|^2) & \text{if } x \in \mathbf{A}_\delta \setminus B^{2n}(0, R)\end{cases} \tag{6.7}$$

where $B^{2n}(0, R) \supseteq \overline{\psi((-\varepsilon, \varepsilon) \times \mathcal{S})}$ (the closure of $\psi((-\varepsilon, \varepsilon) \times \mathcal{S})$), $f_{C,\eta} : (-\varepsilon, \varepsilon) \to \mathbb{R}$ and $h : [0, \infty) \to \mathbb{R}$ are smooth functions satisfying

$$\begin{aligned}&f_{C,\eta}|_{[-\eta,\eta]} \equiv 0, \quad f_{C,\eta}(s) = C \text{ if } |s| \geqslant 2\eta,\\ &f'_{C,\eta}(s)s > 0 \quad \text{if } \eta < |s| < 2\eta,\\ &f'_{C,\eta}(s) - f_{C,\eta}(s) > c^{n,k}(\mathcal{S}) + 1 \quad \text{if } s > 0 \text{ and } \eta < f_{C,\eta}(s) < C - \eta,\\ &h_{C,\eta}(s) = a_H s + b \quad \text{for } s > 0 \text{ large enough}, \ a_H = C/R^2 > \frac{\pi}{2}, \ a_H \notin \frac{\pi}{2}\mathbb{N},\\ &sh'_{C,\eta}(s) - h_{C,\eta}(s) \leqslant 0 \quad \forall\, s \geqslant 0.\end{aligned}$$

We can choose such a family $H_{C,\eta}$ ($C \to +\infty$, $\eta \to 0$) to be cofinal in $\mathcal{F}^{n,k}(\mathbb{R}^{2n}, \mathcal{S})$ defined by (3.16) and also to have the property that

$$C \leqslant C' \Rightarrow H_{C,\eta} \leqslant H_{C',\eta}, \quad \eta \leqslant \eta' \Rightarrow H_{C,\eta} \geqslant H_{C,\eta'}. \tag{6.8}$$

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It follows that

$$c^{n,k}(\mathcal{S}) = \lim_{\eta\to 0, C\to+\infty} c^{n,k}(H_{C,\eta}).$$

By proposition 3.5(i) and (6.8), $\eta \leqslant \eta'$ implies that $c^{n,k}(H_{C,\eta}) \leqslant c^{n,k}(H_{C,\eta'})$, and hence

$$\Upsilon(C) := \lim_{\eta\to 0} c^{n,k}(H_{C,\eta}) \tag{6.9}$$

exists, and

$$\Upsilon(C) = \lim_{\eta\to 0} c^{n,k}(H_{C,\eta}) \geqslant \lim_{\eta\to 0} c^{n,k}(H_{C',\eta}) = \Upsilon(C'),$$

i.e. $C \mapsto \Upsilon(C)$ is non-increasing. We claim

$$c^{n,k}(\mathcal{S}) = \lim_{C\to+\infty} \Upsilon(C). \tag{6.10}$$

In fact, for any $\epsilon > 0$ there exists $\eta_0 > 0$ and $C_0 > 0$ such that $|c^{n,k}(H_{C,\eta}) - c^{n,k}(\mathcal{S})| < \epsilon$ for all $\eta < \eta_0$ and $C > C_0$. Letting $\eta \to 0$ leads to $|\Upsilon(C) - c^{n,k}(\mathcal{S})| \leqslant \epsilon$ for all $C > C_0$. (6.10) holds.

Claim 6.4. Let $\overline{\Sigma_{\mathcal{S}}}$ be the closure of $\Sigma_{\mathcal{S}}$. Then $\overline{\Sigma_{\mathcal{S}}} \subset \Sigma_{\mathcal{S}} \cup \{0\}$.

*Proof.* In fact, let $\varphi^t$ denote the flow of $X_{K_\psi}$. It is not hard to prove

$$\Sigma_{\mathcal{S}} = \{T > 0 \,|\, \exists z \in \mathcal{S}\cap\mathbb{R}^{n,k} \text{such that } \varphi^T(z) \in \mathcal{S}\cap\mathbb{R}^{n,k} \ \&\ \varphi^T(z) \sim z\}.$$

Suppose that $(T_k) \subset \Sigma_{\mathcal{S}}$ satisfy $T_k \to T_0 \geqslant 0$. Then there exists a sequence $(z_k) \subset \mathcal{S}\cap\mathbb{R}^{n,k}$ such that $\varphi^{T_k}(z_k) \in \mathcal{S}\cap\mathbb{R}^{n,k}$ and $\varphi^{T_k}(z_k) \sim z_k$ for $k = 1, 2, \ldots$. Define $\gamma_k(t) = \varphi^{T_k t}(z_k)$ for $t \in [0,1]$ and $k \in \mathbb{N}$. Then $\dot\gamma_k(t) = T_k X_{K_\psi}(\gamma_k(t))$. By the Arzelá-Ascoli theorem $(\gamma_k)$ has a subsequence converging to some $\gamma_0$ in $C^\infty([0,1],\mathcal{S})$, which satisfies the following relations

$$\begin{aligned}
&\dot\gamma_0(t) = T_0 X_{K_\psi}(\gamma_0(t)) \text{ for all } t \in [0,1],\\
&\gamma_0(0) = \lim_{k\to\infty} \gamma_k(0) = \lim_{k\to\infty} z_k \in \mathcal{S}\cap\mathbb{R}^{n,k},\\
&\gamma_0(1) = \lim_{k\to\infty} \gamma_k(1) = \lim_{k\to\infty} \varphi^{T_k}(z_k) \in \mathcal{S}\cap\mathbb{R}^{n,k},\\
&\gamma_0(1) - \gamma_0(0) = \lim_{k\to\infty} (\gamma_k(1) - \gamma_k(0)) \in V_0^{n,k}, \text{i.e.} \gamma_0(1) \sim \gamma_0(0).
\end{aligned}$$

Hence $\gamma_0(t) = \varphi^{T_0 t}(z_0)$ and $T_0 \in \Sigma_{\mathcal{S}}$ if $T_0 > 0$. It follows that $\overline{\Sigma_{\mathcal{S}}} \subset \Sigma_{\mathcal{S}} \cup \{0\}$. □

Note that so far we do not use the assumption $a_H \notin \mathbb{N}\pi/2$.

