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TRANSFORMATIONS OF n-SPACE WHICH PRESERVE A FIXED SQUARE-DISTANCE

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1. Introduction. Our interest here lies in the following theorem:

THEOREM 1. *Assume there is defined on* $\mathbf{R}^n$ ($n \geqq 3$) *a "square-distance" of the form*

$$d(x, y) = \sum_{i,j=1}^{n} g_{ij}(x^i - y^i)(x^j - y^j)$$

where (g_{ij}) *is a given symmetric non-singular matrix over the reals and* $x = (x^1, \ldots, x^n)$, $y = (y^1, \ldots, y^n) \in \mathbf{R}^n$. *Assume further that* f *is a bijection of* $\mathbf{R}^n$ *which preserves a given fixed square-distance* ρ, *i.e.* $d(x, y) = \rho$ *if and only if* $d(f(x), f(y)) = \rho$. *Then (unless* $\rho = 0$ *and* (g_{ij}) *is positive or negative definite)* $f(x) = Lx + f(0)$, *where* L *is a linear bijection of* R^n *satisfying* $d(Lx, Ly) = \pm d(x, y)$ *for all* $x, y \in R^n$ *(the* $-$ *sign is possible if and only if* $\rho = 0$ *and* (g_{ij}) *has signature* 0).

Several special cases of this theorem are known; see for example [**1**]–[**5**]. We establish its full generality by proving the following theorem:

THEOREM 2. *If the square-distance function is not Euclidean and* f *preserves a fixed square-distance* ρ, *then* f *preserves the square-distance* 0.

Since [**1**] covers the Euclidean case and [**5**] the non-Euclidean case with $\rho = 0$, Theorem 2 in conjunction with [**1**] and [**5**] establishes Theorem 1.

Before proceeding further, some terminology and notation are in order.

The symmetric bilinear form (,) defined by

$$(x, y) = \textstyle\sum_1^n g_{ij}x^i y^j$$

for $x, y \in \mathbf{R}^n$ makes $\mathbf{R}^n$ into a *metric vector space*; an exposition of the geometry of such spaces appears in [**6**]. If (g_{ij}) is congruent to $\pm I_n$, both the space and the square-distance function d are called *Euclidean*. When (g_{ij}) is congruent to $\pm\text{diag}\,(+1, -1, \ldots, -1)$, the space and d are called *Minkowskian*. In this case, if $(x, x) = \lambda \neq 0$ for some $x \in \mathbf{R}^n$, $\lambda \in R$, both x and λ are called *timelike* if λ and $g_{11} = \pm 1$ have the same sign, and *spacelike* if their signs differ (the terminology is borrowed from special relativity theory).

Finally, some notation: for $x, y, z, \ldots \in \mathbf{R}^n$, $\langle x, y, z, \ldots \rangle$ denotes the subspace spanned by $x, y, z, \ldots$. Also, for any subspace U of $\mathbf{R}^n$, $U^\perp$ denotes the orthogonal complement of U.

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2. Proof of theorem 2. We now assume that d is not Euclidean and that $\rho \neq 0$. For arbitrary $a \in \mathbf{R}^n$, define

$$Q(a) = \{x \mid x \in \mathbf{R}^n, d(x, a) = \rho\};$$

then $f[Q(a)] = Q[f(a)]$, i.e. "f preserves Q's".

If $Q(a) \cap Q(b) \neq \emptyset$ for all $a, b \in \mathbf{R}^n$ we may, with sufficient attention to details, generalize the basic method of [**2**]. We then show that if $Q(a) \cap Q(b) = \emptyset$ for some $a, b \in \mathbf{R}^n$, the square-distance function must be Minkowskian and the square-distance ρ timelike; the proof of Theorem 2 in this case may then be found in [**4**]. We proceed now to the details.

