A NOTE ON CONSTRUCTIBLE LATTICES

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It is our purpose to show that the constructible orthomodular lattices defined by Janowitz in [2], are embeddable into Boolean lattices. In fact they are subdirect products of Boolean lattices, where the subdirect products are taken in the class of orthomodular posets. We shall make these notions precise. Other concepts, such as disjoint sum and constructible lattice, are defined in [1] and [2].

DEFINITION 1. If L_1 and L_2 are two orthomodular posets, then a map ϕ: $L_1 \to L_2$ is called a *homomorphism* if, for all x and y in L_1,

(i) $\phi(x^{\perp}) = \phi(x)^{\perp}$, $\phi(0) = 0$;
(ii) $x \perp y$ implies that $\phi(x \vee y) = \phi(x) \vee \phi(y)$.

If, in addition

(iii) $x \leqq y$ if and only if $\phi(x) \leqq \phi(y)$

then ϕ is called an *embedding*.

Note that this definition is stronger than Definition 1.4 of Zierler and Schlessinger [4], but that it is the same as the notion used by Kochen and Specker [3].

DEFINITION 2. Let $\{L_\alpha : \alpha \in A\}$ be a family of orthomodular posets, let $\Pi\{L_\alpha : \alpha \in A\}$ denote the direct product of this family and let i_α: $\Pi\{L_\alpha: \alpha \in A\} \to L_\alpha$ denote the natural projection of the direct product onto its component L_α. Then an orthomodular poset L is called a *subdirect product* of the family $\{L_\alpha: \alpha \in A\}$ if

(i) there is an embedding $\phi: L \to \Pi\{L_\alpha: \alpha \in A\}$;
(ii) i_α, restricted to the range of ϕ, is still onto L_α.

THEOREM 1. *Let $\{L_\alpha: \alpha \in A\}$ be a family of orthomodular posets such that, for any α in A, there exists a homomorphism h_α of L_α onto* **2**, *the two-element Boolean lattice. Then $L = DS\{L_\alpha: \alpha \in A\}$, the disjoint sum of the family $\{L_\alpha$: $\alpha \in A\}$, is a subdirect product of $\{L_\alpha: \alpha \in A\}$.*

PROOF. We define an embedding ϕ of L into $\Pi\{L_\alpha: \alpha \in A\}$.

Suppose $a \in L$; if $a = 0,1$ we put $\phi(a) = 0,1$ in $\Pi\{L_\alpha: \alpha \in A\}$ respectively.

If $a \neq 0, 1$ then it belongs to a unique L_α. If $h_\alpha(a) = 0$, we define $\phi(a)$ as a function on A by, if $\beta \in A$,

$$\phi(a)(\beta) = \begin{cases} 0_\beta & (\beta \neq \alpha), \\ a & (\beta = \alpha). \end{cases}$$

If, on the other hand, $h_\alpha(a) = 1$ then we define $\phi(a)$ by

$$\phi(a)(\beta) = \begin{cases} 1_\beta & (\beta \neq \alpha), \\ a & (\beta = \alpha). \end{cases}$$

It is routine to verify that ϕ is an embedding, and the the natural projections onto the components of the direct product are still onto when restricted to the range of ϕ.

The next theorem is our main result.

THEOREM 2. *A constructible lattice is a subdirect product of the Boolean lattices used to construct it.*

PROOF. The theorem is trivial for 0-constructible lattices as they are Boolean. We proceed by induction on the degree of constructibility: suppose the theorem true for i-constructible lattices where $i < 2n$.

First we observe that a subdirect product L of the subdirect products $\{L_\alpha: \alpha \in A\}$ is clearly a subdirect product of the components of the L_α. The direct product is obviously a subdirect product, hence the theorem is true for $(2n)$ – constructible lattices. A subdirect product of Boolean lattices admits plenty of homomorphisms onto **2**, hence the theorem is true for $(2n+1)$ – constructible lattices by Theorem 1. The proof is complete.

COROLLARY 3. *A constructible lattice is embeddable into a Boolean lattice.*

PROOF. The direct product of Boolean lattices is Boolean.

References

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