Claim 6.5. If $a_H \notin \mathbb{N}\pi/2$ then either $\Upsilon(C) \in \overline{\Sigma_{\mathcal{S}}}$ or

$$\Upsilon(C) + C \in \overline{\Sigma_{\mathcal{S}}}. \tag{6.11}$$

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*Proof.* Since $a_H \notin \mathbb{N}\pi/2$, by theorem 3.8 we get that $c^{n,k}(H_{C,\eta})$ is a positive critical value of $\Phi_{H_{C,\eta}}$ and the associated critical point $x \in E$ gives rise to a nonconstant leafwise chord sitting in the interior of $U$. Then lemma 6.2(i) yields

$$c^{n,k}(H_{C,\eta}) = \Phi_{H_{C,\eta}}(x) = f'_{C,\eta}(\tau) - f_{C,\eta}(\tau),$$

where $f'_{C,\eta}(\tau) \in \mathrm{e}^{\tau}\Sigma_{\mathcal{S}}$ and $\eta < |\tau| < 2\eta$. Choose $C > 0$ so large that $c^{n,k}(H_{C,\eta}) < c^{n,k}(\mathcal{S}) + 1$. By the choice of $f$ below (6.7) we get either $f_{C,\eta}(\tau) < \eta$ or $f_{C,\eta}(\tau) > C - \eta$. Moreover $c^{n,k}(H_{C,\eta}) > 0$ implies $f'_{C,\eta}(\tau) > f_{C,\eta}(\tau) \geqslant 0$ and so $\tau > 0$.

Take a sequence of positive numbers $\eta_n \to 0$. By the arguments above, passing to a subsequence we have the following two cases.

*Case 1.* For each $n \in \mathbb{N}$, $c^{n,k}(H_{C,\eta_n}) = f'_{C,\eta_n}(\tau_n) - f_{C,\eta_n}(\tau_n) = \mathrm{e}^{\tau_n}a_n - f_{C,\eta_n}(\tau_n)$, where $a_n \in \Sigma_{\mathcal{S}}$, $0 \leqslant f_{C,\eta_n}(\tau_n) < \eta_n$ and $\eta_n < \tau_n < 2\eta_n$.
*Case 2.* For each $n \in \mathbb{N}$, $c^{n,k}(H_{C,\eta_n}) = f'_{C,\eta_n}(\tau_n) - f_{C,\eta_n}(\tau_n) = \mathrm{e}^{\tau_n}a_n - f_{C,\eta_n}(\tau_n) = \mathrm{e}^{\tau_n}a_n - C - (f_{C,\eta_n}(\tau_n) - C)$, where $a_n \in \Sigma_{\mathcal{S}}$, $C - \eta_n < f_{C,\eta_n}(\tau_n) \leqslant C$ and $\eta_n < \tau_n < 2\eta_n$.

In case 1, since $c^{n,k}(H_{C,\eta_n}) \to \Upsilon(C)$ by (6.9), the sequence $a_n = \mathrm{e}^{-\tau_n}(c^{n,k}(H_{C,\eta_n}) + f_{C,\eta_n}(\tau_n))$ is bounded. Passing to a subsequence we may assume $a_n \to a_C \in \overline{\Sigma_{\mathcal{S}}}$. Then

$$a_C = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\mathrm{e}^{-\tau_n}(c^{n,k}(H_{C,\eta_n}) + f_{C,\eta_n}(\tau_n))\right) = \Upsilon(C)$$

because $\mathrm{e}^{-\tau_n} \to 1$ and $f_{C,\eta_n}(\tau_n) \to 0$.

Similarly, we can prove $\Upsilon(C) + C = a_C \in \overline{\Sigma_{\mathcal{S}}}$ in case 2. □

*Step 1. Prove $c^{n,k}(\mathcal{S}) \in \overline{\Sigma_{\mathcal{S}}}$.* Suppose that there exists an increasing sequence $C_n$ tending to $+\infty$ such that $C_n/R^2 \notin \mathbb{N}\pi/2$ and $\Upsilon(C_n) \in \Sigma_{\mathcal{S}}$ for each $n$. Since $(\Upsilon(C_n))$ is non-increasing we conclude

$$c^{n,k}(\mathcal{S}) = \lim_{n\to\infty} \Upsilon(C_n) \in \overline{\Sigma_{\mathcal{S}}}. \tag{6.12}$$

Otherwise, we have

$$\left.\begin{array}{l}\text{there exists } \bar{C} > 0 \text{ such that (6.11) holds}\\ \text{for each } C \in (\bar{C}, +\infty) \text{ satisfying } C/R^2 \notin \mathbb{N}\pi/2.\end{array}\right\} \tag{6.13}$$