For distinct $a, b \in \mathbf{R}^n$, define the hyperplane $H(a, b)$ by

$$H(a, b) = \{x \mid 2(x, b - a) = (b, b) - (a, a), x \in \mathbf{R}^n\}.$$

LEMMA 1. i) (Benz [**2**]) *For distinct* $a, b \in \mathbf{R}^n$,

$$Q(a) \cap Q(b) = Q(a) \cap H(a, b) = Q(b) \cap H(a, b).$$

ii) *For distinct* $a, b \in \mathbf{R}^n$,

$$Q(a) \cap Q(b) = \{\tfrac{1}{2}(a + b) + k \mid k \in \langle b - a \rangle^{\perp},\quad 4(k, k) = 4\rho - (b - a, b - a)\}.$$

Proof. i) Any two of the equations of $Q(a)$, $Q(b)$, $H(a, b)$ imply the third.

ii) A straightforward calculation verifies that any $\frac{1}{2}(a + b) + k$ for k as described is in $Q(a) \cap Q(b)$. Conversely, given any $x \in Q(a) \cap Q(b)$ define $k = x - \frac{1}{2}(a + b) = (x - a) - \frac{1}{2}(b - a)$. Then $x \in H(a, b)$ implies $k \in \langle b - a \rangle^{\perp}$, and $x \in Q(a)$ implies $4(k, k) = 4\rho - (b - a, b - a)$.

LEMMA 2. i) (Benz [**2**]) *If* $b - a$ *is null for distinct* $a, b \in \mathbf{R}^n$, *then for some* $c \neq a, b$ *in* $\mathbf{R}^n$, $Q(a) \cap Q(c) = Q(a) \cap Q(b)$.

ii) (generalization of Benz [**2**]) *If for distinct* $a, b, c \in \mathbf{R}^n$, $Q(a) \cap Q(b) = Q(a) \cap Q(c)$, *then either* $b - a$ *is null or* $Q(a) \cap Q(b) = \emptyset$.

Proof i) Define $c = \frac{1}{2}(a + b)$; then $H(b, a) = H(c, a)$, and Lemma 1, i) completes the proof.

ii) Assume that $Q(a) \cap Q(b) = Q(a) \cap Q(c) \neq \emptyset$ for distinct a, b, c, and that $b - a$ is not null.

If $Q(a) \cap Q(b)$ is a point, Lemma 1, ii) implies that exactly one $k \in \langle b - a \rangle^{\perp}$ satisfies $4(k, k) = 4\rho - (b - a, b - a)$. Since $-k$ also satisfies this condition, $k = -k = 0$, and $Q(a) \cap Q(b) = \{(a + b)/2\}$. Similarly, $Q(a) \cap Q(c) = \{(a + c)/2\}$; thus $b = c$, a contradiction.

If $Q(a) \cap Q(b)$ is more than a point, then, since $\langle b - a \rangle^{\perp}$ is non-singular, the set of all k's in $\langle b - a \rangle^{\perp}$ with $(k, k) = \rho - \frac{1}{4}(b - a, b - a)$ is a cone or non-degenerate quadric in $\langle b - a \rangle^{\perp}$. It follows that there exist $k_1, \ldots, k_n \in \langle b - a \rangle^{\perp}$ with $(k_i, k_i) = \rho - \frac{1}{4}(b - a, b - a)$ whose endpoints do not all lie on any hyperplane in $\langle b - a \rangle^{\perp}$.

Define $y_i = \frac{1}{2}(b - a) + k_i$, $i = 1, \ldots, n$; then we easily verify that $y_i \in Q(0) \cap H(0, b - a)$. We next prove the y_i's to be linearly independent: assume that $\sum_1^n \rho_i y_i = 0$ for some $\rho_1, \ldots, \rho_n \in R$. Then

$$\tfrac{1}{2}(\textstyle\sum_1^n \rho_i)(b - a) + \sum_1^n \rho_i k_i = 0;$$

thus if $\sum_1^n \rho_i \neq 0$, $b - a \in \langle k_1, \ldots, k_n \rangle \subseteq \langle b - a \rangle^\perp$, an impossibility since $b - a$ is not null. It follows that $\sum_1^n \rho_i = 0$ and $\sum_1^n \rho_i k_i = 0$; if some $\rho_i \neq 0$, then these two conditions then imply that the endpoints of the k_i's lie on a hyperplane in $\langle b - a \rangle^\perp$, another impossibility. Thus $y_1, \ldots, y_n$ are linearly independent.