Claim 6.6. Let $\bar{C} > 0$ be as in (6.13). Then for any $C < C'$ in $(\bar{C}, +\infty)$ there holds

$$\Upsilon(C) + C \geqslant \Upsilon(C') + C'.$$

Its proof is carried out later. Since $\Xi := \{C > \bar{C} \,|\, C \text{ satisfying } C/R^2 \notin \mathbb{N}\pi/2\}$ is dense in $(\bar{C}, +\infty)$, it follows from claim 6.6 that $\Upsilon(C') + C' \leqslant \Upsilon(C) + C$ if $C' > C$ are in $\Xi$. Fix a $C^* \in \Xi$. Then $\Upsilon(C') + C' \leqslant \Upsilon(C^*) + C^*$ for all $C' \in \{C \in \Xi \,|\, C > C^*\}$. Taking a sequence $(C'_n) \subset \{C \in \Xi \,|\, C > C^*\}$ such that $C'_n \to +\infty$, we deduce that $\Upsilon(C'_n) \to -\infty$. This contradicts the fact that $\Upsilon(C'_n) \to c^{n,k}(\mathcal{S}) > 0$. Hence (6.13) does not hold! (6.12) is proved.

*Proof of claim 6.6.* By contradiction we assume that for some $C' > C > \overline{C}$,

$$\Upsilon(C) + C < \Upsilon(C') + C'. \tag{6.14}$$

Let us prove that (6.14) implies:

$$\left.\begin{array}{l}\text{for any given } d \in (\Upsilon(C) + C, \Upsilon(C') + C') \\ \text{there exists } C_0 \in (C, C') \text{ such that } \Upsilon(C_0) + C_0 = d.\end{array}\right\} \tag{6.15}$$

Clearly, this contradicts the facts that $\mathrm{Int}(\Sigma_{\mathcal{S}}) = \emptyset$ and (6.11) holds for all large $C$ satisfying $C/R^2 \notin \mathbb{N}\pi/2$.

It remains to prove (6.15). Put $\Delta_d = \{C'' \in (C, C') \,|\, C'' + \Upsilon(C'') > d\}$. Since $\Upsilon(C') + C' > d$ and $\Upsilon(C') \leqslant \Upsilon(C'') \leqslant \Upsilon(C)$ for any $C'' \in (C, C')$ we obtain $\Upsilon(C'') + C'' > d$ if $C'' \in (C, C')$ is sufficiently close to $C'$. Hence $\Delta_d \neq \emptyset$. Set $C_0 = \inf \Delta_d$. Then $C_0 \in [C, C')$.

Let $(C''_n) \subset \Delta_d$ satisfy $C''_n \downarrow C_0$. Since $\Upsilon(C''_n) \leqslant \Upsilon(C_0)$, we have $d < C''_n + \Upsilon(C''_n) \leqslant \Upsilon(C_0) + C''_n$ for each $n \in \mathbb{N}$, and thus $d \leqslant \Upsilon(C_0) + C_0$ by letting $n \to \infty$.

We conclude $d = \Upsilon(C_0) + C_0$, and so (6.15) is proved. By contradiction suppose that

$$d < \Upsilon(C_0) + C_0. \tag{6.16}$$

Since $d > C + \Upsilon(C)$, this implies $C \neq C_0$ and so $C_0 > C$. For $\hat{C} \in (C, C_0)$, as $\Upsilon(\hat{C}) \geqslant \Upsilon(C_0)$ we derive from (6.16) that $\Upsilon(\hat{C}) + \hat{C} > d$ if $\hat{C}$ is close to $C_0$. Hence such $\hat{C}$ belongs to $\Delta_d$, which contradicts $C_0 = \inf \Delta_d$. $\square$

*Step 2. Prove* $c^{n,k}(U) = c^{n,k}(\mathcal{S})$. Note that $c^{n,k}(U) = \inf_{\eta>0, C>0} c^{n,k}(\hat{H}_{C,\eta})$, where

$$\hat{H}_{C,\eta}(x) = \begin{cases} 0 & \text{if } x \in \mathbf{B}_\delta, \\ \hat{f}_{C,\eta}(\tau) & \text{if } x = \psi(\tau, y), \quad y \in \mathcal{S}, \quad \tau \in [-\delta, \delta], \\ C & \text{if } x \in \mathbf{A}_\delta \cap B^{2n}(0, R), \\ \hat{h}(|x|^2) & \text{if } x \in \mathbf{A}_\delta \setminus B^{2n}(0, R) \end{cases}$$

where $B^{2n}(0,R) \supseteq \overline{\psi((-\varepsilon,\varepsilon) \times \mathcal{S})}$, $\hat{f}_{C,\eta} : (-\varepsilon, \varepsilon) \to \mathbb{R}$ and $\hat{h} : [0, \infty) \to \mathbb{R}$ are smooth functions satisfying the following conditions

$$\hat{f}_{C,\eta}|_{(-\infty,\eta]} \equiv 0, \quad \hat{f}_{C,\eta}(s) = C \quad \text{if } s \geqslant 2\eta,$$

$$\hat{f}'_{C,\eta}(s)s > 0 \quad \text{if } \eta < s < 2\eta,$$

$$\hat{f}'_{C,\eta}(s) - \hat{f}_{C,\eta}(s) > c^{n,k}(\mathcal{S}) + 1 \quad \text{if } s > 0 \quad \text{and} \quad \eta < \hat{f}_{C,\eta}(s) < C - \eta,$$

$$\hat{h}_{C,\eta}(s) = a_H s + b \quad \text{for } s > 0 \text{ large enough}, \; a_H = C/R^2 > \frac{\pi}{2}, \; a_H \notin \frac{\pi}{2}\mathbb{N},$$

$$s\hat{h}'_{C,\eta}(s) - \hat{h}_{C,\eta}(s) \leqslant 0 \quad \forall\, s \geqslant 0.$$

For $H_{C,\eta}$ in (6.7), choose an associated $\hat{H}_{C,\eta}$, where $\hat{f}_{C,\eta}|_{[0,\infty)} = f_{C,\eta}|_{[0,\infty)}$ and $\hat{h}_{C,\eta} = h_{C,\eta}$. Consider $H_s = sH_{C,\eta} + (1-s)\hat{H}_{C,\eta}$, $0 \leqslant s \leqslant 1$, and put $\Phi_s(x) := \Phi_{H_s}(x)$ for $x \in E$.

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It suffices to prove $c^{n,k}(H_0) = c^{n,k}(H_1)$. If $x$ is a critical point of $\Phi_s$ with $\Phi_s(x) > 0$, as in lemma 6.2, we have $x([0,1]) \in \mathcal{S}_\tau = \psi(\{\tau\} \times \mathcal{S})$ for some $\tau \in (\eta, 2\eta)$. The choice of $\hat{H}_{C,\eta}$ shows $H_s(x(t)) \equiv H_{C,\eta}(x(t))$ for $t \in [0,1]$. This implies that each $\Phi_s$ has the same positive critical value as $\Phi_{H_{C,\eta}}$. By the continuity in proposition 3.5(ii), $s \mapsto c^{n,k}(H_s)$ is continuous and takes values in the set of positive critical value of $\Phi_{H_{C,\eta}}$ (which has measure zero by Sard's theorem). Hence $s \mapsto c^{n,k}(H_s)$ is constant. We get $c^{n,k}(\hat{H}_{C,\eta}) = c^{n,k}(H_0) = c^{\Psi}_{\mathrm{EH}}(H_1) = c^{n,k}(H_{C,\eta})$.

Summarizing the above arguments we have proved that $c^{n,k}(\mathcal{S}) = c^{n,k}(U) \in \overline{\Sigma_{\mathcal{S}}}$. Noting that $c^{n,k}(U) > 0$, we deduce $c^{n,k}(\mathcal{S}) = c^{n,k}(U) \in \Sigma_{\mathcal{S}}$ by claim 6.4.

## 7. Proof of theorem 1.8

For $W^{2n}(1)$ in (1.3), note that $W^{2n}(1) \equiv \mathbb{R}^{2n-2} \times W^2(1) \supseteq \mathbb{R}^{2n-2} \times U^2(1)$ via the identification under (1.12). For each integer $0 \leqslant k < n$, (1.14) and (1.11) yield

$$c^{n,k}(W^{2n}(1)) \geqslant \min\{c^{n-1,k}(\mathbb{R}^{2n-2}), c^{1,0}(U^2(1))\} = \frac{\pi}{2}.$$

We only need to prove the inverse direction of the inequality.

Fix a number $0 < \varepsilon < \frac{1}{100}$. For $N > 2$ define

$$W^2(1,N) := \left\{(x_n, y_n) \in W^2(1) \,|\, |x_n| < N,\ |y_n| < N\right\}.$$

Let us smoothen $W^2(1)$ and $W^2(1,N)$ in the following way. Choose positive numbers $\delta_1, \delta_2 \ll 1$ and a smooth even function $g : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

(i) $g(t) = \sqrt{1-t^2}$ for $0 \leqslant t \leqslant 1 - \delta_1$,

(ii) $g(t) = 0$ for $t \geqslant 1 + \delta_2$,

(iii) $g$ is strictly monotone decreasing, and $g(t) \geqslant \sqrt{1-t^2}$ for $1 - \delta_1 \leqslant t \leqslant 1$.

Denote by

$$W^2_g(1) := \{(x_n, y_n) \in \mathbb{R}^2 \,|\, y_n < g(x_n)\},$$

and by $W^2_g(1,N)$ the open subset in $\mathbb{R}^2(x_n, y_n)$ surrounded by curves $y_n = g(x_n)$, $y_n = -N$, $x_n = N$ and $x_n = -N$ (see figure 2). Then $W^2_g(1,N)$ contains $W^2(1,N)$, and we can require $\delta_1, \delta_2$ so small that

$$0 < \mathrm{Area}(W^2_g(1,N)) - \mathrm{Area}(W^2(1,N)) < \frac{\varepsilon}{2}. \tag{7.1}$$

Take another smooth function $h : [0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(iv) $h(0) = \varepsilon/2$ and $h(t) = 0$ for $t > \varepsilon/2$,

(v) $h'(t) < 0$ and $h''(t) > 0$ for any $t \in (0, \varepsilon/2)$,

(vi) the curve $\{(t, h(t)) \,|\, 0 \leqslant t \leqslant \varepsilon/2\}$ is symmetric with respect to line $s = t$ in $\mathbb{R}^2(s,t)$.

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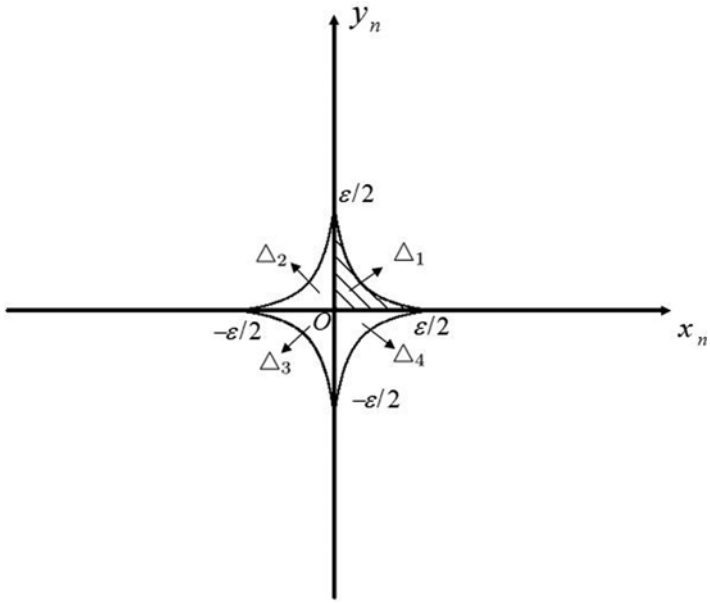


Figure 1. Domains $\triangle_i$, $i = 1, 2, 3, 4$.