Since translations preserve Q's, we obtain from our original assumptions $Q(0) \cap Q(b - a) = Q(0) \cap Q(c - a)$; Lemma 1, i) then yields $Q(0) \cap H(0, b - a) = Q(0) \cap H(0, c - a)$. Then both $H(0, b - a)$ and $H(0, c - a)$ contain the endpoints of the linearly independent $y_1, \ldots, y_n$, and hence are equal, which implies that $b - a = \mu(c - a)$ for some $\mu \in R$. But the equations of $H(0, b - a)$ and $H(0, c - a)$ then imply that $\mu^2 - \mu = 0$, which is impossible since a, b and c are distinct.

LEMMA 3. *Assume that for distinct $a, b \in R^n$ with $b - a$ not null, $Q(a) \cap Q(b) \neq \emptyset$. Then Theorem 2 holds, i.e. $d(p, q) = 0$ if and only if $d[f(p), f(q)] = 0$.*

Proof. If $d(p, q) = 0$, $p - q$ is null; thus for some $r \neq p, q$, $Q(p) \cap Q(q) = Q(r) \cap Q(q)$. Since f preserves Q's, we have $Q[f(p)] \cap Q[f(q)] = Q[f(r)] \cap Q[f(q)]$, thus by part ii) of the previous lemma, $f(p) - f(q)$ is null, i.e. $d[f(p), f(q)] = 0$. The proof of the converse is identical.

We may now assume that for some $a \neq b$ in $\mathbf{R}^n$, $Q(a) \cap Q(b) = \emptyset$ and $b - a$ is not null.

LEMMA 4. *The square-distance d is Minkowskian and ρ is timelike.*

Proof. Assume that $Q(a) \cap Q(b) = \emptyset$ and $b - a$ is not null. Lemma 1, ii) implies that the non-singular hyperspace $\langle b - a \rangle^\perp$ contains no k with $(k, k) = \rho - \frac{1}{4}(b - a, b - a)$; thus $\langle b - a \rangle^\perp$ must be Euclidean, since any non-singular non-Euclidean space contains k's with $(k, k) = \lambda$ for any $\lambda \in R$. Hence our space is the orthogonal direct sum of a line and a Euclidean hyperspace, and is therefore Minkowskian.

Assume ρ is spacelike; then ρ and $(b - a, b - a)$ have opposite signs. But then $\rho - \frac{1}{4}(b - a, b - a)$ has sign opposite that of $(b - a, b - a)$, so some $k \in \langle b - a \rangle^\perp$ satisfies $(k, k) = \rho - \frac{1}{4}(b - a, b - a)$. Since $Q(a) \cap Q(b) = \emptyset$, Lemma 1, ii) shows that such k's do not exist, a contradiction. Thus ρ is timelike, as required.

As mentioned previously, the proof of Theorem 2 for the Minkowskian case may be found in [**4**]; we have thus demonstrated the Theorem's full generality.

One final note: with sufficient attention to the algebraic details we can easily show that the method of Benz [**2**] also generalizes to square-distances over more arbitrary fields, provided all pairs of Q's intersect. If not, then either the space is anisotropic (contains no non-zero null vectors) or it has Witt index 1 (i.e. its largest totally null subspace has dimension 1). For these cases the results of [**1**], [**4**] or herein do not generalize, since they all use the order properties of $\mathbf{R}$; thus, for fields other than $\mathbf{R}$, the complete validity of Theorem 1 remains an open question.

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