Let $\triangle_1$ be the closed domain in $\mathbb{R}^2(x_n, y_n)$ surrounded by curves $y_n = h(x_n)$, $y_n = 0$ and $x_n = 0$ (see figure 1). Denote by

$$\triangle_2 = \{(x_n, y_n) \in \mathbb{R}^2 \,|\, (-x_n, y_n) \in \triangle_1\}, \quad \triangle_3 = -\triangle_1, \quad \triangle_4 = -\triangle_2.$$

Let $p_1 = (N, 0)$, $p_2 = (-N, 0)$, $p_3 = (-N, -N)$, $p_4 = (N, -N)$. Define

$$W^2_{g,\varepsilon}(1, N) = W^2_g(1, N) \setminus ((p_1 + \triangle_3) \cup (p_2 + \triangle_4) \cup (p_3 + \triangle_1) \cup (p_4 + \triangle_2)).$$

Then $W^2_{g,\varepsilon}(1, N)$ has smooth boundary (see figure 2) and

$$0 < \mathrm{Area}(W^2_g(1, N)) - \mathrm{Area}(W^2_{g,\varepsilon}(1, N)) = 4\mathrm{Area}(\triangle_1) < 4\left(\frac{\varepsilon}{2}\right)^2 = \varepsilon^2 < \frac{\varepsilon}{2}.$$

For $n > 1$ and $N > 2$ we define

$$\begin{aligned}
W^{2n}_g(1) &:= \{(x, y) \in \mathbb{R}^{2n} \,|\, (x_n, y_n) \in W^2_g(1)\} = \mathbb{R}^{2n-2} \times W^2_g(1),\\
W^{2n}(1, N) &:= \left\{(x, y) \in W^{2n}(1) \,|\, |x_n| < N,\ |y_n| < N\right\} = \mathbb{R}^{2n-2} \times W^2(1, N),\\
W^{2n}_g(1, N) &:= \{(x, y) \in \mathbb{R}^{2n} \,|\, (x_n, y_n) \in W^2_g(1, N)\} = \mathbb{R}^{2n-2} \times W^2_g(1, N),\\
W^{2n}_{g,\varepsilon}(1, N) &:= \{(x, y) \in \mathbb{R}^{2n} \,|\, (x_n, y_n) \in W^2_{g,\varepsilon}(1, N)\} = \mathbb{R}^{2n-2} \times W^2_{g,\varepsilon}(1, N).
\end{aligned}$$

Clearly, $W^{2n}_{g,\varepsilon}(1, N) \subset W^{2n}_{g,\varepsilon}(1, M)$ for any $M > N > 2$, and each bounded subset of $W^{2n}_g(1)$ can be contained in $W^{2n}_{g,\varepsilon}(1, N)$ for some large $N > 2$. It follows that

$$c^{n,k}(W^{2n}_g(1)) = \sup_{N>2}\{c^{n,k}(W^{2n}_{g,\varepsilon}(1, N))\} = \lim_{N\to+\infty} c^{n,k}(W^{2n}_{g,\varepsilon}(1, N)). \tag{7.2}$$

Let us estimate $c^{n,k}(W^{2n}_{g,\varepsilon}(1, N))$ with theorem 1.7. Regrettably, $W^2_{g,\varepsilon}(1, N)$ is not star-shaped with respect to the origin. Fortunately, it can be approximated

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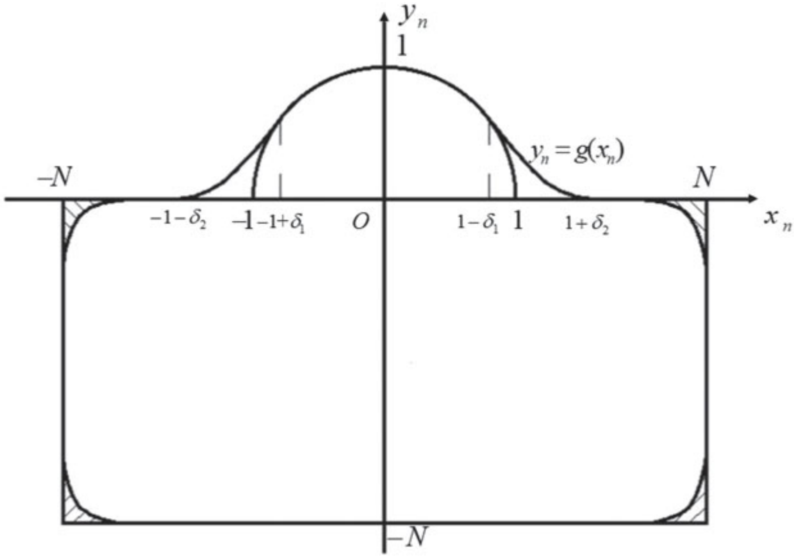


Figure 2. Domain $W^2_{g,\varepsilon}(1, N)$.

arbitrarily by star-shaped domains with respect to the origin and with smooth boundary. Indeed, for a very small $0 < \eta < \varepsilon$ the set

$$W^2_{g,\varepsilon}(1, N, \eta) := W^2_{g,\varepsilon}(1, N) \cup (W^2_{g,\varepsilon}(1, N) + (0, \eta))$$

is the desired one.

Define $j_{g,N,\epsilon,\eta} : \mathbb{R}^2 \to \mathbb{R}$ by

$$j_{g,N,\epsilon,\eta}(z_n) := \inf \left\{ \lambda > 0 \,\middle|\, \frac{z_n}{\lambda} \in W^2_{g,\varepsilon}(1, N, \eta) \right\}, \quad \forall\, z_n = (x_n, y_n) \in \mathbb{R}^2.$$

Then $j_{g,N,\epsilon,\eta}$ is positively homogeneous, and smooth in $\mathbb{R}^2 \setminus \{0\}$. For $(x, y) \in \mathbb{R}^{2n}$ we write $(x, y) = (\hat{z}, z_n)$ and define

$$W^{2n}_{g,\varepsilon,R}(1, N, \eta) := \left\{ \frac{|\hat{z}|^2}{R^2} + j^2_{g,N,\epsilon,\eta}(z_n) < 1 \right\}, \quad \forall\, R > 0.$$

Then we have $W^{2n}_{g,\varepsilon,R_1}(1, N, \eta) \subset W^{2n}_{g,\varepsilon,R_2}(1, N, \eta)$ for $R_1 < R_2$, and

$$W^{2n}_{g,\varepsilon}(1, N, \eta) = \bigcup_{R>0} W^{2n}_{g,\varepsilon,R}(1, N, \eta),$$

which implies by (3.18) that

$$c^{n,k}(W^{2n}_{g,\varepsilon}(1, N, \eta)) = \lim_{R\to+\infty} c^{n,k}(W^{2n}_{g,\varepsilon,R}(1, N, \eta)). \tag{7.3}$$

Observe that for arbitrary $N > 2$ and $R > 0$ we can shrink $0 < \eta < \varepsilon$ so that there holds

$$W^{2n}_{g,\varepsilon,R}(1, N, \eta) \subset W^{2n}_{g,\varepsilon}(1, N, \eta) \subset U^{2n}(N),$$

where for $r > 0$,

$$U^{2n}(r) := \{(x, y) \in \mathbb{R}^{2n} \,|\, x_n^2 + y_n^2 < r^2\} \cup \{(x, y) \in \mathbb{R}^{2n} \,|\, |x_n| < r \text{ and } y_n < 0\}.$$

We obtain

$$c^{n,k}(W^{2n}_{g,\varepsilon,R}(1, N, \eta)) \leqslant c^{n,k}(W^{2n}_{g,\varepsilon}(1, N, \eta)) \leqslant c^{n,k}(U^{2n}(N)) = \frac{\pi}{2}N^2. \tag{7.4}$$

Note that $W^{2n}_{g,\varepsilon,R}(1, N, \eta)$ is a star-shaped domain with respect to the origin and with smooth boundary $\mathcal{S}_{N,g,\varepsilon,R,\eta}$ transversal to the globally defined Liouville vector field $X(z) = z$. Since the flow $\phi^t$ of $X$, $\phi^t(z) = \mathrm{e}^t z$, maps $\mathbb{R}^{n,k}$ to $\mathbb{R}^{n,k}$ and preserves the leaf relation of $\mathbb{R}^{n,k}$, by theorem 1.7 we obtain

$$c^{n,k}(W^{2n}_{g,\varepsilon,R}(1, N, \eta)) \in \Sigma_{\mathcal{S}_{N,g,\varepsilon,R,\eta}}$$

where

$$\Sigma_{\mathcal{S}_{g,N,\varepsilon,R,\eta}} = \{A(x) > 0 \,|\, x \text{ is a leafwise chord on } \mathcal{S}_{N,g,\varepsilon,R,\eta} \text{ for } \mathbb{R}^{n,k}\}.$$

Arguing as in the proof of (5.9) we get that

$$\Sigma_{\mathcal{S}_{g,N,\varepsilon,R,\eta}} \subset \Sigma_{\partial W^2_{g,\varepsilon}(1,N,\eta)} \bigcup \frac{\pi R^2}{2}\mathbb{N}.$$

Hence for $R > N$, by (7.4) we have

$$c^{n,k}(W^{2n}_{g,\varepsilon,R}(1, N, \eta)) \in \Sigma_{\partial W^2_{g,\varepsilon}(1,N,\eta)}. \tag{7.5}$$

Let us compute $\Sigma_{\partial W^2_{g,\varepsilon}(1,N,\eta)}$. Note that the part of $\partial W^2_{g,\varepsilon}(1, N)$ over the line $y_n = -\frac{\varepsilon}{2}$ and between lines $x_n = -N$ and $x_n = N$ is $\{(x_n, f(x_n)) \in \mathbb{R}^2 \,|\, |x_n| \leqslant N\}$, where

$$f(x_n) = \begin{cases} -h(x_n + N) & \text{if } -N \leqslant x_n \leqslant -N + \dfrac{\varepsilon}{2}, \\ g(x_n) & \text{if } -N + \dfrac{\varepsilon}{2} < x_n < N - \dfrac{\varepsilon}{2}, \\ -h(-x_n + N) & \text{if } N - \dfrac{\varepsilon}{2} < x_n \leqslant N. \end{cases}$$

Let $t_0 \in (0, \varepsilon/2)$ be the unique number satisfying $h(t_0) = \eta$. Then there only exist two leafwise chords on $\partial W^2_{g,\varepsilon}(1, N, \eta)$ for $\mathbb{R}^{1,0}$. One is the curve in $\mathbb{R}^2(x_n, y_n)$,

$$\gamma_1 := \{(x_n, \eta + f(x_n)) \in \mathbb{R}^2 \,|\, t_0 - N \leqslant x_n \leqslant N - t_0\},$$

and the other is $\gamma_2 := \partial W^2_{g,\varepsilon}(1, N, \eta) \setminus \gamma_1$. Then $A(\gamma_1)$ is equal to the area of the domain in $\mathbb{R}^2(x_n, y_n)$ surrounded by curves $\gamma_1$ and $x_n$-axis, that is,

$$\begin{aligned} A(\gamma_1) &= \int_{t_0-N}^{N-t_0} (\eta + f(x_n))\,\mathrm{d}x_n \\ &= 2(N - t_0)\eta + \mathrm{Area}(W^2_g(1, N)) - 2N^2 - 2\int_{t_0}^{\varepsilon/2} h(t)\,\mathrm{d}t, \end{aligned} \tag{7.6}$$

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and

$$\begin{aligned} A(\gamma_2) &= 2N^2 - 2\mathrm{Area}(\triangle_1) - 2\int_0^{t_0} h(t)\,\mathrm{d}t \\ &\geqslant 2N^2 - 4\mathrm{Area}(\triangle_1) \\ &> 2N^2 - \varepsilon. \end{aligned} \tag{7.7}$$

Hence $\Sigma_{\partial W^2_{g,\epsilon}(1,N,\eta)} = \{A(\gamma_1), A(\gamma_2)\}$. Let us choose $N > 2$ so large that $(\pi/2)N^2 < 2N^2 - \varepsilon$. Then (7.4), (7.5) and (7.7) lead to

$$c^{n,k}(W^{2n}_{g,\varepsilon,R}(1,N,\eta)) = A(\gamma_1). \tag{7.8}$$

Note that $2N^2 - 4\mathrm{Area}(\triangle_1) > 2N^2 - \varepsilon$ and that (7.1) implies

$$\mathrm{Area}(W^2_g(1,N)) - 2N^2 < \mathrm{Area}(W^2(1,N)) + \frac{\varepsilon}{2} - 2N^2 = \frac{\pi}{2} + \frac{\varepsilon}{2}.$$

It follows from this, (7.6) and (7.8) that

$$c^{n,k}(W^{2n}_{g,\varepsilon,R}(1,N,\eta)) = A(\gamma_1) < \frac{\pi}{2} + \frac{\varepsilon}{2} + 2(N - t_0)\eta - 2\int_{t_0}^{\varepsilon/2} h(t)\,\mathrm{d}t.$$

For fixed $N$ and $\varepsilon$ we may choose $0 < \eta < \varepsilon$ so small that $2(N - t_0)\eta < \varepsilon/2$. Then

$$c^{n,k}(W^{2n}_{g,\varepsilon,R}(1,N,\eta)) < \frac{\pi}{2} + \varepsilon.$$

From this and (7.2)–(7.3) we derive

$$c^{n,k}(W^{2n}(1)) \leqslant c^{n,k}(W^{2n}_g(1)) \leqslant \frac{\pi}{2} + \varepsilon$$

and hence $c^{n,k}(W^{2n}(1)) \leqslant \pi/2$ by letting $\varepsilon \to 0+$.

## 8. Comparison to symmetrical Ekeland–Hofer capacities

For each $i = 1, \ldots, n$, let $e_i$ be the vector in $\mathbb{R}^{2n}$ with 1 in the $i$th position and 0s elsewhere. Then $\{e_i\}_{i=1}^n$ is an orthonormal basis for $L^n_0 := V^{n,0}_0 = \{x \in \mathbb{R}^{2n} \,|\, x = (q_1, \ldots, q_n, 0, \ldots, 0)\} = \mathbb{R}^{n,0}$. It was proved in [**26**, corollary 2.2] that $L^2([0,1], \mathbb{R}^{2n})$ has an orthogonal basis

$$\{\mathrm{e}^{m\pi t J_{2n}} e_i\}_{1 \leqslant i \leqslant n,\ m \in \mathbb{Z}},$$

and every $x \in L^2([0,1], \mathbb{R}^{2n})$ can be uniquely expanded as form $x = \sum_{m\in\mathbb{Z}} \mathrm{e}^{m\pi t J_{2n}} x_m$, where $x_m \in L^n_0$ for all $m \in \mathbb{Z}$ and satisfies $\sum_{m\in\mathbb{Z}} |x_m|^2 < \infty$.

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Noting that $V_1^{n,0} = \{0\}$, the spaces in (2.1) and (2.2) become, respectively,

$$L^2_{n,0} = \left\{ x \in L^2([0,1], \mathbb{R}^{2n}) \,\middle|\, x \overset{L^2}{=} \sum_{m\in\mathbb{Z}} \mathrm{e}^{m\pi t J_{2n}} a_m, a_m \in L_0^n,\ \sum_{m\in\mathbb{Z}} |a_m|^2 < \infty \right\}$$
$$= L^2([0,1], \mathbb{R}^{2n})$$

and

$$H^s_{n,0} = \left\{ x \in L^2([0,1], \mathbb{R}^{2n}) \,\middle|\, x \overset{L^2}{=} \sum_{m\in\mathbb{Z}} \mathrm{e}^{m\pi t J_{2n}} a_m,\ a_m \in L_0^n, \sum_{m\in\mathbb{Z}} |m|^{2s} |a_m|^2 < \infty \right\}$$

for any real $s \geqslant 0$. It follows that the space $\mathbb{E}$ in [**25**, § 1.2] is a subspace of $E = H^{1/2}_{n,0}$ in (2.3). Denote by $\widehat{\Gamma}$ the set of the admissible deformations on $\mathbb{E}$ (see [**25**, § 1.2]) and $\widehat{S}^+$ the unit sphere in $\mathbb{E}$. Then $\Gamma_{n,0}|_{\mathbb{E}} \subset \widehat{\Gamma}$ and $\widehat{S}^+ \subset S^+_{n,0}$. Note that each function $H \in C^0(\mathbb{R}^{2n}, \mathbb{R}_{\geqslant 0})$ satisfying the conditions (H1), (H2) and (H3) below [**25**, definition 1.4] is naturally $\mathbb{R}^{n,0}$-admissible. Then

$$\begin{aligned} c^{n,0}(H) &= \sup_{\gamma\in\Gamma_{n,0}} \inf_{x\in\gamma(S^+_{n,0})} \Phi_H(x) \\ &\leqslant \sup_{\gamma\in\Gamma_{n,0}} \inf_{x\in\gamma(\widehat{S}^+)} \Phi_H(x) \\ &\leqslant \sup_{\gamma\in\widehat{\Gamma}} \inf_{x\in\gamma(\widehat{S}^+)} \Phi_H(x) = c_{\mathrm{EH},\tau_0}(H). \end{aligned}$$

It follows that $c^{n,0}(B) \leqslant c_{\mathrm{EH},\tau_0}(B)$ for each $B \subset \mathbb{R}^{2n}$ intersecting with $\mathbb{R}^{n,0}$.

## Appendix A. Connectedness of the subgroup $\mathrm{Sp}(2n, k) \subset \mathrm{Sp}(2n)$ (by Kun Shi[1])

Let $e_1, \ldots, e_{2n}$ be the standard symplectic basis in the standard symplectic Euclidean space $(\mathbb{R}^{2n}, \omega_0)$. Then $\omega_0(e_i, e_j) = \omega_0(e_{n+i}, e_{n+j}) = 0$ and $\omega_0(e_i, e_{n+j}) = \delta_{ij}$ for all $1 \leqslant i, j \leqslant n$.

Claim A.1. $A \in \mathrm{Sp}(2n)$ belongs to $\mathrm{Sp}(2n, k)$ if and only if

$$A = \begin{pmatrix} I_{n+k} & \begin{pmatrix} O_{k\times(n-k)} \\ B_{(n-k)\times(n-k)} \\ O_{k\times(n-k)} \end{pmatrix} \\ O_{(n-k)\times(n+k)} & I_{n-k} \end{pmatrix} \tag{A.1}$$

for some $B_{(n-k)\times(n-k)} = (B_{(n-k)\times(n-k)})^t \in \mathbb{R}^{(n-k)\times(n-k)}$. Consequently, $tA_0 + (1-t)A_1 \in \mathrm{Sp}(2n, k)$ for any $0 \leqslant t \leqslant 1$ and $A_i \in \mathrm{Sp}(2n, k)$, $i = 0, 1$. Specially, $\mathrm{Sp}(2n, k)$ is a connected subgroup of $\mathrm{Sp}(2n)$.

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The following proof of this claim is presented by Kun Shi.

Let $A \in \mathrm{Sp}(2n,k)$. Then $Ae_i = e_i$ for $i = 1,\dots,n+k$. For $k < j \leqslant n$, suppose $Ae_{n+j} = \sum_{s=1}^{2n} a_{s(n+j)}e_s$, where $a_{st} \in \mathbb{R}$. For $1 \leqslant j \leqslant k$ and $k < l \leqslant n$, we may obtain

$$\begin{aligned}0 = \omega_0(e_{n+l}, e_{n+j}) &= \omega_0(Ae_{n+l}, Ae_{n+j}) = \omega_0(Ae_{n+l}, e_{n+j})\\ &= \sum_{s=1}^{2n} a_{s(n+l)}\omega_0(e_s, e_{n+j}) = \sum_{s=1}^{2n} a_{s(n+l)}\delta_{sj} = a_{j(n+l)} \end{aligned}\tag{A.2}$$

by a straightforward computation. Similarly, for $1 \leqslant j \leqslant n$ and $k < l \leqslant n$, we have

$$\begin{aligned}-\delta_{jl} &= \omega_0(e_{n+l}, e_j) = \omega_0(Ae_{n+l}, Ae_j) = \omega_0(Ae_{n+l}, e_j) = \sum_{s=1}^{2n} a_{s(n+l)}\omega_0(e_s, e_j)\\ &= \sum_{s=n+1}^{2n} a_{s(n+l)}\omega_0(e_s, e_j) = \sum_{i=1}^{n} a_{(n+i)(n+l)}(-\delta_{ji}) = -a_{(n+j)(n+l)}.\end{aligned}$$

It follows from this and (A.2) that $Ae_{n+l} = e_{n+l} + \sum_{j=k+1}^{n} a_{j(n+l)}e_j$. By substituting this and $Ae_{n+s} = e_{n+s} + \sum_{j=k+1}^{n} a_{j(n+s)}e_j$ into $\omega_0(e_{n+l}, e_{n+s}) = \omega_0(Ae_{n+l}, Ae_{n+s})$ we obtain $a_{j(n+l)} = a_{l(n+j)}$ for all $k < j, l \leqslant n$.

Conversely, suppose that $A \in \mathrm{Sp}(2n)$ has form (A.1), that is, $A$ satisfies: $Ae_i = e_i$ for $i = 1,\dots,n+k$, and $Ae_{n+l} = e_{n+l} + \sum_{j=k+1}^{n} a_{j(n+l)}e_j$ for $k < l \leqslant n$, where $a_{j(n+l)} = a_{l(n+j)} \in \mathbb{R}$ for $k < j, l \leqslant n$. Then it is easy to check that $A \in \mathrm{Sp}(2n,k)$.

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